Note added in proof

Ken Ribet has pointed out to me that the extension L/F in Theorem 4.1 is in fact unramified at the discrete places of semistable reduction for A (and at the discrete places of semistable reduction for $A \times B$ in Theorem 4.2). If v is a discrete place of F of semistable reduction for A, choose distinct primes p and q greater than 2 and not equal to char(F) or to the residue characteristic of v. By the Galois criterion of semistable reduction (Proposition 3.5 on p. 350 of SGA 7I, Lecture Notes in Mathematics, Vol. 288 (Springer, Berlin)), the inertia group for v acts on A_p by a unipotent matrix of echelon 2, and therefore the ramification index at v of $F(A_p)$ over F is a power of p. Similarly for q, and therefore v is unramified in $F(A_p) \cap F(A_q)$.

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Higher algebraic K-theory of admissible abelian categories and localization theorems

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Abstract

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We define admissible abelian categories and compute the K-theory of such categories, with the aim to study and compute the K-groups of noncommutative rings and other noncommutative situations. One of the main results of this dissertation is the localization theorem.

Introduction

The purpose of this paper is to generalize the recent results of Thomason and Trobaugh on algebraic K-theory of schemes to certain noncommutative situations, that is, to establish a localization theorem and related results for algebraic K-theory of noncommutative rings and other noncommutative situations.

A localization theorem is a theorem on the local and global relationships which helps one to reduce a global problem to a local one which is usually less difficult. Quillen [11] established a localization theorem for G-theory (or K'-theory) which became a main support for his many results about G-theory of noetherian schemes. Thomason and Trobaugh [14] recently succeeded in establishing a localization theorem for K-theory of commutative rings and quite general schemes and thereby being able to give proofs of many basic results about K-theory of commutative rings and schemes. The attempt to establish localization theorems for K-theory of noncommutative rings started as early as the attempt for the commutative cases. For the story of this, see [1], [2], [4–7], [16], [17], etc.

Since the main results of this paper require more definitions, they will appear in Section 5. Instead we present two applications of the main results here:

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Theorem 6.1. *Let R be an arbitrary ring, $t_1, \ldots, t_n$ be n elements in the center of R or if not, there are $\varphi_1, \ldots, \varphi_n \in \mathrm{Aut}(R)$ such that for any $a \in R$, $t_i a = \varphi_i(a)t_i$, $\varphi_i(t_i) = t_i$, $\varphi_i \circ \varphi_j = \varphi_j \circ \varphi_i$ for all $i, j = 1, \ldots, n$. If $Rt_1 + \cdots + Rt_n = R$, then we have a homotopy equivalence*

$$K^{\mathrm{B}}(R) \to \mathrm{holim}\left(\prod_{j=1}^{n} K^{\mathrm{B}}(R[t_j^{-1}]) \rightrightarrows \prod_{i,j=1}^{n} K^{\mathrm{B}}(R[t_i^{-1}t_j^{-1}]) \substack{\leftarrow\\ \rightarrow\\ \rightarrow} \cdots\right)$$

and therefore a strongly convergent spectral sequence

$$E_2^{p,q} = H^p\left(\prod_{j=1}^{n} K_q^{\mathrm{B}}(R[t_j^{-1}]) \to \prod_{i,j=1}^{n} K_q^{\mathrm{B}}(R[t_i^{-1}t_j^{-1}]) \to \cdots\right)$$
$$\Rightarrow \mathrm{K}_{q-p}^{\mathrm{B}}(R).$$

Actually this paper was partially motivated by the above result which first appeared in correspondence between C. Weibel and T. Hodges where it was raised as kind of a conjecture and all t_i's were assumed in the center of R; and C. Weibel pointed out a proof when R is regular.

Combined with Quillen's results on filtered rings (see [11, Section 6, Theorem 7]), our main results imply:

Proposition 6.2. *Let X be a smooth variety over a field k; then the embedding from the structure sheaf $\mathcal{O}_X$ of X to the sheaf $\mathcal{D}_X$ of germs of differential operators on X induces isomorphisms of K-groups:*

$$K_n(X) \cong K_n(\mathcal{D}_X) \text{ for all } n.$$

The reader may find a brief sketch of the paper is helpful.

For greater generality and wider applications, in Section 1 we introduce the concept of admissible abelian categories (Definition 1.6.1), which is motivated by the way to glue sheaves given over a covering (cf. Proposition 1.5.1). We establish basic properties of such categories, among which an interesting one is Proposition 1.6.9.

Instead of considering locally projective objects, in Section 2 we consider perfect complexes which are a generalization of usual perfect complexes over a scheme to our context of admissible abelian categories. The category of perfect complexes will be the category to define K-theory. This technique was originally developed by A. Grothendieck and greatly exploited in [14]. One main advantage in choosing perfect complexes instead of locally projective objects is that the category of perfect complexes is a biWaldhausen category with cylinder and cocylinder functors. We are then able to take advantage of the powerful results in Waldhausen's construction of K-theory where the existence of cylinder and cocylinder functors is required, while usually the category of locally projective objects does not have cylinder and cocylinder functors. The basic characterization of a perfect complex is Proposition 2.4(a).

Section 3 consists of the definition of K-theory of admissible abelian categories and the proofs of the two basic results of this paper: the excision theorem (Theorem 3.2) and the localization theorem (Theorem 3.3) in proto-form. The K-theory defined here coincides with the usual K-theory in most cases people are interested in, for example, when the admissible abelian category is the category of all R-modules, with R an arbitrary ring; or the admissible abelian category is the category P_R^1 of all quasi-coherent sheaves over the projective line over a ring (cf. [11, Section 8.2]); or the admissible abelian category is the category of all quasi-coherent sheaves of $\mathcal{R}_X$-modules where $\mathcal{R}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras, and X is a scheme with an ample family of line bundles (for example, X is a quasi-projective scheme over an affine scheme S), etc.

Section 4 contains the proof of the projective line bundle theorem (Theorem 4.0.1) for an admissible abelian category. Section 5 follows [14] to construct negative degree K-groups, corresponding to which is the nonconnected K-theory spectrum; and extends the results obtained in Sections 3 and 4 to negative degrees and lists the main theorems this paper has obtained which are Theorem 5.2 (Bass fundamental theorem), Theorem 5.3 (excision), Theorem 5.4 (projective line bundle theorem), Theorem 5.5 (localization) and Theorem 5.7 (Mayer–Vietoris).

Section 6 contains the two applications mentioned above.

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1. Admissible abelian categories

1.0. In this section we define admissible categories and establish basic properties of such categories. One keeps in mind as a naive example the category of all quasi-coherent sheaves of $\mathcal{O}_X$-modules over a scheme X with an ample family of line bundles. The main reference for the part of category theory is [10], also [12] for torsion theories and localization.

A category is called locally small if for every object in the category, the collection of all its subobjects is a set. In this paper, we assume all the categories considered to be locally small.

First we briefly review the torsion theories and localizations over an Ab5 category. An Ab5 category is an abelian category which is cocomplete and has exact colimits. Let $\mathcal{A}$ be an Ab5 category; a torsion theory on $\mathcal{A}$ is a pair of collections of objects in $\mathcal{A}$, $\tau = (\mathcal{T}, \mathcal{F})$, such that (a) if $A \in \mathcal{T}$, $B \in \mathcal{F}$, then

$\mathrm{Hom}_{\mathscr{A}}(A, B) = 0$; (b) if $A \in \mathscr{A}$ is such that for any $B \in \mathscr{F}$, $\mathrm{Hom}_{\mathscr{A}}(A, B) = 0$, then $A \in \mathscr{T}$; if $B \in \mathscr{A}$ is such that for any $A \in \mathscr{T}$, $\mathrm{Hom}_{\mathscr{A}}(A, B) = 0$, then $B \in \mathscr{F}$. Let $\tau = (\mathscr{T}, \mathscr{F})$ be a torsion theory on $\mathscr{A}$, then $\mathscr{T}$ is closed under quotients, extensions and direct sums. Conversely, if we have a collection $\mathscr{T}$ of objects in $\mathscr{A}$ such that $\mathscr{T}$ is closed under quotients, extensions and direct sums, then we have a uniquely determined torsion theory $\tau = (\mathscr{T}, \mathscr{F})$, where $\mathscr{F} = \{B \in \mathscr{A} \mid \mathrm{Hom}_{\mathscr{A}}(A, B) = 0$, for any $A \in \mathscr{T}\}$. A torsion theory $\tau = (\mathscr{T}, \mathscr{F})$ is called hereditary if $\mathscr{T}$ is also closed under subobjects. In this paper we consider only hereditary torsion theories, and omit writing 'hereditary'. We call objects in $\mathscr{T}$ to be τ-torsion and objects in $\mathscr{F}$ to be τ-torsion free. Also, for the sake of explicitly, we use τ-Tor to denote the collection of all τ-torsion objects and τ-Free to denote the collection of all τ-torsion free objects, instead of $\mathscr{T}$ and $\mathscr{F}$. A morphism $f: A \to B$ in $\mathscr{A}$ is called a τ-isomorphism if $\ker(f)$ and $\mathrm{coker}(f)$ are τ-torsion. An object C in $\mathscr{A}$ is called τ-closed if for any τ-isomorphism $f: A \to B$, $f^*: \mathrm{Hom}_{\mathscr{A}}(B, C) \to \mathrm{Hom}_{\mathscr{A}}(A, C)$ is an isomorphism. We denote by $\mathscr{A}_\tau$ the full subcategory of $\mathscr{A}$ of all τ-closed objects. If we let τ-Tor also denote the full subcategory of $\mathscr{A}$ of all τ-torsion objects, then τ-Tor is a thick subcategory of $\mathscr{A}$, so we have the quotient functor $j_\tau^*: \mathscr{A} \to \mathscr{A}/(\tau\text{-Tor})$, which is exact. If j_τ^* has a right adjoint functor $j_{\tau*}: \mathscr{A}/(\tau\text{-Tor}) \to \mathscr{A}$, then we call τ a localizing torsion theory. In that case, we have a category equivalence between $\mathscr{A}/(\tau\text{-Tor})$ and $\mathscr{A}_\tau$, then we can let $j_{\tau*}: \mathscr{A}_\tau \hookrightarrow \mathscr{A}$ be the embedding, and choose a functor still denoted by $j_\tau^*: \mathscr{A} \to \mathscr{A}_\tau$ uniquely up to natural isomorphism, such that $j_\tau^* \circ j_{\tau*} = \mathrm{Id}_{\mathscr{A}_\tau}$, $(j_\tau^*, j_{\tau*})$ are adjoint and j_τ^* is exact. Notice that then $\mathscr{A}_\tau$ is also an Ab5 category, $j_{\tau*}$ is left exact, and the adjunction map $\mathrm{Id} \to j_{\tau*} \circ j_\tau^*$ is a τ-isomorphism. A torsion theory is a localizing torsion theory iff for any object $A \in \mathscr{A}$, there is an object $C \in \mathscr{A}_\tau$ and a τ-isomorphism $A \to C$, or iff the embedding $j_{\tau*}: \mathscr{A}_\tau \to \mathscr{A}$ has an exact adjoint functor $j_\tau^*: \mathscr{A} \to \mathscr{A}_\tau$. From now on, for a localizing torsion theory τ, we always let $j_{\tau*}: \mathscr{A}_\tau \to \mathscr{A}$ denote the embedding. Choose the adjoint functor $j_\tau^*: \mathscr{A} \to \mathscr{A}_\tau$ to be such that $j_\tau^* \circ j_{\tau*} = \mathrm{Id}_{\mathscr{A}_\tau}$, then $(j_\tau^*, j_{\tau*})$ is called a localizing adjoint pair of functors for τ.

Let $X = \{\tau, \sigma, \ldots\}$ be a set of torsion theories on $\mathscr{A}$, $\tau \leq \sigma$ means $(\tau\text{-Tor}) \subseteq (\sigma\text{-Tor})$; the union $\bigcup_{\tau \in X} \tau$ is the smallest torsion theory $\geq \tau$, for all $\tau \in X$; the intersection $\bigcap_{\tau \in X} \tau$ is the biggest torsion theory $\leq \tau$, for all $\tau \in X$, and then

$$\Big(\bigcap_{\tau \in X} \tau\Big)\text{-Tor} = \bigcap_{\tau \in X} (\tau\text{-Tor}).$$

1.0.1. Lemma. *Let $\tau \leq \sigma$ be two localizing torsion theories on $\mathscr{A}$; then:*

(a) *$\mathscr{A}_\tau \supset \mathscr{A}_\sigma$ as subcategories of $\mathscr{A}$.*

(b) *$\mathscr{A}_\tau \cap (\sigma\text{-Tor})$ is closed under subobjects, quotients, extensions, and direct sums in $\mathscr{A}_\tau$. So $\mathscr{A}_\tau \cap (\sigma\text{-Tor})$ determines a torsion theory $\bar{\sigma}$ on $\mathscr{A}_\tau$ with $\bar{\sigma}\text{-Tor} = \mathscr{A}_\tau \cap (\sigma\text{-Tor})$. Then $(\mathscr{A}_\tau)_{\bar{\sigma}} = \mathscr{A}_\sigma$, and $(j_\sigma^*|_{\mathscr{A}_\tau}, j_{\sigma*})$ becomes a localizing adjoint pair for $\bar{\sigma}$ on $\mathscr{A}_\tau$.*

Proof. Omitted. □

Thus if $\tau \leq \sigma$, we will use $\bar{\sigma}$ to denote the induced localizing torsion theory on $\mathscr{A}_\tau$, and choose $(j_\sigma^*|_{\mathscr{A}_\tau}, j_{\sigma*})$ to be the localizing adjoint pair of functors for $\bar{\sigma}$.

1.0.2. Lemma. *Let τ,σ be two localizing torsion theories on $\mathscr{A}$. If there is a natural isomorphism $(j_{\tau*} \circ j_\tau^*) \circ (j_{\sigma*} \circ j_\sigma^*) \to (j_{\sigma*} \circ j_\sigma^*) \circ (j_{\tau*} \circ j_\tau^*)$, then $\mathscr{A}_\tau \cap (\sigma\text{-Tor})$ is closed under subobjects, quotients, extensions, and direct sums in $\mathscr{A}_\tau$. So $\mathscr{A}_\tau \cap (\sigma\text{-Tor})$ determines a torsion theory $\bar{\sigma}$ on $\mathscr{A}_\tau$ with $\bar{\sigma}\text{-Tor} = \mathscr{A}_\tau \cap (\sigma\text{-Tor})$. Then $(\mathscr{A}_\tau)_{\bar{\sigma}} = \mathscr{A}_{\tau \cup \sigma}$, $j_\sigma^*|_{\mathscr{A}_\tau}: \mathscr{A}_\tau \to \mathscr{A}_{\tau \cup \sigma}$ and $j_{\sigma*}|_{\mathscr{A}_{\tau \cup \sigma}}: \mathscr{A}_{\tau \cup \sigma} \to \mathscr{A}_\tau$ become a localizing adjoint pair of functors $\bar{\sigma}$ on $\mathscr{A}_\tau$.*

Proof. Notice that $\mathscr{A}_{\tau \cup \sigma} \subseteq \mathscr{A}_\tau$ and $\mathscr{A}_{\tau \cup \sigma} \subseteq \mathscr{A}_\sigma$, then the proof is easy. □

Thus if τ,σ are two localizing torsion theories on an Ab5 category $\mathscr{A}$ such that there is a natural isomorphism $(j_{\tau*} \circ j_\tau^*) \circ (j_{\sigma*} \circ j_\sigma^*) \to (j_{\sigma*} \circ j_\sigma^*) \circ (j_{\tau*} \circ j_\tau^*)$, then $\tau \cup \sigma$ is also a localizing torsion theory on $\mathscr{A}$, and we will use $\bar{\sigma}$ to denote the localizing torsion theory on $\mathscr{A}_\tau$ induced from σ, and choose $(j_\sigma^*|_{\mathscr{A}_\tau}, j_{\sigma*}|_{\mathscr{A}_{\tau \cup \sigma}})$ to be the localizing adjoint pair of functors for $\bar{\sigma}$. See Corollary 1.5.2 for the more delicate problem of intersections of torsion theories.

1.1.0. Definition. Let $\mathscr{A}$ be an Ab5 category; a *line bundle on* $\mathscr{A}$ is an endo-equivalence of the category $F: \mathscr{A} \to \mathscr{A}$. Notice that F preserves exactness, colimits, etc. A *section* of F is a natural transformation $s: \mathrm{Id} \to F$ such that $Fs = sF: F \to F^2$. A *divisor on* $\mathscr{A}$ is a pair (s, F) with s a section of F.

Given a divisor (s, F) on $\mathscr{A}$ we construct an endo-functor $s^{-1}: \mathscr{A} \to \mathscr{A}$ with $s^{-1}A = \varinjlim (A \xrightarrow{s} FA \xrightarrow{sF} F^2A \xrightarrow{sF^2} \cdots)$ for any $A \in \mathscr{A}$. Because F and $\varinjlim$ are exact, so is s^{-1}. It is easy to see that $s^{-1} \circ s^{-1} = s^{-1}$ (notice that here we use a convention: if the morphism $A \to \varinjlim (A \xrightarrow{s} FA \xrightarrow{sF} F^2A \xrightarrow{sF^2} \cdots)$ is an isomorphism, we let $s^{-1}A = A$). Let s also denote the torsion theory on $\mathscr{A}$ with s-Tor = all those $A \in \mathscr{A}$ with $s^{-1}A = 0$. Then the embedding $\mathscr{A}_s \to \mathscr{A}$ is right adjoint to $s^{-1}: \mathscr{A} \to \mathscr{A}_s$. So s is a localizing torsion theory on $\mathscr{A}$, and we let $j_s^* = s^{-1}$.

1.1.1. Lemma. *Let (s, F) be a divisor on an* Ab5 *category $\mathscr{A}$, then the embedding $j_{s*}: \mathscr{A}_s \to \mathscr{A}$ is exact and commutes with colimits.*

Proof. First notice that

$$j_{s*} \circ j_s^* = s^{-1} = \varinjlim (\mathrm{Id} \xrightarrow{s} F \xrightarrow{sF} F^2 \xrightarrow{sF^2} \cdots)$$

commutes with colimits. Let $\{B_\alpha\}$ be a diagram in $\mathscr{A}_s$, then

$$j_{s*}\varinjlim B_\alpha = j_{s*}\varinjlim j_s^* j_{s*}B_\alpha = j_{s*}j_s^*\varinjlim j_{s*}B_\alpha$$
$$= \varinjlim j_{s*}j_s^* j_{s*}B_\alpha = \varinjlim j_{s*}B_\alpha .$$

The exactness of j_{s*} is easy to see by considering the pushout of the diagram:

$$A \hookleftarrow B \to 0 .$$

Remember that $j_{\tau*}$ is always left exact for any localizing torsion theory τ. □

1.1.2. Example. Let $\mathscr{A} = R$-Mod, the category of all left R-modules, where R is a ring, $F = \mathrm{Id}$, let s be a central element of R, and also let $s : \mathrm{Id} \to \mathrm{Id}$ be the multiplication by s. Then (s, F) is a divisor on R-Mod, $s^{-1}M = M[s^{-1}]$ for any $M \in R$-Mod, and $\mathscr{A}_s = R[s^{-1}]$-Mod.

1.1.3. Example. Let $\mathscr{A} = R$-Mod, s be an element of R such that there is an automorphism φ of R with $sx = \varphi(x)s$, $\varphi(s) = s$, for any $x \in R$. Let ${}_\varphi R$ be the R–R bimodule R with multiplication $r \otimes a \otimes b \to \varphi(r)ab$ and let $F = {}_\varphi R \otimes -$, $s : \mathrm{Id} \to F$ be induced by the bimodule morphism $R \to {}_\varphi R$ sending x to sx for any $x \in R$. Then (s, F) is a divisor on R-Mod, $s^{-1}M = M[s^{-1}]$ for any $M \in R$-Mod, and $\mathscr{A}_s = R[s^{-1}]$-Mod.

By Morita theory, if $\mathscr{A} = R$-Mod, any line bundle F on $\mathscr{A}$ is of the form $P \otimes -$ for some finitely generated projective R–R bimodule P which is invertible in the sense that there is another R–R bimodule Q such that $P \otimes Q \cong Q \otimes P \cong R$ as bimodules. A section is a bimodule morphism $s : R \to P$ such that $s \otimes 1_P = 1_P \otimes s : P \to P \otimes P$.

1.1.4. Example. Let $\mathscr{A} = \mathrm{Qcoh}(X)$, the category of all quasi-coherent sheaves of $\mathcal{O}_X$-modules over a scheme X, $\mathscr{L}$ be a line bundle on X, $s : \mathcal{O}_X \to \mathscr{L}$ a section. Set $F = \mathscr{L} \otimes -$. Then (s, F) is a divisor on $\mathrm{Qcoh}(X)$ and $\mathscr{A}_s = \mathrm{Qcoh}(X_s)$, where X_s is the nonvanishing locus for s. Let F be a line bundle on $\mathscr{A}$, $\mathscr{L} = F(\mathcal{O}_X)$. Then $\mathscr{L}$ is a line bundle in the usual sense because $\mathscr{L}|_U = P^\sim$, where U is any open affine subscheme of X and P is a projective invertible $\Gamma(U, \mathcal{O}_X)$-module and $F = \mathscr{L} \otimes -$.

1.1.5. Example. Let X be a scheme, $\mathscr{R}_X$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras, $\mathscr{A}$ = category of all sheaves of $\mathscr{R}_X$-modules $\in \mathrm{Qcoh}(X)$. Then a line bundle $\mathscr{L}$ on X induces a line bundle F on $\mathscr{A}$ with $F(\mathscr{M}) = L \otimes_{\mathcal{O}_X} \mathscr{M}$, for any $\mathscr{M} \in \mathscr{A}$. A section of $\mathscr{L}$ on X induces a section on F, and $\mathscr{A}_s$ = the category of sheaves of $\mathscr{R}_{X_s}$-modules $\in \mathrm{Qcoh}(X_s)$. In particular, if X_s is affine, then $\mathscr{A}_s = \Gamma(X_s, \mathscr{R}_X)$-Mod.

1.1.6. Example. Let $\mathscr{A} = P_R^1$, the category of modules on the projective line over R (recall [11]). An object in P_R^1 is a triple (M, θ, N), where $M \in R[T]$-Mod, $N \in R[T^{-1}]$-Mod and $\theta : M[T^{-1}] \to N[T]$ is an isomorphism of $R[T, T^{-1}]$-modules. Let $F = (\ )(1)$, i.e. for any $(M, \theta, N) \in P_R^1$, $F((M, \theta, N)) = (M, \theta, N)(1) = (M, T^{-1}\theta, N)$, and let

$$s = (1, T^{-1}) : (M, \theta, N) \to F((M, \theta, N)) = (M, T^{-1}\theta, N) .$$

Then (s, F) is a divisor on P_R^1, and

$$s^{-1}(M, \theta, N) = \varinjlim ((M, \theta, N) \xrightarrow{(1, T^{-1})} (M, T^{-1}\theta, N) \to \cdots)$$
$$= (M, \theta, N[T^{-1}]) .$$

Thus $\mathscr{A}_s$ is equivalent to the category of all $(M, \theta, M[T^{-1}])$, where $M \in R[T]$-Mod, so is equivalent to $R[T]$-Mod.

Before going ahead, we establish the following facts:

1.2.1.1. Let G_1, G_2, H_1 and H_2 be endo-functors on a category $\mathscr{A}$, $\lambda_1 : G_1 \to H_1$, $\lambda_2 : G_2 \to H_2$ be natural transformations. Then we have the canonical natural transformation $\lambda_1\lambda_2 : G_1G_2 \to H_1H_2$, where $\lambda_1\lambda_2 = H_1\lambda_2 \circ \lambda_1 G_2 = \lambda_1 H_2 \circ G_1\lambda_2$.

More generally, let $G_1, \ldots, G_n$, $H_1, \ldots, H_n$ be endo-functors on $\mathscr{A}$, $\lambda_i : G_i \to H_i$ be natural transformations. By induction we have a natural transformation

$$\lambda_1 \cdots \lambda_n : G_1 \cdots G_n = (\cdots((G_1G_2)G_3)\cdots G_n)$$
$$\to H_1 \cdots H_n = (\cdots((H_1H_2)H_3)\cdots H_n) .$$

These natural isomorphisms are subject to coherent conditions that certain diagrams commute, therefore $\lambda_1 \cdots \lambda_n$ thus defined is independent of the parentheses.

1.2.1.2. If we have a third set of endo-functors of $\mathscr{A}$, $P_1, \ldots, P_n$, and natural transformations $\mu_i : H_i \to P_i$, then the composite

$$G_1 \cdots G_n \xrightarrow{\lambda_1 \cdots \lambda_n} H_1 \cdots H_n \xrightarrow{\mu_1 \cdots \mu_n} P_1 \cdots P_n$$

is equal to

$$G_1 \cdots G_n \xrightarrow{(\mu \circ \lambda)_1 \cdots (\mu \circ \lambda)_n} P_1 \cdots P_n ,$$

where $(\mu \circ \lambda)_i = \mu_i \circ \lambda_i$. We can use induction to prove it easily.

1.2.1.3. Let $G_1, \ldots, G_n$ be a set of endo-functors on $\mathscr{A}$, natural isomorphisms $\beta_{ij} : G_iG_j \to G_jG_i$, $i, j = 1, \ldots, n$ are said to satisfy the coherent commutative condition if $\beta_{ij} \circ \beta_{ji} = \mathrm{id}$ and the following diagram commutes:

$$\begin{array}{ccccc} G_iG_jG_k & \xrightarrow{G_i\beta_{jk}} & G_iG_kG_j & \xrightarrow{\beta_{ik}G_j} & G_kG_iG_j \\ \downarrow{\scriptstyle \beta_{ij}G_k} & & & & \downarrow{\scriptstyle G_k\beta_{ij}} \\ G_jG_iG_k & \xrightarrow[G_j\beta_{ik}]{} & G_jG_kG_i & \xrightarrow[\beta_{jk}G_i]{} & G_kG_jG_i\,. \end{array}$$

If $G_1, \ldots, G_n$ and $H_1, \ldots, H_n$ are two sets of endo-functors on $\mathscr{A}$, and natural transformations $\beta_{ij} : G_iG_j \to G_jG_i$ and $\gamma_{ij} : H_iH_j \to H_jH_i$ satisfy the coherent commutative condition, and natural transformations $\lambda_i : G_i \to H_i$ are such that the diagram

$$\begin{array}{ccc} G_iG_j & \xrightarrow{\beta_{ij}} & G_jG_i \\ \downarrow{\scriptstyle \lambda_i\lambda_j} & & \downarrow{\scriptstyle \lambda_j\lambda_i} \\ H_iH_j & \xrightarrow[\gamma_{ij}]{} & H_jH_i \end{array}$$

commutes, then by coherence for any permutation ρ of $\{1, \ldots, n\}$, there is a unique natural transformation

$$\Lambda_\rho : G_1 \cdots G_n \to H_{\rho(1)} \cdots H_{\rho(n)}$$

induced by $\lambda_i\beta_{ij}$ and γ_{ij} which is independent of intermediate steps. The reader who wants to see details about coherence may consult [9].

1.2.2. Definition. A finite set of divisors $(s_1, F_1), \ldots, (s_n, F_n)$ on an Ab5 category $\mathscr{A}$ is called *compatible* if there are natural isomorphisms $\gamma_{ij} : F_iF_j \to F_jF_i$, $i, j = 1, \ldots, n$, which satisfy the coherent commutative condition as in 1.2.1.3, and further, $s_jF_i = \gamma_{ij} \circ F_is_j : F_i \to F_jF_i$.

1.2.3. Example. Let $\mathscr{A} = R$-Mod, $s_1, \ldots, s_n$ be n elements in the ring R, $\varphi_1, \ldots, \varphi_n$ be n automorphisms of R, such that for any $a \in R$, $s_ia = \varphi_i(a)s_i$, $\varphi_j(s_i) = s_i$, $\varphi_i\varphi_j = \varphi_j\varphi_i$; set $F_i = {}_{\varphi_i}R \otimes -$. Then $\{(s_1, F_1), \ldots, (s_n, F_n)\}$ becomes a set of divisors on $\mathscr{A}$ as described in Example 1.1.3, and they are compatible. In particular, if $s_1, \ldots, s_n$ are n elements in the center of a ring R, then we can let $\varphi_i = \mathrm{Id}$ for all i and get a compatible set of divisors.

1.3.1. Definition. If (s, F), (t, G) are two divisors on an Ab5 category $\mathscr{A}$ and there is a natural isomorphism $\gamma : F \to G$ such that $\gamma \circ t = s$, then we say (s, F) and (t, G) are *isomorphic*.

1.3.2. Lemma. *For two isomorphic divisors* (s, F) *and* (t, G) *with isomorphism* γ, γ *induces a natural isomorphism* $s^{-1} \to t^{-1}$.

Proof. In the diagram

$$\begin{array}{ccccc} \mathrm{Id} & \xrightarrow{s} & F & \xrightarrow{sF} & F^2 \cdots \\ \| & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} \\ \mathrm{Id} & \xrightarrow[t]{} & G & \xrightarrow[tG]{} & G^2 \cdots \end{array}$$

the vertical natural transformations are from 1.2.1.1, and the commutativity of the diagram is from 1.2.1.2. Because all the vertical maps are isomorphisms, we get a natural isomorphism on the colimits: $s^{-1} \to t^{-1}$. □

1.3.3. Let $(s_1, F_1), \ldots, (s_n, F_n)$ be compatible divisors on $\mathscr{A}$ with natural isomorphisms $\gamma_{ij} : F_iF_j \to F_jF_i$. By 1.2.1.1 we have natural transformations $s_is_j : \mathrm{Id} \to F_iF_j$; then (s_is_j, F_iF_j) is a divisor on $\mathscr{A}$ by 1.2.1.3. We call (s_is_j, F_iF_j) the intersection of (s_i, F_i) and (s_j, F_j). It is a straightforward check that (s_is_j, F_iF_j) is isomorphic to (s_js_i, F_jF_i) through γ_{ij}. More generally, let $I = \{i_1, \ldots, i_p\}$ be a p-tuple of elements of $\{1, \ldots, n\}$; then we have a divisor $(\prod_{k=1}^{p} s_{i_k}, \prod_{k=1}^{p} F_{i_k})$ on $\mathscr{A}$. If J is a q-tuple, then $\{\gamma_{ij}\}$ induce a unique isomorphism $\gamma_{IJ} : (\prod_k F_{i_k})(\prod_l F_{j_l}) \to (\prod_l F_{j_l})(\prod_k F_{i_k})$.

1.3.4. Lemma. $\{(\prod s_i, \prod F_i)\}_I$ *are compatible divisors on* $\mathscr{A}$. *In particular, for arbitrary natural numbers* $k_1, \ldots, k_n$, $\{(s_i^{k_i}, F_i^{k_i}),\ i = 1, \ldots, n\}$ *are compatible divisors on* $\mathscr{A}$.

Proof. Obvious. □

1.3.5. Lemma. *Let* $\{(s_i, F_i), i = 1, \ldots, n\}$ *be compatible divisors on* $\mathscr{A}$; *then*:

(a) *There are natural isomorphisms of functors* $(s_is_j)^{-1} \to s_i^{-1}s_j^{-1}$ *induced by* $\{\gamma_{ij}\}$.

(b) *There are natural isomorphisms* $\beta_{ij} : s_i^{-1}s_j^{-1} \to s_j^{-1}s_i^{-1}$ *induced by* $\{\gamma_{ij}\}$ *which satisfy the coherent commutative condition in* 1.2.1.3. *If*

$$\lambda_i : \mathrm{Id} \to s_i^{-1} = \varinjlim\,(\mathrm{Id} \xrightarrow{s_i} F^i \xrightarrow{s_iF_i} F^2 \to \cdots)$$

denotes the natural transformation induced by $\mathrm{Id} \to \mathrm{Id}$, *then we have*

$$\lambda_js_i^{-1} = \beta_{ij} \circ s_i^{-1}\lambda_j\,.$$

(c) $F_i(s_j\text{-Tor}) \subset s_j\text{-Tor}$. $s_i^{-1}(s_j\text{-Tor}) \subset s_j\text{-Tor}$.

Proof. (a) Let $\mathbf{Z}^+$ be the ordered set of all nonnegative integers, $G_{ij} : \mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathrm{Cat}(\mathscr{A}, \mathscr{A})$ be the functor with $G_{ij}(p, q) = F_i^p F_j^q$; then (a) is from the commutativity of colimits:

$$\varinjlim_q (\varinjlim_p G_{ij}(p, q)) \cong \varinjlim_p (\varinjlim_q G_{ij}(p, q)) \cong \varinjlim_p G_{ij}(p, p) .$$

(b) Let G^{T}_{ji} be the transpose of G_{ji}, i.e., $G^{\mathrm{T}}_{ji}(p, q) = G_{ji}(q, p)$. Then let β_{ij} be the natural isomorphism induced from the natural isomorphism $\Gamma_{ij} : G_{ij} \to G^{\mathrm{T}}_{ji}$ induced by γ_{ij} in an obvious way. The other statements are from 1.2.1.

(c) Let $A \in (s_j\text{-Tor})$. Then $s_j^{-1}(s_i^{-1}A) \cong s_i^{-1}(s_j^{-1}A) = s_i^{-1}(0) = 0$, so $s_i^{-1}(A) \in s_j\text{-Tor}$, thus $s_i^{-1}(s_j\text{-Tor}) \subset s_j\text{-Tor}$. Similarly, $s_j^{-1}(F_iA) \cong F_i(s_j^{-1}A) = 0$, so $F_i(A) \in s_j\text{-Tor}$, thus $F_i(s_j\text{-Tor}) \subset s_j\text{-Tor}$. □

1.4.1. A complex of functors $(F., d)$ on $\mathscr{A}$ is a chain (or cochain) complex in $\mathrm{Cat}(\mathscr{A}, \mathscr{A})$. We call $(F., d)$ acyclic on a subcategory $\mathscr{B} \subset \mathscr{A}$ if for all $B \in \mathscr{B}$,

$$(F.(B), d) =: \cdots \to F_n(B) \xrightarrow{d} F_{n-1}(B) \to \cdots$$

is acyclic in $\mathscr{A}$. We call $(F., d)$ acyclic mod $\mathscr{B}$ if $H_*(F.(A), d) \in \mathscr{B}$ for all $A \in \mathscr{A}$. If G is another endo-functor on $\mathscr{A}$, then $G(F., d) = (GF., Gd)$ and $(F., d)G = (F.G, dG)$ are complexes of functors on $\mathscr{A}$. If we have two complexes of functors $(F., d)$ and $(H., \partial)$ on $\mathscr{A}$, we define $(F., d)(H., \partial)$ to be the direct sum total complex of the double complex $(F_pH_q, \tilde{d})$ of functors.

1.4.2. Lemma. *If $(F., d)$ or $(H., \partial)$ is bounded, or they both are bounded from one side, then $(F., d)(H., \partial)$ is acyclic if one of the following is assumed:*

(a) *$(H., \partial)$ is acyclic and all F_n's are exact.*

(b) *$(F., d)$ is acyclic.*

Proof. Obvious. □

1.4.3. Definition. Let $(s_1, F_1), \ldots, (s_n, F_n)$ be compatible divisors on an Ab5 category $\mathscr{A}$; then

$$\prod_{i=1}^{n} (\mathrm{Id} \xrightarrow{s_i} F_i) = \mathrm{Id} \xrightarrow{\epsilon} K^{\cdot}(s_1, \ldots, s_n)$$

is a complex of functors on $\mathscr{A}$, where

$$K^{\cdot}(s_1, \ldots, s_n) = \bigoplus_{i=1}^{n} F_i \to \bigoplus_{i<j} F_iF_j \to \cdots .$$

We call $K^{\cdot}(s_1, \ldots, s_n)$ the *Koszul complex of functors on $\mathscr{A}$ with respect to* $(s_1, F_1), \ldots, (s_n, F_n)$, and call $\epsilon = (s_i)$ the *augmentation* of the Koszul complex.

1.4.4. Lemma. $\mathrm{Id} \xrightarrow{\epsilon} K^{\cdot}(s_1, \ldots, s_n)$ *is acyclic* mod $(\bigcap_{i=1}^n (s_i\text{-Tor}))$.

Proof. Because $s_i^{-1}(\mathrm{Id} \xrightarrow{s_j} F_j) \cong (\mathrm{Id} \xrightarrow{s_j} F_j)s_i^{-1}$, and $s_i^{-1}(\mathrm{Id} \xrightarrow{s_i} F_i) \cong s_i^{-1} \xrightarrow{\mathrm{id}} s_i^{-1}$ is acyclic,

$$s_i^{-1}\Big(\prod_i (\mathrm{Id} \xrightarrow{s_i} F_i)\Big) \cong \prod_{k=1}^{i-1} (\mathrm{Id} \to F_k)(s_i^{-1} \to s_i^{-1}) \prod_{k=i+1}^{n} (\mathrm{Id} \to F_k)$$

is acyclic by Lemma 1.4.2, i.e.,

$$H^*\Big(\prod_i (\mathrm{Id} \xrightarrow{s_i} F_i)\Big) \in s_i\text{-Tor} ,$$

for all i, so $\prod_i (\mathrm{id} \xrightarrow{s_i} F_i)$ is acyclic mod $(\bigcap_{i=1}^n (s_i\text{-Tor}))$. □

1.4.5. Definition. Let $(s_1, F_1), \ldots, (s_n, F_n)$ be compatible divisors on an Ab5 category $\mathscr{A}$; then

$$\prod_{i=1}^{n} (\mathrm{Id} \xrightarrow{\lambda_i} s_i^{-1}) = \mathrm{Id} \xrightarrow{\delta} \check{c}(s_1^{-1}, \ldots, s_n^{-1})$$

is a complex of functors on $\mathscr{A}$, where λ_i is as in Lemma 1.3.5(b) and

$$\check{c}(s_1^{-1}, \ldots, s_n^{-1}) = \bigoplus_{i=1}^{n} s_i^{-1} \to \bigoplus_{i<j} s_i^{-1}s_j^{-1} \to \cdots .$$

We call $\check{c}(s_1^{-1}, \ldots, s_n^{-1})$ *Čech complex of functors on $\mathscr{A}$ with respect to* $(s_1, F_1), \ldots, (s_n, F_n)$, and call $\delta = (\lambda_i)$ the *augmentation* of the Čech complex.

1.4.6. Lemma $\mathrm{Id} \xrightarrow{\delta} \check{c}(s_1^{-1}, \ldots, s_n^{-1})$ *is acyclic* mod $(\bigcap_{i=1}^n (s_i\text{-Tor}))$.

Proof. Similar to the proof of Lemma 1.4.4. □

1.5.1. Proposition. *Let $(s_1, F_1), \ldots, (s_n, F_n)$ be compatible divisors on an* Ab5 *category $\mathscr{A}$, denote by Σ the category of data $(M_i, \theta_{ij})_{i,j}$, where $M_i \in \mathscr{A}_{s_i}$, $\theta_{ij} : s_i^{-1}(M_j) \xrightarrow{\cong} s_j^{-1}(M_i)$ for $i < j$, such that for any $i < j < k$, the following diagram commutes:*

$$\begin{array}{ccccc}
s_i^{-1}s_j^{-1}M_k & \xrightarrow{\beta_{ij}} & s_j^{-1}s_i^{-1}M_k & \xrightarrow{s_j^{-1}\theta_{ik}} & s_j^{-1}s_k^{-1}M_i \\
\downarrow{\scriptstyle s_i^{-1}\theta_{jk}} & & & & \downarrow{\scriptstyle \beta_{jk}} \\
s_i^{-1}s_k^{-1}M_j & \xrightarrow[\beta_{ik}]{} & s_k^{-1}s_i^{-1}M_j & \xrightarrow[s_k^{-1}\theta_{ij}]{} & s_k^{-1}s_j^{-1}M_i ;
\end{array}$$

a morphism in Σ is $f = (f_i) : (M_i, \theta_{ij}) \to (N_i, n_{ij})$, where $f_i : M_i \to N_i$ such that $s_j^{-1} f_i \circ \theta_{ij} = \eta_{ij} \circ s_i^{-1} f_j$.

Then there is a category equivalence between $\mathscr{A}_\sigma$ and Σ, where $\sigma = \bigcap s_i$.

Proof. Define $G : \mathscr{A} \to \Sigma$ by $G(A) = (s_i^{-1}A, \beta_{ij})$ for any $A \in \mathscr{A}$, where β_{ij} is as in Lemma 1.3.5(b), and $H : \Sigma \to \mathscr{A}$ by $H((M_i, \theta_{ij})) = P$ for any $(M_i, \theta_{ij}) \in \Sigma$, where P is the equalizer of the following fork:

$$\prod_i M_i \underset{g=(g_{ij})_{i<j}}{\overset{f=(f_{ij})_{i<j}}{\rightrightarrows}} \prod_{i<j} s_i^{-1}(M_j)\,,$$

where f_{ij} is the composite

$$\prod_i M_i \xrightarrow{\pi_j} M_j \xrightarrow{\lambda_i} s_i^{-1}(M_j) \xrightarrow{\theta_{ij}} s_j^{-1}(M_i)$$

and g_{ij} is the composite

$$\prod_i M_i \xrightarrow{\pi_i} M_i \xrightarrow{\lambda_j} s_j^{-1}(M_i)\,,$$

and the π_i's are projections. In order to prove $\mathscr{A}_\sigma \xrightarrow{\cong} \Sigma$, since $\mathscr{A}_\sigma \xrightarrow{\cong} \mathscr{A}/(\sigma\text{-Tor})$, we need to prove $\mathscr{A}/(\sigma\text{-Tor}) \xrightarrow{\cong} \Sigma$. By [10, Theorem 4.9], we need to show that G is exact, which is obvious, and that H is a full and faithful right adjoint functor of G. By the universal property of an equalizer, for any $A \in \mathscr{A}$ and $(M_i, \theta_{ij}) \in \Sigma$, we have a natural isomorphism

$$\mathrm{Hom}_{\mathscr{A}}(A, P) \xrightarrow{\cong} \mathrm{Hom}_{\Sigma}((s_i^{-1}A, \beta_{ij}), (M_i, \theta_{ij}))\,,$$

i.e., H is right adjoint to G. To prove H is full and faithful, we need to prove that the adjunction morphism $GH \to \mathrm{Id}$ is an isomorphism, i.e., $s_i^{-1}(P) \xrightarrow{\cong} M_i$ for all i. Since s_l^{-1} is exact, we get the exact sequence

$$0 \to s_l^{-1}(P) \to \prod_i s_l^{-1}(M_i) \underset{s_l^{-1}g}{\overset{s_l^{-1}f}{\rightrightarrows}} \prod_{i<j} s_l^{-1} s_i^{-1}(M_j)\,.$$

Consider M_l as an object in $\mathscr{A}$; then $G(M_l) = (s_i^{-1}(M_l), \beta_{il}) \in \Sigma$. We claim that the equalizer of the fork from $G(M_l)$ is M_l itself, i.e.,

$$0 \to M_l \xrightarrow{(s_i^{-1})} \prod_i s_i^{-1}(M_l) \underset{g}{\overset{f}{\rightrightarrows}} \prod_{i<j} s_i^{-1} s_j^{-1}(M_l)$$

is exact. In fact, let $h = (h_i) : N \to \prod_i s_i^{-1}(M_l)$ be such that $f \circ h = g \circ h$, where $N \in \mathscr{A}$; then $\theta_{ij} \circ \lambda_i \circ h_j = \lambda_j \circ h_i$ for all i, j; in particular, $\theta_{il} \circ \lambda_i \circ h_l = \lambda_l \circ h_i$ for all i. Because

$$\lambda_l : s_i^{-1}(M_l) \to s_l^{-1}(s_i^{-1}(M_l)) \cong s_i^{-1} s_l^{-1}(M_l) = s_i^{-1}(M_l)$$

is an isomorphism, we have $h_i = \lambda_l^{-1} \circ \theta_{il} \circ \lambda_i \circ h_l$ for all i, i.e., h_i is uniquely

determined by h_l. Let $k = h_l : N \to M_l = s_l^{-1}(M_l)$; then $((s_i^{-1})) \circ k = h$, so M_l is indeed the equalizer of the fork.

Because the following diagram commutes:

$$\begin{array}{ccc} \prod_i s_i^{-1}(M_l) & \rightrightarrows & \prod_{i<j} s_i^{-1}(s_j^{-1}(M_l)) \\ \downarrow{\scriptstyle (\theta_{il})} & & \downarrow{\scriptstyle (\beta_{il} \circ s_i^{-1}(\theta_{jl}))} \\ \prod_i s_l^{-1}(M_i) & \rightrightarrows & \prod_{i<j} s_l^{-1}(s_i^{-1}(M_j))\,, \end{array}$$

and the vertical morphisms are isomorphisms, the rows have isomorphic kernels, i.e., $s_l^{-1}(P) \cong M_l$. □

1.5.2. Corollary. *Let $(s_1, F_1), \ldots, (s_n, F_n)$ be compatible divisors on an* Ab5 *category $\mathscr{A}$, then $\sigma = \bigcap_{i=1}^n s_i$ is also a localizing torsion theory on $\mathscr{A}$, and the functor $j_\sigma^* : \mathscr{A} \to \mathscr{A}_\sigma$ can be chosen to send $A \in \mathscr{A}$ to the equalizer of $(s_i^{-1}(A), \beta_{ij})$.* □

1.6.1. Definition. An *admissible abelian category* is an Ab5 category $\mathscr{A}$ provided with a finite set of compatible divisors $\{(s_i, F_i),\ i = 1, \ldots, n\}$ such that:

(a) Each $\mathscr{A}_{s_i}$ has a set of small projective generators.

(b) $\bigcap_{i=1}^n (s_i\text{-Tor}) = 0$.

Recall that an object $A \in \mathscr{A}$ is called small if for any morphism $A \to \coprod_{i \in I} B_i$, where $B_i \in \mathscr{A}$, the image is contained in $\coprod_{j \in J} B_j$ for some finite subset J of I.

1.6.2. Example. In Example 1.2.3, if we further assume that $Rs_1 + \cdots + Rs_n = R$, then $\{R\text{-Mod}, (s_i, F_i), i = 1, \ldots, n\}$ becomes an admissible abelian category.

1.6.3. Example. Let $\mathscr{A} = P_R^1$, the category of modules on the projective line bundle over a ring R, and $F = (\ )(1) : P_R^1 \to P_R^1$ as in Example 1.1.6 be the functor sending (M, θ, N) to $F(M, \theta, N) = (M, T^{-1}\theta, N)$; then

$$s_1 = (1, T^{-1}) : (M, \theta, N) \to F(M, \theta, N)\,,$$

and

$$s_2 = (T, 1) : (M, \theta, N) \to F(M, \theta, N)$$

are two sections of F, and (s_1, F) and (s_2, F) are compatible. Since

$$s_1\text{-Tor} = \{(0, 0, N) \in P_R^1 \mid N \in R[T^{-1}]\text{-Mod with } N[T] = 0\}\,,$$

$$s_2\text{-Tor} = \{(M, 0, 0) \in P_R^1 \mid M \in R[T]\text{-Mod with } M[T^{-1}] = 0\}\,.$$

so $(s_1\text{-Tor}) \cap (s_2\text{-Tor}) = 0$, and because $(P^1_R)_{s_1} \cong R[T]$-Mod and $(P^1_R)_{s_2} \cong R[T^{-1}]$-Mod, both of which have a set of small projective generators, $\{P^1_R, (s_1, F), (s_2, F)\}$ is admissible.

The following lemma gives us more examples of admissible abelian categories:

1.6.4. Lemma. (a) *Let $(s_1, F_1), \dots, (s_n, F_n)$ be compatible divisors on an* Ab5 *category $\mathscr{A}$, such that $\mathscr{A}_{s_i}$ has a set of small projective generators for all i, $\sigma = \bigcap_{i=1}^n s_i$. Then $\{\mathscr{A}_\sigma, (\bar{s}_i, \bar{F}_i), i = 1, \dots, n\}$ is an admissible abelian category, where $(\bar{s}_i, \bar{F}_i)$ is the divisor on $\mathscr{A}_\sigma$ induced by (s_i, F_i).*

(b) *Let $\{\mathscr{A}, (s_i, F_i), i = 1, \dots, n\}$ be an admissible abelian category, $\{(t_j, G_j), j = 1, \dots, m\}$ be another finite set of divisors on $\mathscr{A}$ such that $\{(s_i, F_i), (t_j, G_j)\}$ are compatible, $\tau = \bigcap_{j=1}^m t_j$. Then $\{\mathscr{A}_\tau, (\bar{s}_i\bar{t}_j, \bar{F}_i\bar{G}_j), i = 1, \dots, n, j = 1, \dots, m\}$ is admissible.*

Proof. (a) From Corollary 1.5.2, σ is a localizing torsion theory on $\mathscr{A}$, so $\mathscr{A}_\sigma$ is also an Ab5 category. We only need to show that F_i induces endo-equivalence on $\mathscr{A}_\sigma$. From Lemma 1.3.5(c), $F_i(s_j\text{-Tor}) \subset s_j\text{-Tor}$, so $F_i(\bigcap_{j=1}^n (s_j\text{-Tor})) \subset \bigcap_{j=1}^n (s_j\text{-Tor})$, that is, $F_i(\sigma\text{-Tor}) \subset \sigma\text{-Tor}$; then $F_i^{-1}(\sigma\text{-Tor}) \subset \sigma\text{-Tor}$ because σ-Tor is closed under isomorphism. Let $f: A \to B$ be a σ-isomorphism; then $F_i^{-1}(f): F_i^{-1}A \to F_i^{-1}B$ is also a σ-isomorphism, so for any $C \in \mathscr{A}_\sigma$,

$$(F_i^{-1}(f))^*: \mathrm{Hom}_{\mathscr{A}}(F_i^{-1}B, C) \to \mathrm{Hom}_{\mathscr{A}}(F_i^{-1}A, C)$$

is an isomorphism. But $\mathrm{Hom}_{\mathscr{A}}(F_i^{-1}B, C) \cong \mathrm{Hom}_{\mathscr{A}}(B, F_iC)$, so we get an isomorphism

$$f^*: \mathrm{Hom}_{\mathscr{A}}(B, F_iC) \to \mathrm{Hom}_{\mathscr{A}}(A, F_iC),$$

and thus $F_iC \in \mathscr{A}_\sigma$, i.e.,

$$F_i|_{\mathscr{A}_\sigma}: \mathscr{A}_\sigma \to \mathscr{A}_\sigma .$$

In the same way, we prove

$$F_i^{-1}|_{\mathscr{A}_\sigma}: \mathscr{A}_\sigma \to \mathscr{A}_\sigma .$$

So $F_i|_{\mathscr{A}_\sigma}$ is an endo-equivalence of $\mathscr{A}_\sigma$.

(b) As in (a), $\{(s_i, F_i)\}$ and $\{(t_j, G_j)\}$ naturally induce compatible divisors $\{(\bar{s}_i, \bar{F}_i)\}$ and $\{(\bar{t}_j, \bar{G}_j)\}$ on $\mathscr{A}_\tau$ by restriction.

First we prove $\bigcap_{i,j} (\bar{s}_i\bar{t}_j\text{-Tor}) = 0$. Let $M \in \bigcap_{i,j} (\bar{s}_i\bar{t}_j)\text{-Tor})$, i.e.,

$$\bar{s}_i^{-1}\bar{t}_j^{-1}(M) \cong \bar{t}_j^{-1}\bar{s}_i^{-1}(M) = 0$$

for all i, j. Let M_i denote the maximal $\bar{s}_i$-torsion subobject of M; then $M/M_i \subset \bar{s}_i^{-1}(M)$. So $\bar{t}_j^{-1}(M/M_i) \subset \bar{t}_j^{-1}(\bar{s}_i^{-1}(M)) = 0$; then $M/M_i \in \bar{t}_j$-Tor for all j, i.e., $M/M_i \in \bigcap_j (\bar{t}_j\text{-Tor})$. But $\bigcap_j (\bar{t}_j\text{-Tor}) = 0$, as is proved in (a) above, so $M/M_i = 0$, i.e., $M \in \bar{s}_i$-Tor for all i. Thus

$$M \in \bigcap_{i=1}^n (\bar{s}_i\text{-Tor}) = \left(\bigcap_{i=1}^n (s_i\text{-Tor})\right) \cap \mathscr{A}_\tau = 0 ,$$

and therefore $\bigcap_{i,j} (\bar{s}_i\bar{t}_j\text{-Tor}) = 0$.

Since $(\mathscr{A}_\tau)_{\bar{s}_i\bar{t}_j} = (\mathscr{A})_{s_it_j} = (\mathscr{A}_{s_i})_{\bar{s}_i\bar{t}_j}$, it remains to prove that if $\mathscr{A}$ has a set of small projective generators, and (t, G) is a divisor on $\mathscr{A}$, then $\mathscr{A}_t$ has a set of small projective generators. It is easy to see that an object P in an Ab5 category is small projective iff the functor $\mathrm{Hom}(P, -)$ commutes with arbitrary colimits. Now let $P \in \mathscr{A}$ be small projective, $\{A_\alpha\}$ be an arbitrary diagram in $\mathscr{A}_t$; then

$$\begin{aligned} \mathrm{Hom}_{\mathscr{A}_t}(t^{-1}(P), \varinjlim A_\alpha) &\cong \mathrm{Hom}_{\mathscr{A}}(P, j_{t*} \varinjlim A_\alpha) \\ &= \mathrm{Hom}_{\mathscr{A}}(P, \varinjlim j_{t*}A_\alpha) \\ &\cong \varinjlim \mathrm{Hom}_{\mathscr{A}}(P, j_{t*}A_\alpha) \\ &\cong \varinjlim \mathrm{Hom}_{\mathscr{A}_t}(t^{-1}P, A_\alpha) , \end{aligned}$$

where the embedding $j_{t*}: \mathscr{A}_t \to \mathscr{A}$ is exact and commutes with colimits by Lemma 1.1.1, so $t^{-1}(P)$ is also small projective in $\mathscr{A}_t$. Obviously the image under t^{-1} of a set of generators in $\mathscr{A}$ is also a set of generators in $\mathscr{A}_t$. Therefore, $\mathscr{A}_t$ also has a set of small projective generators. This finishes the proof of (b). □

Next, we proceed to prove that an admissible abelian category is a grothendieck category with a set of locally finitely generated generators. Recall that a grothendieck category is an Ab5 category with a set of generators.

1.6.5. Definition. Let $\mathscr{A}$ be an Ab5 category with a set of small projective generators. An object A in $\mathscr{A}$ is called *finitely generated* (f.g.) if there is an epimorphism $\coprod_{i=1}^q P_i \to A$, where $\{P_i\}$ are small projective objects in $\mathscr{A}$. Let $\{\mathscr{A}, (s_i, F_i), i = 1, \dots, n\}$ be an admissible abelian category; an object A in $\mathscr{A}$ is called *locally* f.g. (with respect to $\{(s_i, F_i)\}$, if specification is needed) if each $s_i^{-1}A$ is f.g. in $\mathscr{A}_{s_i}$.

1.6.6. Lemma. *Let $\{\mathscr{A}, (s_i, F_i), i = 1, \dots, n\}$ be an admissible abelian category; then:*

(a) *If $A \in \mathscr{A}$ is locally f.g., then for any directed inductive system $\{A_\alpha\}$ in $\mathscr{A}$,* *and any epimorphism* $\varinjlim (A_\alpha) \to A$, *there is some* A_{α_0} *such that*

$$A_{\alpha_0} \to \varinjlim (A_\alpha) \to A$$

is an epimorphism.

(b) *If* $\{(s'_j, F'_j), j = 1, \ldots, m\}$ *is another finite set of compatible divisors on* $\mathcal{A}$ *such that* $\{\mathcal{A}, (s'_j, F'_j), i = 1, \ldots, m\}$ *becomes an admissible abelian category, and* $\{(s_i, F_i), (s'_j, F'_j)\}$ *are compatible, then an object* $A \in \mathcal{A}$ *is locally f.g. with respect to* $\{(s_i, F_i)\}$ *iff* A *is locally f.g. with respect to* $\{(s'_j, F'_j)\}$.

Proof. (a) By definition, for each i, we have an epimorphism $\coprod_{k=1}^{q} P_k \to s_i^{-1} A$, where $\{P_k\}$ are small projective objects in $\mathcal{A}_{s_i}$; then we have the lifting

$$\begin{array}{ccc} & & \coprod_{k=1}^{q} P_k \\ & \swarrow & \downarrow \\ \varinjlim s_i^{-1}(A_\alpha) = s_i^{-1}(\varinjlim A_\alpha) & \to & s_i^{-1}(A) \end{array}$$

But $\coprod_{k=1}^{q} P_k$ is also small projective in $\mathcal{A}_{s_i}$, so the lifting can be factorized as

$$\coprod_{k=1}^{q} P_k \to s_i^{-1}(A_{\alpha_0}) \to \varinjlim s_i^{-1}(A_\alpha)$$

for some α_0. Thus the composite

$$s_i^{-1}(A_{\alpha_0}) \to \varinjlim s_i^{-1}(A_\alpha) \to s_i^{-1}(A)$$

is an epimorphism. Choose α_0 big enough such that we have epimorphisms

$$s_i^{-1}(A_{\alpha_0}) \to \varinjlim s_i^{-1}(A_\alpha) \to s_i^{-1}(A)$$

for all $i = 1, \ldots, n$; then we have an epimorphism $A_{\alpha_0} \to \varinjlim (A_\alpha) \to A$.

(b) Because $\{(s_i, F_i), (s'_j, F'_j)\}$ are compatible, $\{\mathcal{A}, (s_i s'_j, F_i F'_j), i, j\}$ is also admissible. Let $A \in \mathcal{A}$ be locally f.g. with respect to $\{(s_i, F_i)\}$, then obviously A is locally f.g. with respect to $\{(s_i s'_j, F_i F'_j), i, j\}$. So $s'^{-1}_j(A)$ is locally f.g. in $\{\mathcal{A}_{s'_j}, (s_i s'_j, F_i F'_j), i = 1, \ldots, n\}$ for each j. Because $A_{s'_j}$ has a set of small projective generators, we have an epimorphism

$$\coprod_{\alpha \in I} P_\alpha = \varinjlim_J \Big(\coprod_{\alpha \in J} P_\alpha\Big) \to s'^{-1}_j(A),$$

where J runs over all finite subsets of I and $\{P_\alpha\}$ are small projective objects in $A_{s'_j}$. But by (a) above, we have J_0 such that the composite

$$\coprod_{\alpha \in J_0} P_\alpha \to \varinjlim_J \Big(\coprod_{\alpha \in J} P_\alpha\Big) \to s'^{-1}_j(A)$$

is an epimorphism, i.e., $s'^{-1}_j(A)$ is f.g. in $A_{s'_j}$ for each j. So A is locally f.g. with respect to $\{(s'_j, F'_j)\}$. □

1.6.7. Lemma. *Lt* $\mathcal{A}$ *be an* Ab5 *category with a set of small projective generators,* $\{(t_j, G_j), j = 1, \ldots, m\}$ *be a finite set of compatible divisors on* $\mathcal{A}$, $\tau = \bigcap_{i=1}^{m} t_j$. *If* $A \in \mathcal{A}_\tau$, $B \in \mathcal{A}$, *such that there is an embedding* $A \hookrightarrow j_\tau^* B$ *and* A *is locally f.g. in* $\mathcal{A}_\tau$, *then there is a f.g. subobject* B_0 *of* B *such that* $A \cong j_\tau^* B_0$.

Proof. We regard $\mathcal{A} = \{\mathcal{A}, (\mathrm{id}, \mathrm{Id})\}$ as an admissible abelian category; then by Lemma 1.6.4(b) $\mathcal{A}_\tau$ is an admissible abelian category, so it makes sense to say A is locally f.g. in $\mathcal{A}_\tau$. Because $j_{\tau*}$ is left exact, we have $j_{\tau*}A \hookrightarrow j_{\tau*} j_\tau^* B$. Let B' be the pullback:

$$\begin{array}{ccc} B' & \hookrightarrow & B \\ \downarrow & & \downarrow \mu_B \\ j_{\tau*}A & \hookrightarrow & j_{\tau*} \circ j_\tau^* B \end{array}$$

where μ_B is the adjunction morphism which is a τ-isomorphism. Because j_τ^* is exact, it preserves pullbacks:

$$\begin{array}{ccc} j_\tau^* B' & \hookrightarrow & j_\tau^* B \\ \downarrow & & \downarrow j_\tau^*(\mu_B) = \mathrm{id} \\ A = j_\tau^* j_{\tau*} A & \hookrightarrow & j_\tau^* j_{\tau*} \circ j_\tau^* B = j_\tau^* B \end{array}$$

So we have $j_\tau^* B' \cong A$. Since $\mathcal{A}$ has a set of small projective generators, we have $\varinjlim B'_\alpha = \bigcup B'_\alpha = B'$, where B'_α runs over all f.g. subobjects of B'. Then

$$\varinjlim j_\tau^* B'_\alpha = j_\tau^* \varinjlim B'_\alpha \cong A.$$

Since A is locally f.g., we have by Lemma 1.6.6(a), an epimorphism $j_\tau^* B'_{\alpha_0} \to A$ for some α_0. On the other hand, $j_\tau^* B'_{\alpha_0} \hookrightarrow j_\tau^* B' \cong A$, so $j_\tau^* B'_{\alpha_0} \cong A$. We take $B_0 = B'_{\alpha_0} \hookrightarrow B' \hookrightarrow B$. □

1.6.8. Lemma. *Let* $\{\mathcal{A}, (s_i, F_i), i = 1, \ldots, n\}$ *be an admissible abelian category,* $\{(t_j, G_j), j = 1, \ldots, m\}$ *be another finite set of divisors on* $\mathcal{A}$ *such that* $\{(s_i, F_i), (t_j, G_j)\}$ *are compatible, and* $\tau = \bigcap_{j=1}^{m} t_j$. *If* $A \in \mathcal{A}_\tau$, $B \in \mathcal{A}$, *such that*

$A \hookrightarrow j_\tau^* B$ and A is locally f.g. in $\mathcal{A}_\tau$, then there is a f.g. subobject B_0 of B such that $A \cong j_\tau^* B_0$.

Proof. First, we may assume $m<n$ and $\{(t_j, G_j), j=1,\ldots,m\}$ is a part of $\{(s_i, F_i), i=1,\ldots,n\}$; for if not so, we let

$$\{(s_i, F_i), (s_it_j, F_iG_j), i=1,\ldots,n, j=1,\ldots,m\}$$

replace $\{(s_i, F_i), i=1,\ldots,n\}$, and

$$\{(s_it_j, F_iG_j), i=1,\ldots,n, j=1,\ldots,m\}$$

replace $\{(t_j, G_j), j=1,\ldots,m\}$. Then because $\bigcap_{i=1}^n (s_i\text{-Tor})=0$, we have $\tau = \bigcap_{j=1}^m t_j = \bigcap_{i,j} s_it_j$. So this replacement does not change $\mathcal{A}_\tau$, and does not change being locally f.g. by Lemma 1.6.6(b).

We use induction on n. When $n=1$, nothing needs to be proved because $0=m<n$. Assume the lemma for $n-1$.

To do the inductive step, let $\sigma = \bigcap_{i=1}^{n-1} s_i$, $B_1 = j_\sigma^* B \in \mathcal{A}_\sigma$. Then

$$A \hookrightarrow j_\tau^*(B) = j_\tau^*(j_\sigma^*(B)) = j_\tau^*(B_1)$$

(notice that $\sigma \le \tau$, for $m \le n-1$). From the induction hypothesis, we have a locally f.g. subobject $B_{10} \hookrightarrow B_1 \in \mathcal{A}_\sigma$ such that $j_\tau^*(B_{10}) \cong A$. Then $j^*_{\overline{\sigma\cup s_n}}(B_{10})$ is also locally f.g. in $\mathcal{A}_{\sigma\cup s_n} \subset \mathcal{A}_\sigma$. Let $B_2 = j_{s_n}^*(B) = s_n^{-1} \in \mathcal{A}_{s_n}$, then

$$j^*_{\overline{\sigma\cup s_n}}(B_{10}) \hookrightarrow j^*_{\overline{\sigma\cup s_n}}(B_1) = j^*_{\overline{\sigma\cup s_n}}(j_\sigma^*(B)) = j^*_{\sigma\cup s_n}(B)$$

$$= j^*_{\overline{\sigma\cup s_n}}(j^*_{s_n}(B)) = j^*_{\overline{\sigma\cup s_n}}(B_2)\,.$$

By Lemma 1.6.7, we have a f.g. subobject $B_{20} \hookrightarrow B_2$ in $\mathcal{A}_{s_n}$ such that $j^*_{\overline{\sigma\cup s_n}}(B_{20}) \cong j^*_{\overline{\sigma\cup s_n}}(B_{10})$. Then by Proposition 1.5.1, there is a $B_0 \in \mathcal{A}$ such that $j_\sigma^*(B_0) \cong B_{10}$ and $j_{s_n}^*(B_0) \cong B_{20}$, so B_0 is locally f.g. in $\mathcal{A}$. But $B_{10} \hookrightarrow B_1 = j_\sigma^*(B)$, $B_{20} \hookrightarrow B_2 = j_{s_n}^*(B)$, so we have $B_0 \hookrightarrow B$, and $j_\tau^*(B_0) = j_\tau^*(j_\sigma^*(B_0)) \cong j_\tau^*(B_0) \cong A$. This finishes the induction. □

1.6.9. Proposition. *Let $\{\mathcal{A}, (s_i, F_i), i=1,\ldots,n\}$ be an admissible abelian category, then $\mathcal{A}$ has a set of locally f.g. generators. Therefore, $\mathcal{A}$ is a grothendieck category.*

Proof. First we prove that for any $A \in \mathcal{A}$, $\bigcup A_\alpha = A$, where A_α runs over all the locally f.g. subobjects of A, which implies that the collection of all locally f.g. objects generates $\mathcal{A}$. We use induction on n. When $n=1$, $\mathcal{A}$ has a set of small projective generators, so obviously $\bigcup A_\alpha = A$. Assume we always have $\bigcup A_\alpha = A$ for $n-1$. To do the induction step, let $\sigma = \bigcap_{i=1}^{n-1} s_i$, $A_i = j_\sigma^*(A)$, $A_2 = j_{s_n}^*(A) = s_n^{-1}(A)$; then by the induction hypothesis, we have $\bigcup A_{1\alpha} = A_1$, $\bigcup A_{2\alpha} = A_2$. By Lemma 1.6.8, for each $A_{1\alpha}$, there is a locally f.g. subobject $A_\alpha \hookrightarrow A$ such that $j_\sigma^*(A_\alpha) \cong A_{1\alpha}$; for each $A_{2\alpha}$, there is a locally f.g. subobject $A' \hookrightarrow A$ such that $j_\sigma^*(A'_\alpha) \cong A_{2\alpha}$. Then $j_\sigma^*(\bigcup A_\alpha) = A_1 = j_\sigma^*(A)$, $j_{s_n}^*(\bigcup A_\alpha) = A_2 = j_{s_n}^*(A)$, so $\bigcup A_\alpha = A$.

Next we need to show that the collection of all isomorphism classes of locally f.g. objects in $\mathcal{A}$ is a set. But this is obvious, because each $\mathcal{A}_{s_i}$ has a set of small projective generators, so the collection of all isomorphic classes of f.g. objects in $\mathcal{A}_{s_i}$ is a set. Then the collection of all isomorphic classes of locally f.g. objects in $\mathcal{A}$ is a set because of Proposition 1.5.1. □

1.6.10. Corollary. *Let $\{\mathcal{A}, (s_i, F_i), i=1,\ldots,n\}$ be an admissible abelian category. Then $A \in \mathcal{A}$ is locally f.g. iff for an arbitrary directed inductive system $\{A_\alpha\}$ and epimorphism $\varinjlim A_\alpha \to A$, there is some α_0 such that the composite*

$$A_{\alpha_0} \to \varinjlim A_\alpha \to A$$

is an epimorphism. Therefore, being locally f.g. is independent of the choice of $\{(s_i, F_i)\}$.

Proof. In Lemma 1.6.6(a) we have proved the 'only if' part of this corollary. For the 'if' part, by Proposition 1.6.9, we have

$$\varinjlim A_\alpha = \bigcup A_\alpha = A\,,$$

where A_α runs over all locally f.g. subobjects of A. Then from the hypothesis, we have some α_0 such that

$$A_{\alpha_0} \to \varinjlim A_\alpha = A$$

is an epimorphism. So $A_{\alpha_0} = A$, and A is locally f.g. □

2. Perfect complexes

2.0. In this section we will generalize the notion of perfect complexes over a scheme to our context of an admissible abelian category and establish the basic characterization of a perfect complex and other properties. The category of perfect complexes will be the category from which we construct our K-theory spectra.

2.1.0. Let $\mathscr{A}$ be an abelian category, we will fix the following notations in this paper:

$C(\mathscr{A})=$: the category of all complexes in $\mathscr{A}$, i.e., objects are complexes whose terms are objects in $\mathscr{A}$, and morphisms are chain maps.

$H(\mathscr{A})=$: the category of all complexes in $\mathscr{A}$, but morphisms are homotopy equivalence classes of chain maps.

$D(\mathscr{A})=$: the derived category, formed from $H(\mathscr{A})$ by formally inverting all quasi-isomorphisms (i.e., chain maps which induce isomorphisms on the homology of the complexes); for details see [8].

Notice that $C(\mathscr{A})$, $H(\mathscr{A})$ and $D(\mathscr{A})$ have the same objects, but different morphisms, and we have the canonical functors $C(\mathscr{A})\to H(\mathscr{A})\to D(\mathscr{A})$.

2.1.1. Definition. Let $\{\mathscr{A},(s_i,F_i),\ i=1,\dots,n\}$ be an admissible abelian category; we call an object $P\in\mathscr{A}$ *locally small projective* if each $s_i^{-1}P$ is small projective in $\mathscr{A}_{s_i}$. A complex $E^\cdot$ in $\mathscr{A}$ is called *strictly perfect* if $E^\cdot$ is bounded and all terms E^n are locally small projective. A complex $E^\cdot$ is called *perfect* if for each i, there is a bounded complex $E_i^\cdot$ in $\mathscr{A}_{s_i}$ with each E_i^n small projective in $\mathscr{A}_{s_i}$ and a quasi-isomorphism $E_i^\cdot\to s_i^{-1}E^\cdot$. We will fix the following notation in this paper:

$\mathbf{P}(\mathscr{A})=$: the full subcategory of $\mathscr{A}$ of all locally small projective objects in $\mathscr{A}$.

We need an inductive resolution lemma which is a special case of [14, Lemma 1.9.5]. Recall that a complex $E^\cdot$ is called cohomologically bounded above if $H^i(E^\cdot)=0$ when $i\geq n$ for some n.

2.1.2. Lemma. *Let* $\mathbf{A}$ *be an abelian category,* $\mathbf{D}$ *be a full subcategory of* $\mathbf{A}$, $C_0(\mathbf{A})$ *be a full subcategory of* $C(\mathbf{A})$ *such that every complex in* $C_0(\mathbf{A})$ *is cohomologically bounded above. Suppose all bounded complexes in* $\mathbf{D}$ *are in* $C_0(\mathbf{A})$, *and* $C_0(\mathbf{A})$ *is closed under mapping cones of morphisms* $D^\cdot\to C^\cdot$, *where* $D^\cdot$ *is any bounded complex in* $\mathbf{D}$ *and* $C^\cdot\in C_0(\mathbf{A})$. *Assume further*

2.1.2.1. *For any integer* n, *any* $C^\cdot\in C_0(\mathbf{A})$ *such that* $H^i(C^\cdot)=0$ *when* $i\geq n$ *and any epimorphism in* $\mathbf{A}$, $A\to H^{n-1}(C^\cdot)$, *where* $A\in\mathbf{A}$, *there exist a* $D\in\mathbf{D}$ *and a map* $D\to A$ *such that the composite* $D\to A\to H^{n-1}(C^\cdot)$ *is an epimorphism in* $\mathbf{A}$.

Then for any cohomologically bounded above complex $D^\cdot$ *in* $\mathbf{D}$, *any* $C^\cdot\in C_0(\mathbf{A})$ *and any chain map* $D^\cdot\xrightarrow{x}C^\cdot$, *there exists a bounded above complex* $D'^\cdot$ *in* $\mathbf{D}$, *a degreewise split monomorphism* $D^\cdot\xrightarrow{b}D'^\cdot$, *and a quasi-isomorphism* $D'^\cdot\xrightarrow{x'}C^\cdot$ *such that* $x=x'\circ b$.

If further x *is already an* n*-quasi-isomorphism, then we may choose* $D'^i=D^i$ *for* $i\geq n$.

Proof. This is given by an inductive construction using 2.1.2.1. For details, see [14, Lemma 1.9.5]. □

2.2.1. Lemma. *Let* $\mathscr{A}$ *be an* Ab5 *category with a set of small projective generators, and regard* $\mathscr{A}=\{\mathscr{A},(\mathrm{id},\mathrm{Id})\}$ *as an admissible abelian category.*

(a) *If* $E^\cdot\in C(\mathscr{A})$, $F^\cdot$ *is strictly perfect and there is a quasi-isomorphism* $E^\cdot\to F^\cdot$, *then there is another strictly perfect complex* $F'^\cdot$ *and a quasi-isomorphism* $F'^\cdot\to E^\cdot$, *thus* $E^\cdot$ *is perfect.*

(b) *If* $E^\cdot\to F^\cdot\to G^\cdot$ *is a homotopy fibre sequence in* $D(\mathscr{A})$, *and any two of* $E^\cdot$, $F^\cdot$ *and* $G^\cdot$ *are perfect, then so is the third one.*

(c) *If* $E^\cdot,F^\cdot\in C(\mathscr{A})$, *then* $E^\cdot\oplus F^\cdot$ *is perfect iff* $E^\cdot$ *and* $F^\cdot$ *are both perfect.*

(d) *For any* $E^\cdot\in C(\mathscr{A})$, *there is a directed inductive system* $\{E_\alpha^\cdot\}$ *of strictly perfect complexes and a quasi-isomorphism* $\varinjlim E_\alpha^\cdot\to E^\cdot$.

(e) $E^\cdot$ *is perfect iff for any directed inductive system* $\{E_\alpha^\cdot\}$ *in* $C(\mathscr{A})$, *we have an isomorphism*

$$\mathrm{Hom}_{D(\mathscr{A})}(E^\cdot,\varinjlim E_\alpha^\cdot)\cong\varinjlim \mathrm{Hom}_{D(\mathscr{A})}(E^\cdot,E_\alpha^\cdot)\,.$$

Proof. The proof is essentially the same as in [14]. The interest reader may follow the ideas in [14] to fill in the details without much trouble.

(a) Confer [14, 2.2.4].

(b), (c) Confer [14, 2.2.13].

(d) Confer [14, 2.3.2].

(e) We can have a quick proof for (e) in our context.

If $E^\cdot$ is perfect, $\{E_\alpha^\cdot\}$ is a directed inductive system in $C(\mathscr{A})$, we want to prove

$$\mathrm{Hom}_{D(\mathscr{A})}(E^\cdot,\varinjlim E_\alpha^\cdot)\cong\varinjlim \mathrm{Hom}_{D(\mathscr{A})}(E^\cdot,E_\alpha^\cdot)\,.$$

Because every perfect complex is quasi-isomorphic to a strictly perfect complex, we may assume $E^\cdot$ is strictly perfect. Let $f\in\mathrm{Hom}_{D(\mathscr{A})}(E^\cdot,\varinjlim E_\alpha^\cdot)$ be represented as

$$E^\cdot\xleftarrow{\sim}F^\cdot\to\varinjlim E_\alpha^\cdot\,.$$

By (a) above we have another strictly perfect complex $E'^\cdot$ and a quasi-isomorphism $E'^\cdot\xrightarrow{\sim}F^\cdot$, so f can be represented as

$$E^\cdot\xleftarrow{\sim}E'^\cdot\to\varinjlim E_\alpha^\cdot\,.$$

Since $E'^\cdot$ is bounded and each term E'^n is small projective, the chain map $E'^\cdot\to\varinjlim E_\alpha^\cdot$ can be factorized as

$$E'^\cdot\to E_{\alpha_0}^\cdot\to\varinjlim E_\alpha^\cdot$$

for some α_0. It is easy to check that this factorization gives us the desired isomorphism.

Conversely, if $E^\cdot$ makes $\mathrm{Hom}_{D(\mathcal{A})}(E^\cdot, -)$ commute with directed inductive systems in $C(\mathcal{A})$. Choose a quasi-isomorphism: $\varinjlim E^\cdot_\alpha \tilde{\to} E^\cdot$, as is insured by (d), where all $E^\cdot_\alpha$'s are strictly perfect. Because

$$\mathrm{Hom}_{D(\mathcal{A})}(E^\cdot, \varinjlim E^\cdot_\alpha) \cong \varinjlim \mathrm{Hom}_{D(\mathcal{A})}(E^\cdot, E^\cdot_\alpha),$$

we see that the inverse of $\varinjlim E^\cdot_\alpha \tilde{\to} E^\cdot$ in $D(\mathcal{A})$ factorizes through some $E^\cdot_\alpha$, so $E^\cdot$ is a direct summand of some $E^\cdot_\alpha$ in $D(\mathcal{A})$. By (c) above, $E^\cdot$ is perfect. □

Next we proceed to give a characterization for the perfect complexes in a general admissible abelian category, analogous to the one given in Lemma 2.2.1(e).

2.3.1. Definition. Let $\mathcal{A}$ be an Ab5 category, $\{(s_i, F_i), i = 1, \dots, n\}$ be compatible divisors on $\mathcal{A}$, $\sigma = \bigcup_{i=1}^n s_i$. Define a functor

$$\hat{R}j_{\sigma*} : C(\mathcal{A}_\sigma) \to C(\mathcal{A}),$$

$$\hat{R}j_{\sigma*}(E^\cdot) = \mathrm{Tot}(\check{c}(s_1^{-1}, \dots, s_n^{-1}) j_{\sigma*}E^\cdot)$$

$$= \mathrm{Tot}(\check{c}(\bar{s}_1^{-1}, \dots, \bar{s}_n^{-1})E^\cdot), \qquad \forall E^\cdot \in C(\mathcal{A}_\sigma),$$

where $\check{c}(s_1^{-1}, \dots, s_n^{1-})$ and $\check{c}(\bar{s}_1^{1-}, \dots, \bar{s}_n^{-1})$ are Čech complexes of functors defined in Definition 1.4.5. There are natural transformations $j_{\sigma*} \to \hat{R}j_{\sigma*}$ and $\mathrm{Id}_{\mathcal{A}_\sigma} = j^*_\sigma j_{\sigma*} \to j^*_\sigma \hat{R}j_{\sigma*}$ induced by the augmentation of the Čech complex $\delta = (\lambda_i) : \mathrm{Id} \to \check{c}(s_1^{-1}, \dots, s_n^{-1})$.

2.3.2. Lemma. (a) $\hat{R}j_{\sigma*} : C(\mathcal{A}_\sigma) \to C(\mathcal{A})$ *preserves quasi-isomorphisms, therefore naturally induces a functor* $D(\mathcal{A}_\sigma) \to D(\mathcal{A})$, *still denoted by* $\hat{R}j_{\sigma*}$.

(b) *For any* $E^\cdot \in C(\mathcal{A}_\sigma)$, $\bar{\delta} = (\bar{\lambda}_i) : E^\cdot \to j^*_\sigma \hat{R}j_{\sigma*}E^\cdot$ *is a quasi-isomorphism.*

(c) $(j^*_\sigma, \hat{R}j\sigma_*)$ *is an adjoint pair of functors between* $D(\mathcal{A})$ *and* $D(\mathcal{A}_\sigma)$.

(d) $\hat{R}j_{\sigma*}$ *commutes with colimits.*

Proof. (a) If $E^\cdot \in C(\mathcal{A}_\sigma)$ is acyclic, then $\bar{s}_i^{-1}E^\cdot$ is acyclic in $\mathcal{A}_{s_i}$, because $\bar{s}_i^{-1}$ is exact. Thus

$$\hat{R}j_{\sigma*}(E^\cdot) = \mathrm{Tot}\Big(\bigoplus_i \bar{s}_i^{-1}E^\cdot \to \bigoplus_{i<j} \bar{s}_i^{-1}\bar{s}_j^{-1}E^\cdot \to \cdots\Big)$$

is acyclic because each column of the bicomplex is acyclic. Clearly $\hat{R}j_{\sigma*}$ preserves mapping cones, so $\hat{R}j_{\sigma*}$ preserves quasi-isomorphisms.

(b) By Lemma 1.4.6,

$$\mathrm{cone}(\bar{\delta}) = \mathrm{cone}(E^\cdot \to j^*_\sigma \hat{R}j_{\sigma*}E^\cdot) = \mathrm{Tot}(E^\cdot \xrightarrow{\bar{\delta}} \check{c}(\bar{s}_1^{-1}, \dots, \bar{s}_n^{-1})E^\cdot)$$

is acyclic because each row of the bicomplex is acyclic mod $(\bigcap_{i=1}^n (\bar{s}_i\text{-Tor}))$ and $\bigcap_{i=1}^n (\bar{s}_i\text{-Tor}) = 0$, so $\bar{\delta}$ is a quasi-isomorphism.

(c) Define natural transformations

$$\mu : \mathrm{Id}_{D(\mathcal{A})} \to \hat{R}j_{\sigma*}j^*_\sigma, \qquad \nu : j^*_\sigma \hat{R}j_{\sigma*} \to \mathrm{Id}_{D(\mathcal{A}_\sigma)}$$

as follows: for any $E^\cdot \in D(\mathcal{A})$, let

$$\mu_{E^\cdot} = \delta : E^\cdot \to \mathrm{Tot}(\check{c}(s_1^{-1}, \dots, s_n^{-1})E^\cdot) = \hat{R}j_{\sigma*}j^*_\sigma E^\cdot;$$

for any $F^\cdot \in D(\mathcal{A}_\sigma)$, let $\nu_{F^\cdot}$ be the morphism represented by

$$j^*_\sigma \hat{R}j_{\sigma*}(F^\cdot) \xleftarrow{\bar{\delta}} F^\cdot \xrightarrow{\mathrm{id}} F^\cdot.$$

In order to prove that $(j^*_\sigma, \hat{R}j_{\sigma*})$ is an adjoint pair between $D(\mathcal{A})$ and $D(\mathcal{A}_\sigma)$, we need to prove that the natural transformations of functors

$$\nu j^*_\sigma \circ j^*_\sigma \mu : j^*_\sigma \to j^*_\sigma$$

and

$$\hat{R}j_{\sigma*}\nu \circ \mu \hat{R}j_{\sigma*} : \hat{R}j_{\sigma*} \to \hat{R}j_{\sigma*}$$

are natural isomorphisms. (Notice that one usually requires $\nu j^*_\sigma \circ j^*_\sigma \mu = \mathrm{Id}_{j^*_\sigma}$ and $\hat{R}j_{\sigma*}\nu \circ \mu \hat{R}j_{\sigma*} = \mathrm{Id}_{\hat{R}j_{\sigma*}}$, but a slight modification of the usual proof shows that under these weaker hypotheses we still have a natural isomorphism

$$\mathrm{Hom}_{D(\mathcal{A}_\sigma)}(j^*_\sigma, -) \cong \mathrm{Hom}_{D(\mathcal{A})}(-, \hat{R}j_{\sigma*})$$

and thus an adjoint pair.)

Let $E^\cdot \in D(\mathcal{A})$, then $(\nu j^*_\sigma \circ j^*_\sigma \mu)_{E^\cdot}$ is the composite in $D(\mathcal{A})$:

$$\begin{array}{c} j^*_\sigma E^\cdot \xrightarrow{j^*_\sigma\delta = \bar{\delta}} j^*_\sigma(\mathrm{Tot}(\check{c}(s_1^{-1}, \dots, s_n^{-1})E^\cdot)) \xleftarrow{\bar{\delta}} j^*_\sigma E^\cdot \xrightarrow{\mathrm{id}} j^*_\sigma E^\cdot \\ \| \\ \mathrm{Tot}(\check{c}(\bar{s}_1^{-1}, \dots, \bar{s}_n^{-1}) j^*_\sigma E^\cdot). \end{array}$$

Because $\bar{\delta}$ is a quasi-isomorphism by (b) above, $\nu j^*_\sigma \circ j^*_\sigma \mu$ is an isomorphism in $D(\mathcal{A}_\sigma)$.

Let $F^\cdot \in D(\mathcal{A}_\sigma)$, then $(\hat{R}Fj_{\sigma*}\nu \circ \mu \hat{R}j_{\sigma*})_{F^\cdot}$ is the composite in $D(\mathcal{A})$:

$$\hat{R}j_{\sigma*}F^\cdot \xrightarrow{\delta_{\hat{R}j_{\sigma*}F^\cdot}} \hat{R}j_{\sigma*}j^*_\sigma \hat{R}j_{\sigma*}(F^\cdot) \xleftarrow{\hat{R}j_{\sigma*}(\bar{\delta})} \hat{R}j_{\sigma*}F^\cdot \xrightarrow{\mathrm{id}} \hat{R}j_{\sigma*}F^\cdot.$$

Because $\bar{\delta}$ is a quasi-isomorphism and $\hat{R}j_{\sigma*}$ preserves quasi-isomorphisms, so

$\hat{R}j_{\sigma *}(\bar{\delta})$ is also a quasi-isomorphism. To see that $\delta_{\hat{R}j_{\sigma *}F^{\cdot}}$ is a quasi-isomorphism, look at

$$\mathrm{cone}(\delta_{\hat{R}j_{\sigma *}F^{\cdot}}) = \mathrm{Tot}\left(\prod_{i=1}^{n} (\mathrm{Id} \to s_i^{-1})\check{c}(s_i^{-1}, \ldots, s_n^{-1}) j_\sigma^* F^{\cdot}\right).$$

It is easy to check that the complex of functors

$$\prod_{i=1}^{n} (\mathrm{Id} \to s_i^{-1})\check{c}(s_1^{-1}, \ldots, s_n^{-1})$$

is acyclic since each

$$\prod_{i=1}^{n} (\mathrm{Id} \to s_i^{-1}) s_{i_1}^{-1} \cdots s_{i_r}^{-1}$$

is acyclic. So $\delta_{\hat{R}j_{\sigma *}F^{\cdot}}$ is a quasi-isomorphism, and $(\hat{R}j_{\sigma *}\nu \circ \mu \hat{R}j_{\sigma *})_{F^{\cdot}}$ is an isomorphism in $D(\mathcal{A})$.

(d) This follows immediately from Lemma 1.1.1. □

2.3.3. Lemma. *Let $\mathcal{A}$ be an* Ab5 *category, $(s_1, F_1), \ldots, (s_{n-1}, F_{n-1})$ and (t, G) be n compatible divisors on $\mathcal{A}$. Denote $(s_i', F_i') = (s_i t, F_i G)$, $i = 1, \ldots, n-1$, $\sigma = \bigcap_{i=1}^{n-1} s_i$, $\sigma' = \bigcap_{i=1}^{n-1} s_i' = \sigma \cup t$. If $(\sigma\text{-Tor}) \cap (t\text{-Tor}) = 0$, then for any $E^{\cdot} \in C(\mathcal{A})$, we have an exact sequence of complexes*

$$0 \to E^{\cdot} \to \hat{R}j_{\sigma *}(j_\sigma^* E^{\cdot}) \oplus j_{t*}(j_t^* E^{\cdot}) \to \hat{R}j_{\sigma' *}(j_{\sigma'}^* E^{\cdot}) \to 0.$$

Proof. Consider the complex of functors

$$\begin{aligned}
&\prod_{i=1}^{n-1} (\mathrm{Id} \to s_i^{-1})(\mathrm{Id} \to t^{-1}) \\
&\quad = (\mathrm{Id} \to \check{c}(s_1^{-1}, \ldots, s_{n-1}^{-1}))(\mathrm{Id} \to t^{-1}) \\
&\quad = \mathrm{Id} \to \check{c}(s_1^{-1}, \ldots, s_{n-1}^{-1}) \oplus t^{-1} \to \check{c}(s_1^{-1}, \ldots, s_{n-1}^{-1}) t^{-1}
\end{aligned}$$

which is acyclic mod $((\bigcap_i (s_i\text{-Tor})) \cap (t\text{-Tor})) = \mathrm{mod}\,((\sigma\text{-Tor}) \cap (t\text{-Tor})) = \mathrm{mod}\,(0)$ by Lemma 1.4.6, so it is acyclic. Then for any $E^{\cdot} \in C(\mathcal{A})$, we get an exact sequence

$$\begin{aligned}
0 \to E^{\cdot} &\to \mathrm{Tot}(\check{c}(s_1^{-1}, \ldots, s_{n-1}^{-1}) E^{\cdot}) \oplus t^{-1} E^{\cdot} \\
&\to \mathrm{Tot}(\check{c}(s_1^{-1}, \ldots, s_{n-1}^{-1}) t^{-1} D^{\cdot}) \to 0,
\end{aligned}$$

which is just the exact sequence required in the lemma. □

Now we can prove the following:

2.4. Proposition. *Let $\{\mathcal{A}, (s_i, F_i), i = 1, \ldots, n\}$ be an admissible abelian category; then:*

(a) *A complex $E^{\cdot} \in C(\mathcal{A})$ is perfect iff $\mathrm{Hom}_{D(\mathcal{A})}(E^{\cdot}, -)$ commutes with directed colimits in $C(\mathcal{A})$. Therefore, being perfect is independent of the choice of the compatible divisors $\{(s_i, F_i), i = 1, \ldots, n\}$.*

(b) *If $E^{\cdot} \to F^{\cdot} \to G^{\cdot}$ is a homotopy fibre sequence in $D(\mathcal{A})$, and any two of $E^{\cdot}$, $F^{\cdot}$ and $G^{\cdot}$ are perfect, then so is the third one.*

(c) *$E^{\cdot} \oplus F^{\cdot}$ is perfect iff $E^{\cdot}, F^{\cdot}$ are both perfect.*

Proof. (a) Let $E^{\cdot}$ be perfect. We use induction on the number n of divisors to prove $\mathrm{Hom}_{D(A)}(E^{\cdot}, -)$ commutes with directed colimits. When $n = 1$, $s_1 : \mathrm{Id} \to F_1$ has to be an isomorphism, and thus $\mathcal{A}_{s_1} = \mathcal{A}$. So $\mathcal{A}$ has a set of small projective generators, and $\mathrm{Hom}_{D(\mathcal{A})}(E^{\cdot}, -)$ commutes with directed colimits by Lemma 2.2.1(e). Assume (a) for $n-1$. To do the inductive step, since $\bigcap_{i=1}^{n} (s_i\text{-Tor}) = 0$, in Lemma 2.3.3 we take $(t, G) = (s_n, F_n)$. Then for any $F^{\cdot} \in C(\mathcal{A})$, we have a short exact sequence of complexes

$$0 \to F^{\cdot} \to \hat{R}j_{\sigma *} j_\sigma^* F^{\cdot} \oplus j_{s_n *} j_{s_n}^* F^{\cdot} \to \hat{R}j_{\sigma' *} j_{\sigma'}^* F^{\cdot} \to 0,$$

where $(s_i', F_i') = (s_i s_n, F_i F_n)$, $\sigma = \bigcap_{i=1}^{n-1} s_i$, and $\sigma' = \bigcap_{i=1}^{n-1} s_i'$. So we have the long exact sequence

$$\begin{aligned}
\cdots &\to \mathrm{Hom}_{D(\mathcal{A})}(E^{\cdot}, F^{\cdot}) \to \mathrm{Hom}_{D(\mathcal{A})}(E^{\cdot}, \hat{R}j_{\sigma *} j_\sigma^* F^{\cdot}) \oplus \mathrm{Hom}_{D(\mathcal{A})}(E^{\cdot}, j_{s_n *} j_{s_n}^* F^{\cdot}) \\
&\to \mathrm{Hom}_{D(\mathcal{A})}(E^{\cdot}, \hat{R}j_{\sigma' *} j_{\sigma'}^* F^{\cdot}) \to \mathrm{Hom}_{D(\mathcal{A})}(E^{\cdot}[-1], F^{\cdot}) \to \cdots.
\end{aligned}$$

By Lemma 2.3.2(c), we have

$$\mathrm{Hom}_{D(\mathcal{A})}(E^{\cdot}, \hat{R}j_{\sigma *} j_\sigma^* F^{\cdot}) \cong \mathrm{Hom}_{D(\mathcal{A}_\sigma)}(j_\sigma^* E^{\cdot}, j_\sigma^* F^{\cdot}).$$

Because j_σ^* commutes with directed colimits and clearly preserves perfectness, and $\mathrm{Hom}_{D(\mathcal{A})}(j_\sigma^* E^{\cdot}, j_\sigma^*(\ ))$ commutes with directed colimits by the induction hypothesis, so does $\mathrm{Hom}_{D(\mathcal{A})}(E^{\cdot}, \hat{R}j_{\sigma *} j_\sigma^*(\ ))$. Similarly $\mathrm{Hom}_{D(\mathcal{A})}(E^{\cdot}, j_{s_n *} j_{s_n}^*(\ ))$ and $\mathrm{Hom}_{D(\mathcal{A})}(E^{\cdot}, \hat{R}j_{\sigma' *} j_{\sigma'}^*(\ ))$ also commute with directed colimits. Then by the five-lemma, $\mathrm{Hom}_{D(\mathcal{A})}(E^{\cdot}, -)$ commutes with directed colimits.

Conversely, if $E^{\cdot} \in C(\mathcal{A})$ such that $\mathrm{Hom}_{D(\mathcal{A})}(E^{\cdot}, -)$ commutes with directed colimits in $C(\mathcal{A})$, then $\mathrm{Hom}_{D(\mathcal{A}_{s_i})}(s_i^{-1} E^{\cdot}, -)$ also commutes with directed colimits in $C(\mathcal{A}_{s_i})$. Because $(s_i^{-1} = j_{s_i}^*, j_{s_i *})$ are an exact adjoint pair of functors between $\mathcal{A}$ and $\mathcal{A}_{s_i}$, they automatically induce an adjoint pair of functors between the derived categories $D(\mathcal{A})$ and $D(\mathcal{A}_{s_i})$; and because $j_{s_i *}$ commutes with directed colimits by Lemma 1.1.1, by Lemma 2.2.1(e), $s_i^{-1} E^{\cdot}$ is perfect in $\mathcal{A}_{s_i}$ for all $i = 1, \ldots, n$. So $E^{\cdot}$ is perfect in $\mathcal{A}$.

(b) Apply (a) above and the five-lemma to the following long exact sequence

$$\cdots \to \mathrm{Hom}_{D(\mathscr{A})}(E^\cdot[-1], -) \to \mathrm{Hom}_{D(\mathscr{A})}(G^\cdot, -)$$
$$\to \mathrm{Hom}_{D(\mathscr{A})}(F^\cdot, -) \to \mathrm{Hom}_{D(\mathscr{A})}(E^\cdot, -) \to \cdots .$$

(c) Because

$$\mathrm{Hom}_{D(\mathscr{A})}(E^\cdot \oplus F^\cdot, -) \cong \mathrm{Hom}_{D(\mathscr{A})}(E^\cdot, -) \oplus \mathrm{Hom}_{D(\mathscr{A})}(F^\cdot, -),$$

we can apply (a) above. □

2.5.0. Let $\{\mathscr{A}, (s_i, F_i), i = 1, \ldots, n\}$ be an admissible abelian category, assume we have another two finite sets of divisors on $\mathscr{A}$, $\{(t_j, G_j), j = 1, \ldots, m\}$ and $\{(u_k, H_k), k = 1, \ldots, r\}$, such that $\{(s_i, F_i), (t_j, G_j), (u_k, H_k), i, j, k\}$ are compatible. Put $\tau = \bigcap_{j=1}^m t_j$, $\mu = \bigcap_{k=1}^r u_k$.

2.5.1. Lemma. *With the assumptions and notations as in 2.5.0, suppose that $E^\cdot \in C(\mathscr{A}_\tau)$ and $j_{\bar\mu}^* E^\cdot \in C((\mathscr{A}_\tau)_{\bar\mu}) = C(\mathscr{A}_{\tau\cup\mu})$ is acyclic in $\mathscr{A}_{\tau\cup\mu}$. Then $j_\mu^*(\hat{R}j_{\tau *}E^\cdot) \in C(\mathscr{A}_\mu)$ is acyclic in $\mathscr{A}_\mu$.*

Proof. Since $j_\mu^*(\hat{R}j_{\tau *}E^\cdot)$ is acyclic in $\mathscr{A}_\mu$ iff

$$\bar\mu_k^{-1}(j_\mu^*(\hat{R}j_{\tau *}E^\cdot)) = \mu_k^{-1}(\hat{R}j_{\tau *}E^\cdot) \in C(\mathscr{A}_{u_k})$$

is acyclic for all $k = 1, \ldots, r$, it suffices to prove $u_k^{-1}(\hat{R}j_{\tau *}E^\cdot)$ is acyclic for all $k = 1, \ldots, r$. Since $j_{\bar\mu}^* E^\cdot$ is acyclic in $(\mathscr{A}_\tau)_{\bar\mu} = \mathscr{A}_{\tau\cup\mu}$, $\tau \cup \mu = (\bigcap_{j=1}^m t_j) \cup (\bigcap_{k=1}^r u_k) = \bigcap_{j,k} t_j u_k$ (Lemma 1.3.5(a)), $t_j^{-1}u_k^{-1}E^\cdot$ is acyclic in $\mathscr{A}_{t_j u_k}$ for all j, k. Since the embedding $\mathscr{A}_{t_j u_k} = (\mathscr{A}_{u_k})_{\bar t_j} \to \mathscr{A}_{u_k}$ is exact, $t_j^{-1}u_k^{-1}E^\cdot$ is acyclic in $\mathscr{A}_{u_k}$. Then

$$u_k^{-1}(\hat{R}j_{\tau *}E^\cdot) = \mathrm{Tot}\Big(\bigoplus_j u_k^{-1}t_j^{-1}E^\cdot \to \bigoplus_{i<j} u_k^{-1}t_i^{-1}t_j^{-1}E^\cdot \to \cdots\Big)$$
$$\cong \mathrm{Tot}\Big(\bigoplus_j t_j^{-1}u_k^{-1}E^\cdot \to \bigoplus_{i<j} t_i^{-1}t_j^{-1}u_k^{-1}E^\cdot \to \cdots\Big)$$

is acyclic in $\mathscr{A}_{u_k}$ because each column of the double complex is acyclic in $\mathscr{A}_{u_k}$. □

2.5.2. Lemma. *With the assumptions and notations as in 2.5.0, if further $(\tau\text{-Tor}) \cap (\mu\text{-Tor}) = 0$, then:*

(a) *If $E^\cdot \in C(\mathscr{A})$ and $j_\mu^* E^\cdot \in C(\mathscr{A}_\mu)$ is acyclic in $\mathscr{A}_\mu$, then $E^\cdot \to \hat{R}j_{\tau *}(j_\tau^* E^\cdot)$ is a quasi-isomorphism in $\mathscr{A}$.*

(b) *If $E^\cdot \in C(\mathscr{A}_\tau)$ is perfect, $j_{\bar\mu}^* E^\cdot \in C(\mathscr{A}_{\tau\cup\mu})$ is acyclic in $\mathscr{A}_{\tau\cup\mu}$, then $\hat{R}j_{\tau *}E^\cdot \in C(\mathscr{A})$ is perfect in $\mathscr{A}$ with $j_\mu^*(\hat{R}j_{\tau *}E^\cdot)$ acyclic in $\mathscr{A}_\mu$.*

Proof. (a) By Lemma 2.3.2(b),

$$j_\tau^*(\mathrm{cone}(E^\cdot \to \hat{R}j_{\tau *}j_\tau^* E^\cdot)) = \mathrm{cone}(j_\tau^* E^\cdot \to j_\tau^* \hat{R}j_{\tau *}(j_\tau^* E^\cdot))$$

is acyclic in $\mathscr{A}_\tau$, i.e., $\mathrm{cone}(E^\cdot \to \hat{R}j_{\tau *}j_\tau^* E^\cdot)$ is acyclic mod(τ-Tor). By assumption, $j_\mu^* E^\cdot$ is acyclic in $\mathscr{A}_\mu$, then $u_k^{-1}E^\cdot = \bar\mu_k^{-1}(j_\mu^* E^\cdot)$ is acyclic in $\mathscr{A}_{u_k}$ for all k. So

$$u_k^{-1}(\mathrm{cone}(E^\cdot \to \hat{R}j_{\tau *}j_\tau^* E^\cdot)) = u_k^{-1}\Big(\mathrm{Tot}\Big(\prod_{j=1}^m (\mathrm{Id} \to t_j^{-1})E^\cdot\Big)\Big)$$
$$\cong \mathrm{Tot}\Big(\prod_{j=1}^m (\mathrm{Id} \to t_j^{-1})u_k^{-1}E^\cdot\Big)$$

is acyclic in $\mathscr{A}_{u_k}$ for all k, that is, cone $(E^\cdot \to \hat{R}j_{\tau *}j_\tau^*(E^\cdot)$ is acyclic mod $(\bigcap_{k=1}^r (u_k\text{-Tor}))$. But

$$\Big(\bigcap_{k=1}^r (u_k\text{-Tor})\Big) \cap (\tau\text{-Tor}) = (\mu\text{-Tor}) \cap (\tau\text{-Tor}) = 0,$$

so $\mathrm{cone}(E^\cdot \to \hat{R}j_{\tau *}j_\tau^* E^\cdot)$ is acyclic in $\mathscr{A}$, that is, $E^\cdot \to \hat{R}j_{\tau *}j_\tau^* E^\cdot$ is a quasi-isomorphism.

(b) Follow the proof of [14, Theorem 2.6.3], but replace Rj_* by $\hat{R}j_*$. □

2.6. Definition. An admissible abelian category $\{\mathscr{A}, (s_i, F_i), i = 1, \ldots, n\}$ is called *strongly admissible* if $\mathbf{P}(\mathscr{A})$ generates $\mathscr{A}$. (Recall that $\mathbf{P}(\mathscr{A})$ denotes the class of all locally small projective objects in $\mathscr{A}$.)

2.6.1. Examples. (a) Example 1.6.2 and Example 1.6.3 are examples of strongly admissible abelian categories.

(b) Let X be a scheme with an ample family of line bundles, $\mathscr{R}_X$ be a quasi-coherent sheaf of $\mathscr{O}_X$-algebras, $\mathscr{A}$ = the category of all sheaves of $\mathscr{R}_X$-modules $\subset \mathrm{Qcoh}(X)$; then $\mathscr{A}$ is a strongly admissible abelian category (cf. Example 1.1.5).

2.6.2. Lemma. *Let $\{\mathscr{A}, (s_i, F_i), i = 1, \ldots, n\}$ be a strongly admissible category, $E^\cdot$ be a bounded perfect complex in $\mathscr{A}$. Then there is a strictly perfect complex $E'^\cdot$ and a quasi-isomorphism $E'^\cdot \to E^\cdot$. Thus, a complex in $\mathscr{A}$ is perfect iff it is isomorphic in $D(\mathscr{A})$ to a strictly perfect complex.*

Proof. Let $E^\cdot$ be bounded and perfect. In Lemma 2.1.2, let $\mathbf{A} = \mathscr{A}$, $\mathbf{D} = \mathbf{P}(\mathscr{A})$, $C_0(\mathscr{A})$ = all perfect complexes in $\mathscr{A}$. Because $\mathbf{P}(\mathscr{A})$ generates $\mathscr{A}$, it is easy to see that all conditions in Lemma 2.1.2 are satisfied. So for every perfect complex $E^\cdot$, there is a bounded above complex $F^\cdot$ with all $F^m \in \mathbf{P}(\mathscr{A})$ and a quasi-isomorphism $F^\cdot \to E^\cdot$. So $F^\cdot$ is also perfect. Then for each $i = 1, \ldots, n$, there is a strictly perfect complex $E_i^\cdot$ in $\mathscr{A}_{s_i}$ and a quasi-isomorphism $E_i^\cdot \xrightarrow{\sim} s_i^{-1}F^\cdot$, so $\text{cone}(E_i^\cdot \to s_i^{-1}F^\cdot)$ is acyclic. Because $E_i^\cdot$ and $s_i^{-1}F^\cdot$ are both bounded above complexes of small projective objects in $\mathscr{A}_{s_i}$, and moreover $E_i^\cdot$ is bounded below, $\text{cone}(E_i^\cdot \to s_i^{-1}F^\cdot)$ splits, and when m small enough,

$$\begin{aligned} s_i^{-1}F^m &= \text{cone}(E_i^\cdot \to s_i^{-1}F^\cdot)^m \\ &= \text{Im}(s_i^{-1}d^m) \oplus \text{Im}(s_i^{-1}d^{m-1}) \\ &= s_i^{-1}\text{Im}(d^m) \oplus s_i^{-1}\text{Im}(d^{m-1}) \end{aligned}$$

where the d^m's are the boundary maps of $F^\cdot$. Thus when m small enough, each $s_i^{-1}\text{Im}(d^m)$ is small projective in $\mathscr{A}_{s_i}$, that is, $\text{Im}(d^m) \in \mathbf{P}(\mathscr{A})$. Choose m small enough so that also $E^k = 0$ when $k < m$ because $E^\cdot$ is bounded; then we have a quasi-isomorphism $\tau^{\geq m}F^\cdot \xrightarrow{\sim} E^\cdot$, and $\tau^{\geq m}F^\cdot$ is strictly perfect. We let $E'^\cdot = \tau^{\geq m}F^\cdot$.

If $E^\cdot$ is isomorphic in $D(\mathscr{A})$ to a strictly perfect complex, then by Proposition 2.4(a), $E^\cdot$ is perfect. If $E^\cdot$ is an arbitrary perfect complex, then $E^\cdot$ is cohomologically bounded below, so when m small enough and n big enough $E^\cdot$ is isomorphic in $D(\mathscr{A})$ to $\tau^{\leq n}(\tau^{\geq m}E^\cdot)$, and $\tau^{\leq n}(\tau^{\geq m}E^\cdot)$ is a bounded perfect complex. From the above, there is a strictly perfect complex $E'^\cdot$ and a quasi-isomorphism $E'^\cdot \xrightarrow{\sim} \tau^{\leq n}(\tau^{\geq m}E^\cdot)$. So $E^\cdot$ is isomorphic in $D(\mathscr{A})$ to $E'^\cdot$ □

3. Excision theorem and proto-localization theorem

3.0. In this section we will use Waldhausen's K-theory construction to define K-theory for admissible abelian categories. It can be regarded as a generalization of K-theory of schemes to certain noncommutative situations. The proto-localization theorem is proved here, which is one of main goals of this paper. For Waldhausen categories, complicial biWaldhausen categories, the derived category of a complicial biWaldhausen category, K-theory spectra of Waldhausen categories, and basic results like the additivity theorem, fibration theorem, approximation theorem, cofinality theorem, etc., we refer to [15] and [14, Section 1].

3.1.0. Definition. Let $\{\mathscr{A}, (s_i, F_i), i = 1, \ldots, n\}$ be an admissible abelian category, $\mathscr{P}(\mathscr{A})$ = the full subcategory of $C(\mathscr{A})$ of all perfect complexes in $\mathscr{A}$. $\mathscr{P}(\mathscr{A})$ becomes a complicial biWaldhausen category with cylinder and cocylinder functors, where weak equivalences are quasi-isomorphisms and cofibrations are degree-wise monomorphisms. We define the K-theory spectrum of $\mathscr{A}$, denoted by $K(\mathscr{A})$, to be the Waldhausen K-theory spectrum of $\mathscr{P}(\mathscr{A})$, i.e., $K(\mathscr{A}) = K^{\mathrm{W}}(\mathscr{P}(\mathscr{A}))$, where K^{W} denote the Waldhausen K-theory functor.

We quote Thomason and Trobaugh's derived category theorem below for easy reference:

3.1.1. Theorem. *Let* **A** *and* **B** *be two complicial biWaldhausen categories, each of which is closed under the formation of canonical pushouts and pullbacks, and let* $F : \mathbf{A} \to \mathbf{B}$ *be a complicial exact functor. Suppose that F induces an equivalence of the derived categories*

$$\mathbf{W}^{-1}F : \mathbf{W}^{-1}\mathbf{A} \to \mathbf{W}^{-1}\mathbf{B}.$$

Then F induces a homotopy equivalence of the K-theory spectra

$$K^{\mathrm{W}}(F) : K^{\mathrm{W}}(\mathbf{A}) \to K^{\mathrm{W}}(\mathbf{B}).$$

Proof. [14, Theorem 1.9.8]. □

As we have shown that perfectness is independent of the choice of the compatible divisors $\{(s_i, F_i), i = 1, \ldots, n\}$ (Proposition 2.4(a)), $\mathscr{P}(\mathscr{A})$ is independent of the choice of $\{(s_i, F_i), i = 1, \ldots, n\}$ and so is $K(\mathscr{A})$.

3.1.2. Example. Let R be a ring; then $\{R\text{-Mod}, (\text{id}, \text{Id})\}$ is a strongly admissible abelian category. Let $\mathscr{P}_0(\mathscr{A})$ denote the full subcategory of $C(\mathscr{A})$ of all strictly perfect complexes in an admissible abelian category $\mathscr{A}$; then by Lemma 2.6.2,

$$\mathbf{W}^{-1}(\mathscr{P}_0(R\text{-Mod})) \to \mathbf{W}^{-1}\mathscr{P}(R\text{-Mod})$$

is a category equivalence, so we have

$$K(R\text{-Mod}) = K^{\mathrm{W}}(\mathscr{P}(R\text{-Mod})) = K^{\mathrm{W}}(\mathscr{P}_0(R\text{-Mod})).$$

According to Gillet and Waldhausen ([3] or [14]), $K^{\mathrm{W}}(\mathscr{P}_0(R\text{-Mod}))$ is homotopy equivalent to $K^{\mathrm{Q}}(\mathbf{P}(R\text{-Mod}))$ (recall that $\mathbf{P}(\mathscr{A})$ is the full subcategory of all locally small projective objects in the admissible abelian category $\mathscr{A}$), where K^{Q} denotes the Quillen K-theory functor for exact categories, and $K^{\mathrm{Q}}(\mathbf{P}(R\text{-Mod}))$ is the usual K-theory spectrum of R, so we have $K(R\text{-Mod}) = K(R)$.

Generally, if $\{\mathscr{A}, (s_i, F_i), i = 1, \ldots, n\}$ is a strongly admissible abelian category, then $\mathbf{P}(\mathscr{A})$ is an exact category, the embedding $i : \mathbf{P}(\mathscr{A}) \to \mathscr{A}$ is exact and reflects exactness, $\mathbf{P}(\mathscr{A})$ is closed under extensions in $\mathscr{A}$, and $\mathbf{P}(\mathscr{A})$ satisfies the following extra condition: if $f : P \to Q$ is a morphism in $\mathbf{P}(\mathscr{A})$ and $i(f)$ is an epimorphism in $\mathscr{A}$, then f is an admissible epimorphism in $\mathbf{P}(\mathscr{A})$. So the theorem of Gillet and Waldhausen applies, and we have

$$K(\mathscr{A}) = K^{\mathrm{W}}(\mathscr{P}(\mathscr{A})) \cong K^{\mathrm{W}}(\mathscr{P}_0(\mathscr{A})) \cong K^{\mathrm{Q}}(\mathbf{P}(\mathscr{A})).$$

If $\{\mathscr{A}, (s_i, F_i), i=1,\dots,n\}$ is only an admissible (not strongly) abelian category, then $K(\mathscr{A})$ need not be homotopy equivalent to $K^Q(\mathbf{P}(\mathscr{A}))$. We will see that it is $K(\mathscr{A})$ that behaves well under localization.

3.1.3. Let $\{\mathscr{A}, (s_i, F_i), i=1,\dots,n\}$ be an admissible abelian category, $\{(t_j, G_j), j=1,\dots,m\}$ and $\{(u_k, H_k), k=1,\dots,r\}$ be another two finite sets of divisors on $\mathscr{A}$ such that $\{(s_i, F_i), (t_j, G_j), (u_k, H_k), i, j, k\}$ are compatible, $\tau = \bigcap_{j=1}^m t_j$, $\mu = \bigcap_{k=1}^r u_k$.

Denote $\mathscr{P}(\mathscr{A} \text{ off } \mathscr{A}_\mu)$ = Waldhausen subcategory of $\mathscr{P}(\mathscr{A})$ of those perfect complexes $E^\cdot$ with $j_\mu^* E^\cdot$ acyclic in $\mathscr{A}_\mu$ ('off' is to suggest that the complexes are supported off $\mathscr{A}_\mu$). Then $\mathscr{P}(\mathscr{A} \text{ off } \mathscr{A}_\mu)$ is also a complicial biWaldhausen category. Define $K(\mathscr{A} \text{ off } \mathscr{A}_\mu) = K^W(\mathscr{P}(\mathscr{A} \text{ off } \mathscr{A}_\mu))$. Clearly,

$$j_\tau^* : \mathscr{P}(\mathscr{A} \text{ off } \mathscr{A}_\mu) \to \mathscr{P}(\mathscr{A}_\tau \text{ off } \mathscr{A}_{\tau\cup\mu})$$

is an exact functor of Waldhausen categories.

3.2. Theorem (excision). *With the assumptions and notations as in* 3.1.3, *assume* $(\tau\text{-Tor}) \cap (\mu\text{-Tor}) = 0$. *Then* j_τ^* *induces a homotopy equivalence of spectra*

$$K(j_\tau^*) : K(\mathscr{A} \text{ off } \mathscr{A}_\mu) \to K(\mathscr{A}_\tau \text{ off } \mathscr{A}_{\tau\cup\mu}) .$$

Proof. By Lemma 2.5.2(b), $\hat{R}j_{\tau*}$ is an exact functor from $\mathscr{P}(\mathscr{A}_\tau \text{ off } \mathscr{A}_{\tau\cup\mu})$ to $\mathscr{P}(\mathscr{A} \text{ off } \mathscr{A}_\mu)$. By Lemma 2.3.2(b), $j_\tau^* \hat{R}j_{\tau*}$ is naturally isomorphic to Id over the derived category $\mathbf{W}^{-1}(\mathscr{P}(\mathscr{A}_\tau \text{ off } \mathscr{A}_{\tau\cup\mu}))$, and by Lemma 2.5.2(a), $\hat{R}j_{\tau*}j_\tau^*$ is naturally isomorphic to Id over the derived category $\mathbf{W}^{-1}(\mathscr{P}(\mathscr{A} \text{ off } \mathscr{A}_\mu))$. So $(j_\tau^*, \hat{R}j_{\tau*})$ induces an equivalence of the two derived categories. Then by Theorem 3.1.1, they induce a homotopy equivalence of the two Waldhausen K-theory spectra. □

3.3. Theorem (proto-localization). *With the assumptions and notations as in* 3.1.3, *we have the following homotopy fibre sequences of spectra*:

(a) (absolute form)

$$K(\mathscr{A} \text{ off } \mathscr{A}_\tau) \to K(\mathscr{A}) \to K(\mathscr{A}_\tau)^\sim ,$$

(b) (relative form)

$$K(\mathscr{A} \text{ off } \mathscr{A}_{\tau\cap\mu}) \to K(\mathscr{A} \text{ off } \mathscr{A}_\mu) \to K(\mathscr{A}_\tau \text{ off } \mathscr{A}_{\tau\cup\mu})^\sim ,$$

where $K(\ )^\sim$ *is some covering spectrum of* $K(\ )$, *i.e., there is a map of spectra* $K(\ )^\sim \to K(\ )$ *such that*

$$\pi_n(K(\ )^\sim) \to \pi_n(K(\ )) = K_n(\ )$$

are isomorphisms for $n \geq 1$ *and a monomorphism for* $n = 0$.

First let us see what will suffice to prove the theorem (cf. [14, 5.2]). For (a), let $\mathscr{P}(\mathscr{A} \text{ for } \mathscr{A}_\tau)$ be the complicial biWaldhausen category which has the same underlying category as $\mathscr{P}(\mathscr{A})$ does, and the same cofibrations as $\mathscr{P}(\mathscr{A})$ does; but the weak equivalences in $\mathscr{P}(\mathscr{A} \text{ for } \mathscr{A}_\tau)$ differ from those in $\mathscr{P}(\mathscr{A})$ and are defined as follows: $f : E^\cdot \to F^\cdot$ in $\mathscr{P}(\mathscr{A} \text{ for } \mathscr{A}_\tau)$ is called a weak equivalence iff $j_\tau^* f : j_\tau^* E^\cdot \to j_\tau^* F^\cdot$ is a quasi-isomorphism in $\mathscr{A}_\tau$. By the fibration theorem [15, 1.6.4], we have the following homotopy fibre sequence:

$$K^W(\mathscr{P}(\mathscr{A} \text{ off } \mathscr{A}_\tau)) \to K^W(\mathscr{P}(\mathscr{A})) \to K^W(\mathscr{P}(\mathscr{A} \text{ for } \mathscr{A}_\tau)) .$$

If we can prove that $K^W(\mathscr{P}(\mathscr{A} \text{ for } \mathscr{A}_\tau))$ is a covering spectrum of $K^W(\mathscr{P}(\mathscr{A}_\tau))$, then the proof of (a) is done. By Thomason and Trobaugh's cofinality theorem [14, Theorem 1.10.1], if we let

$$G = \operatorname{coker}(K_0(\mathscr{A}) \to K_0(\mathscr{A}_\tau)) ,$$

then we have a homotopy fibre sequence of spectra

$$K^W(\mathscr{B}) \to K^W(\mathscr{P}(\mathscr{A})) \to \text{“}G\text{”} ,$$

where $\mathscr{B}$ is the full Waldhausen subcategory of $\mathscr{P}(\mathscr{A}_\tau)$ whose objects are those perfect complexes $E^\cdot$ such that the class $[E^\cdot] \in K_0(\mathscr{A}_\tau)$ is in the image of $K_0(\mathscr{A}) \to K_0(\mathscr{A}_\tau)$, and "$G$" is the spectrum with $\pi_0\text{“}G\text{”} = G$, , $\pi_i\text{“}G\text{”} = 0$, $i \geq 1$. So we need to prove $K^W(\mathscr{P}(\mathscr{A} \text{ for } \mathscr{A}_\tau)) \cong K^W(\mathscr{B})$. By Theorem 3.1.1, we need to prove that the derived category $\mathbf{W}^{-1}(\mathscr{P}(\mathscr{A} \text{ for } \mathscr{A}_\tau))$ of $\mathscr{P}(\mathscr{A} \text{ for } \mathscr{A}_\tau)$ is equivalent to the derived category $\mathbf{W}^{-1}(\mathscr{B})$ of $\mathscr{B}$ through an exact functor. Thus after all, to prove Theorem 3.3(a), it will suffice to prove that

$$\mathbf{W}^{-1}(j_\tau^*) : \mathbf{W}^{-1}(\mathscr{P}(\mathscr{A} \text{ for } \mathscr{A}_\tau)) \to \mathbf{W}^{-1}(\mathscr{B})$$

is an equivalence of categories. Notice that for an admissible abelian category $\mathscr{A}$, $\mathbf{W}^{-1}(\mathscr{P}(\mathscr{A}))$ is a full subcategory of $D(\mathscr{A}) = \mathbf{W}^{-1}(C(\mathscr{A}))$, so it will suffice to prove the following:

3.3.1(a) For a perfect complex $E^\cdot$ in $\mathscr{A}_\tau$, its class $[E^\cdot] \in K_0(\mathscr{A}_\tau)$ is in the image of $K_0(\mathscr{A}) \to K_0(\mathscr{A}_\tau)$ iff there is a perfect complex $F^\cdot$ in $\mathscr{A}$ such that $j_\tau^* F^\cdot$ is isomorphic to $E^\cdot$ in $D(\mathscr{A}_\tau)$.

3.3.2(a) For any two perfect complexes $E_1^\cdot, E_2^\cdot$ in $\mathscr{A}$, and a morphism $f : j_\tau^* E_1^\cdot \to j_\tau^* E_2^\cdot$ in $D(\mathscr{A}_\tau)$, there is a third perfect complex $E'^\cdot$ in $\mathscr{A}$ and morphisms $b : E_1^\cdot \to E'^\cdot$ and $c : E_2^\cdot \to E'^\cdot$ in $D(\mathscr{A})$ such that $j_\tau^*(b) = j_\tau^*(c) \circ f$ in $D(\mathscr{A}_\tau)$ and $j_\tau^*(c)$ is an isomorphism in $D(\mathscr{A}_\tau)$.

3.3.3(a) For any two perfect complexes $E_1^\cdot, E_2^\cdot$ in $\mathscr{A}$, and two morphisms $f_1, f_2 : E_1^\cdot \to E_2^\cdot$ in $D(\mathscr{A})$, if $j_\tau^*(f_1) = j_\tau^*(f_2)$ in $D(\mathscr{A}_\tau)$, then there is a third perfect complex $E'^\cdot$ in $\mathscr{A}$ and a morphism $e : E_2^\cdot \to E'^\cdot$ in $D(\mathscr{A})$ such that $e \circ f_1 = e \circ f_2$ in $D(\mathscr{A})$ and $j_\tau^*(e)$ is an isomorphism in $D(\mathscr{A}_\tau)$.

From the same consideration as above, to prove Theorem 3.3(b), it will suffice to prove the following:

3.3.1(b) For a perfect complex $E^\cdot$ in $\mathscr{A}_\tau$ with $j_\mu^*(E^\cdot)$ acyclic in $\mathscr{A}_{\tau\cup\mu}$, its class $[E^\cdot] \in K_0(\mathscr{A}_\tau \text{ off } \mathscr{A}_{\tau\cup\mu})$ is in the image of $K_0(\mathscr{A} \text{ off } \mathscr{A}_\mu) \to K_0(\mathscr{A}_\tau \text{ off } \mathscr{A}_{\tau\cup\mu})$ iff there is a perfect complex $F^\cdot$ in $\mathscr{A}$ with $j_\mu^*(F^\cdot)$ acyclic in $\mathscr{A}_\mu$ such that $j_\tau^* F^\cdot$ is isomorphic to $E^\cdot$ in $D(\mathscr{A}_\tau)$.

3.2.2(b) For any two perfect complexes $E_1^\cdot, E_2^\cdot$ in $\mathscr{A}$ with $j_\mu^*(E_1^\cdot)$ and $j_\mu^*(E_2^\cdot)$ acyclic in $\mathscr{A}_\mu$, and a morphism $f : j_\tau^* E_1^\cdot \to j_\tau^* E_2^\cdot$ in $D(\mathscr{A}_\tau)$, there is a third perfect complex $E'^\cdot$ in $\mathscr{A}$ with $j_\mu^*(E'^\cdot)$ acyclic in $\mathscr{A}_\mu$ and two morphisms $b : E_1^\cdot \to E'^\cdot$ and $c : E_2^\cdot \to E'^\cdot$ in $D(\mathscr{A})$ such that $j_\tau^*(b) = j_\tau^*(c) \circ f$ in $D(\mathscr{A}_\tau)$ and $j_\tau^*(c)$ is an isomorphism in $D(\mathscr{A}_\tau)$.

3.3.3(b) For any two perfect complexes $E_1^\cdot, E_2^\cdot$ in $\mathscr{A}$ with $j_\mu^*(E_1^\cdot)$ and $j_\mu^*(E_2^\cdot)$ acyclic in $\mathscr{A}_\mu$, and two morphisms $f_1, f_2 : E_1^\cdot \to E_2^\cdot$ in $D(\mathscr{A})$, if $j_\tau^*(f_1) = j_\tau^*(f_2)$ in $D(\mathscr{A}_\tau)$, then there is a third perfect complex $E'^\cdot$ in $\mathscr{A}$ with $j_\mu^*(E'^\cdot)$ acyclic in $\mathscr{A}_\mu$ and a morphism $e : E_2^\cdot \to E'^\cdot$ in $D(\mathscr{A})$ such that $e \circ f_1 = e \circ f_2$ in $D(\mathscr{A})$ and $j_\tau^*(e)$ is an isomorphism in $D(\mathscr{A}_\tau)$.

The rest of this section will be given to the proof of 3.3.1(a),(b), 3.3.2(a),(b) and 3.3.3(a),(b). But [14, 5.2.6] shows that 3.3.2 in fact implies 3.3.3, and the proof can be taken over to our context. So what we need to show is 3.3.1 and 3.3.2. We start with the following:

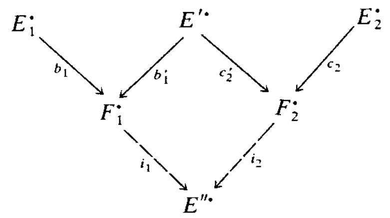
3.3.4. Remark. If 3.3.2 is true, then actually we can choose b and c to be chain maps in the following way: Let b be represented as

$$E_1^\cdot \xrightarrow{b_1} F_1^\cdot \xleftarrow{b_1'} E'^\cdot$$

and c be represented as

$$E_2^\cdot \xrightarrow{c_2} F_2^\cdot \xleftarrow{c_2'} E'^\cdot .$$

Let $E''^\cdot$ be the homotopy pushout



Then

$$j_\tau^* E_1^\cdot \xrightarrow{j_\tau^*(i_1 \circ b_1)} j_\tau^* E''^\cdot \xleftarrow{j_\tau^*(i_2 \circ c_2)} j_\tau^* E_2^\cdot$$

represents $j_\tau^*(c)^{-1} \circ j_\tau^*(b) = f$, i.e., $j_\tau^*(i_1 \circ b_1) = j_\tau^*(i_2 \circ c_2) \circ f$ in $D(\mathscr{A}_\tau)$. We replace $E'^\cdot$, b and c by $E''^\cdot$, $i_1 \circ b_1$ and $i_2 \circ c_2$, respectively. If $j_\mu^*(E'^\cdot)$ is acyclic in $\mathscr{A}_\mu$, then so is $j_\mu'(F_1^\cdot)$ and $j_\mu^*(F_1^\cdot)$, and so is $j_\mu^*(E''^\cdot)$.

Also notice that in $K_0(\mathscr{A})$ or $K_0(\mathscr{A} \text{ off } \mathscr{A}_\mu)$, we always have $[E''^\cdot] = [F_1^\cdot] + [F_2^\cdot] - [E'^\cdot] = [E'^\cdot] + [E'^\cdot] - [E'^\cdot] = [E'^\cdot]$.

3.4.1. Lemma (cf. [14, 5.4.1]). *Let $\{\mathscr{A}, (s_i, F_i), i = 1, \ldots, n\}$ be an admissible abelian category, (t, G) be a divisor on $\mathscr{A}$, $E^\cdot$ be a strictly perfect complex in $\mathscr{A}$ and $F^\cdot$ be an arbitrary complex in $\mathscr{A}$. Then:*

(a) *If $f : t^{-1}E^\cdot \to t^{-1}F^\cdot$ is a chain map, then there is an integer $p > 0$, a chain map $b : E^\cdot \to G^pF^\cdot$ such that $t^{-1}(b) = t^{-1}(t^p) \circ f$ (recall from 1.3.3 that t^p is the section of G^p induced by $t : \mathrm{id} \to G$, and we have the equality $G^pt^q \circ t^p = t^{p+q} = t^qG^p \circ t^p$).*

(b) *If $f_1, f_2 : E^\cdot \to F^\cdot$ are two chain maps such that $t^{-1}(f_1) = t^{-1}(f_2)$, then there is an integer $p > 0$ such that $t^p \circ f_1 = t^p \circ f_2$.*

(c) *If $f_1, f_2 : E^\cdot \to F^\cdot$ are two chain maps such that $t^{-1}(f_1) \overset{h}{=} t^{-1}(f_2)$, where '$\overset{h}{=}$' stands for 'being homotopy equal to', then there is an integer $p > 0$ such that $t^p \circ f_1 \overset{h}{=} t^p \circ f_2$.*

Proof. If Q is a small projective object in an Ab5 category, then $\mathrm{Hom}(Q, -)$ commutes with directed colimits. If Q is a locally small projective object in an admissible abelian category $\mathscr{A}$, by Proposition 1.5.1, $\mathscr{A}$ is equivalent to Σ as defined in Proposition 1.5.1, then $\mathrm{Hom}_{\mathscr{A}}(Q, -) \cong \mathrm{Hom}_\Sigma((s_i^{-1}(Q), \beta_{ij}), -)$ commutes with directed colimits because each $s_i^{-1}(Q)$ does. Therefore, for the strictly perfect complex $E^\cdot$, the mapping complex $\mathrm{Hom}^\cdot(E^\cdot, -)$ commutes with directed colimits, and then

$$\begin{aligned}
&\mathrm{Hom}^{\cdot}_{\mathscr{A}_\tau}(t^{-1}(E^{\cdot}), t^{-1}(F^{\cdot}))\\
&\quad\cong \mathrm{Hom}^{\cdot}_{\mathscr{A}}(E^{\cdot}, j_{t*}t^{-1}(F^{\cdot}))\\
&\quad= \mathrm{Hom}^{\cdot}_{\mathscr{A}}(E^{\cdot}, \varinjlim\,(F^{\cdot} \xrightarrow{t} GF^{\cdot} \to \cdots))\\
&\quad\cong \varinjlim\,(\mathrm{Hom}^{\cdot}_{\mathscr{A}}(E^{\cdot}, F^{\cdot}) \to \mathrm{Hom}^{\cdot}_{\mathscr{A}}(E^{\cdot}, GF^{\cdot}) \to \cdots)\,.
\end{aligned}$$

Since the cycle group $Z^0(\mathrm{Hom}^{\cdot})$ is the group of all chain maps and the cohomology groups $H^0(\mathrm{Hom}^{\cdot})$ is the group of all homotopy classes of chain maps, applying these functors, we get the lemma. □

We will first prove 3.3.1 and 3.3.2 for a special case: $\mathscr{A}$ has a set of small projective generators.

3.4.2. Lemma. *Let $\mathscr{A}$ be an* Ab5 *category with a set of small projective generators. We regard $\mathscr{A} = \{\mathscr{A}, (\mathrm{Id}, \mathrm{id})\}$ as a (strongly) admissible abelian category. Let $\{(t_j, G_j), j = 1, \dots, m\}$ be a finite set of compatible divisors on $\mathscr{A}$, $\tau = \bigcap_{j=1}^m t_j$, $E^{\cdot}$ be a strictly perfect complex in $\mathscr{A}$, and $F^{\cdot}$ be an arbitrary complex in $\mathscr{A}$. If $f : j_\tau^* E^{\cdot} \to j_\tau^* F^{\cdot}$ is a chain map, then there is an integer $p > 0$ and a chain map $b : E^{\cdot} \to K^{\cdot}(t_1^p, \dots, t_m^p)F^{\cdot}$ such that $j_\tau^*(b) = j_\tau^*(t) \circ f$ in $C(\mathscr{A}_\tau)$, where $K^{\cdot}(t_1^p, \dots, t_m^p)$ is the Koszul complex and $t = (t_1^p, \dots, t_m^p) : \mathrm{Id} \to K^{\cdot}(t_1^p, \dots, t_m^p)$ is the augmentation, as in Definition* 1.4.3.

Proof. First we prove that there is an integer $q > 0$ and chain maps $b_j : E^{\cdot} \to G_j^q F^{\cdot}$ such that $j_\tau^*(b_j) = j_\tau^*(t_j^q) \circ f$ for all $j = 1, \dots, m$. Applying Lemma 3.4.1(a) to

$$\bar{t}_j^{-1}(f) : \bar{t}_j^{-1}(j_\tau^* E^{\cdot}) = t_j^{-1}E^{\cdot} \to t_j^{-1}F^{\cdot} = \bar{t}_j^{-1}(j_\tau^* F^{\cdot})\,,$$

there is a $q_1 > 0$ big enough and $b'_j : E^{\cdot} \to G_j^{q_1}F^{\cdot}$ such that $t_j^{-1}(b'_j) = t_j^{-1}(t_j^{q_1}) \circ \bar{t}_j^{-1}(f)$ for all $j = 1, \dots, m$. Since $\{\mathscr{A}_\tau, (\bar{t}_j, \bar{G}_j), j = 1, \dots, m\}$ is an admissible abelian category from Lemma 1.6.4(b), and $j_\tau^* E^{\cdot}$ is a strictly perfect complex in $\mathscr{A}_\tau$, by applying Lemma 3.4.1(b) to $j_\tau^*(b'_j)$ and $j_\tau^*(t_j^{q_1}) \circ f$, we see that there is an integer $q_2 > 0$ such that

$$\bar{t}_j^{q_2}\bar{G}_j^{q_1} \circ j_\tau^*(b'_j) = \bar{t}_j^{q_2}\bar{G}_j^{q_1} \circ j_\tau^*(t_j^{q_1}) \circ f = \bar{t}_j^{q_1+q_2} \circ f$$

in $\mathscr{A}_\tau$. Let $b_j = t_j^{q_2}G_j^{q_1} \circ b'_j$, $q = q_1 + q_2$, then we have $j_\tau^*(b_j) = j_\tau^*(t_j^q) \circ f$.

According to Lemma 1.3.4, $\{(t_j^q, G_j^q), j = 1, \dots, m\}$ are still compatible divisors on $\mathscr{A}$. Obviously $(t_j^q)^{-1} = t_j^{-1}$, to simplify notation, we write (t_j, G_j) for (t_j^q, G_j^q). Then we have $b_j : E^{\cdot} \to G_jF^{\cdot}$ and $j_\tau^*(b_j) = j_\tau^*(t_j) \circ f$.

Let (t_it_j, G_iG_j) be the intersection of (t_i, G_i) and (t_j, G_j), as defined in 1.3.3. A straightforward check shows that

$$(t_it_j)^{-1}(t_iG_j \circ b_j) = (t_it_j)^{-1}(G_it_j \circ b_i)\,.$$

By Lemma 3.4.1(b) there is a $p_1 > 0$ such that

$$(t_it_j)^{p_1}G_iG_j \circ t_iG_j \circ b_j = (t_it_j)^{p_1}G_iG_j \circ G_it_j \circ b_i\,.$$

From the coherent commutativity of $\{(t_j, G_j)\}$, it is easy to check that

$$t_i^{p_1+1}G_j^{p_1+1} \circ t_j^{p_1}G_j \circ b_j = G_i^{p_1+1}t_j^{p_1+1} \circ t_i^{p_1}G_i \circ b_i\,.$$

Choose p_1 big enough so that this holds for all i, j. Let $p = p_1 + 1$, and

$$b = (t_1^{p_1}G_1 \circ b_1, \dots, t_m^{p_1}G_m \circ b_m) : E^{\cdot} \to K^{\cdot}(t_1^p, \dots, t_m^p)F^{\cdot}$$

be induced from the chain map of the double complexes:

$$\begin{array}{ccccccc}
E^{\cdot} & =: & E^{\cdot} & \longrightarrow & 0 & \cdots\longrightarrow & 0\\
\big\downarrow & & \big\downarrow{\scriptstyle b} & & \big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle 0}\\
K^{\cdot}(t_1^p, \dots, t_m^p)F^{\cdot} & =: & \bigoplus_j G_j^pF^{\cdot} & \longrightarrow & \bigoplus_{i<j} G_i^pG_j^pF^{\cdot} & \cdots\longrightarrow & G_1^p \cdots G_m^pF^{\cdot}
\end{array}$$

Then the equality $j_\tau^*(b) = j_\tau^*(t) \circ f$ follows from the equalities

$$j_\tau^*(t_j^{p_1}G_j \circ b_j) = j_\tau^*(t_j^{p_1}G_j) \circ j_\tau^*(b_j) = j_\tau^*(t_j^{p_1}G_j) \circ j_\tau^*(t_j) \circ f = j_\tau^*(t_j^{p_1+1}) \circ f$$

for all j. This finishes the proof of the lemma. □

Notice that $j_\tau^*(t)$ is a quasi-isomorphism by Lemma 1.4.4, so the above lemma extends a chain map in $\mathscr{A}_\tau$ to a morphism in $D(\mathscr{A})$. In order to extend a morphism in $D(\mathscr{A}_\tau)$ to $D(\mathscr{A})$, as is needed in 3.3.2, we need to work harder.

3.4.3. Lemma. *Let $\mathscr{A}$ be an* Ab5 *category with a set of small projective generators, $\{(t_j, G_j), j = 1, \dots, m\}$ be a finite set of compatible divisors on $\mathscr{A}$, $\tau = \bigcap_{j=1}^m t_j$; then:*

(a) *$j_\tau^*(\mathbf{P}(\mathscr{A}))$ generates $\mathscr{A}_\tau$.*

(b) *For any perfect complex $E^{\cdot}$ in $\mathscr{A}_\tau$, there is a bounded above complex $F^{\cdot}$ in $\mathscr{A}_\tau$ with all $F^n \in j_\tau^*(\mathbf{P}(\mathscr{A}))$, and a quasi-isomorphism $F^{\cdot} \to E^{\cdot}$.*

(c) *If $F^{\cdot}$ is a bounded above complex in $\mathscr{A}_\tau$ with all $F^n \in j_\tau^*(\mathbf{P}(\mathscr{A}))$, then for any given integer r, there is a strictly perfect complex $P^{\cdot}$ in $\mathscr{A}$ with $P^i = 0$ when $i \le r$, a complex $M^{\cdot}$ in $\mathscr{A}_\tau$ with $M^j = 0$ when $j > r$, and a morphism $d^r : M^r \to j_\tau^* P^{r-1}$ such that*

$$C_r(M^\cdot, d^r, P^\cdot) =: \cdots M^{r-1} \to M^r \xrightarrow{d^r} j_\tau^* P^{r+1} \to j_\tau^* P^{r+2} \to \cdots$$

becomes a complex in $\mathscr{A}_\tau$, *for which there is a quasi-isomorphism*

$$F^\cdot \to C_r(M^\cdot, d^r, P^\cdot).$$

(d) *If* $E^\cdot$ *is a strictly perfect complex in* $\mathscr{A}$ *with* $E^i = 0$ *when* $i \le n$, *and*

$$f : j_\tau^* E^\cdot \to C_r(M^\cdot, d^r, P^\cdot)$$

is a quasi-isomorphism with $r \ll n$, *then there is a* $p > 0$ *and a chain map*

$$g : C_r(M^\cdot, d^r, P^\cdot) \to j_\tau^*(K^\cdot(t_1^p, \ldots, t_m^p)F^\cdot)$$

which is a quasi-isomorphism.

Proof. (a) Let $A \in \mathscr{A}_\tau$, because $\mathbf{P}(\mathscr{A})$ generates $\mathscr{A}$, we have a surjection $\coprod P_\alpha \to j_{\tau *}A$ with $P_\alpha \in \mathbf{P}(\mathscr{A})$. Then

$$j_\tau^*\left(\coprod P_\alpha\right) = \coprod j_\tau^*(P_\alpha) \to j_\tau^* j_{\tau *} A = A$$

is a surjection because j_τ^* is exact. So $j_\tau^*(\mathbf{P}(\mathscr{A}))$ generates $\mathscr{A}_\tau$.

(b) In Lemma 2.1.2, let $\mathbf{A} = \mathscr{A}_\tau$, $\mathbf{D}$ be the full subcategory of $\mathscr{A}_\tau$ with $\mathrm{Ob}(\mathbf{D}) = j_\tau^*(\mathbf{P}(\mathscr{A}))$, $C_0(\mathscr{A}_\tau)$ = all perfect complexes in $\mathscr{A}_\tau$. To check condition 2.1.2.1, let $E^\cdot \in C_0(\mathscr{A}_\tau)$, and suppose $H^i(E^\cdot) = 0$ when $i \ge n$, and $M \to H^{n-1}(E^\cdot)$ is an epimorphism in $\mathscr{A}_\tau$. Since $j_\tau^*(\mathbf{P}(\mathscr{A}))$ generates $\mathscr{A}_\tau$, we have an epimorphism $\coprod_{\alpha \in I} j_\tau^*(P_\alpha) \to M$, so the composite

$$\coprod_{\alpha \in I} j_\tau^*(P_\alpha) \to M \to H^{n-1}(E^\cdot)$$

is also an epimorphism. We claim that $H^{n-1}(E^\cdot)$ is locally f.g. In fact, since $E^\cdot$ is perfect, for each $j = 1, \ldots, m$, we have a strictly perfect complex $E_j^\cdot$ in $\mathscr{A}_{t_j}$ and a quasi-isomorphism $E_j^\cdot \to \bar{t}_j^{-1}E^\cdot$. So

$$H^i(E_j^\cdot) \cong H^i(\bar{t}_j^{-1}E^\cdot) = \bar{t}_j^{-1}H^i(E^\cdot) = 0$$

for $i \ge n$, thus $E_j^i = B^i(E_j^\cdot) \oplus Z^i(E_j^\cdot)$ for $i \ge n-1$. In particular, $E_j^{n-1} = B^{n-1}(E_j^\cdot) \oplus Z^{n-1}(E_j^\cdot)$, so $Z^{n-1}(E_j^\cdot)$ is a small projective object in $\mathscr{A}_{t_j}$ for each $j = 1, \ldots, m$, and we have epimorphisms

$$Z^{n-1}(E_j^\cdot) \to H^{n-1}(E_j^\cdot) \cong H^{n-1}(\bar{t}_j^{-1}E^\cdot) = \bar{t}_j^{-1}H^{n-1}(E^\cdot)$$

for all j. Thus by definition, $H^{n-1}(E^\cdot)$ is locally f.g. in $\mathscr{A}_\tau$. From Corollary 1.6.10, we have a finite subset $J \subset I$ such that the composite

$$\coprod_{\alpha \in J} j_\tau^*(P_\alpha) \to \coprod_{\alpha \in I} j_\tau^*(P_\alpha) \to H^{n-1}(E^\cdot)$$

is an epimorphism. But $\coprod_{\alpha \in J} j_\tau^*(P_\alpha) = j_\tau^*(\coprod_{\alpha \in J} P_\alpha) \in \mathbf{D}$, so condition 2.1.2.1 is met. We can apply Lemma 2.1.2 to get a bounded above complex $F^\cdot$ with all $F^n \in j_\tau^*(\mathbf{P}(\mathscr{A}))$ and a quasi-isomorphism $F^\cdot \to E^\cdot$.

(c) Let $F^\cdot$ be a bounded above complex in $\mathscr{A}_\tau$ with all $F^n \in j_\tau^*(\mathbf{P}(\mathscr{A}))$. We use induction. Since $F^\cdot$ is bounded above, we may assume $F^\cdot$ is already of the form of $C_n(M^\cdot, d^n, P^\cdot)$ for some integer n, i.e.,

$$F^\cdot =: \cdots \to M^{n-1} \xrightarrow{d^{n-1}} M^n \xrightarrow{d^n} j_\tau^* P^{n+1} \xrightarrow{j_\tau^*(\partial)} j_\tau^* p^{n-2} \to \cdots$$

with $(P^\cdot, \partial)$ a strictly perfect complex in $\mathscr{A}$. Let $M^n = j_\tau^* Q$, with $Q \in \mathrm{P}(\mathscr{A})$ regarded as a complex; then we have the chain map $j_\tau^* Q \xrightarrow{d^n} j_\tau^* P^\cdot$. Applying Lemma 3.4.2, we have a chain map

$$b : Q \to K^\cdot(t_1^p, \ldots, t_m^p)P^\cdot$$

such that $j_\tau^*(b) = j_\tau^*(t) \circ d^n$ with $j_\tau^*(t)$ a quasi-isomorphism. Now let

$$P'^\cdot =: \mathrm{cone}(Q \xrightarrow{b} K^\cdot(t_1^p, \ldots, t_m^p)P^\cdot),$$
$$M'^\cdot = \sigma^{\le n-1}M^\cdot =: \cdots \to M^{n-2} \to M^{n-1} \to 0 \cdots.$$

Then $C_{n-1}(M'^\cdot, d^{n-1}, P'^\cdot)$ becomes a complex, and $\mathrm{cone}(j_\tau^* b) = C_{n-1}(M'^\cdot, d^{n-1}, P'^\cdot)$. We have the following commutative diagram of chain maps and complexes:

$$\begin{array}{ccccc}
M^\cdot & \xrightarrow{d^n} & j_\tau^*(P^\cdot) & \longrightarrow & \mathrm{cone}(d^n) = F^\cdot \\
\| & & \downarrow{\scriptstyle j_\tau^*(t)} & & \\
(M'^\cdot \to j_\tau^* Q) & \xrightarrow{j_\tau^*(b)} & j_\tau^* K^\cdot(t_1^p, \ldots, t_m^p)P^\cdot & \longrightarrow & \mathrm{cone}(j_\tau^* b)
\end{array}$$

Since $j_\tau^*(t)$ is a quasi-isomorphism, we get a quasi-isomorphism $F^\cdot \to C_{n-1}(M'^\cdot, d^{n-1}, P'^\cdot)$. We iterate this procedure to construct $C_r(M^\cdot, d^r, P^\cdot)$.

(d) Because $f : j_\tau^* E^\cdot \to C_r(M^\cdot, d^r, P^\cdot)$ is a quasi-isomorphism and $\hat{R}j_{\tau *}$ preserves quasi-isomorphisms (Lemma 2.3.2(a)), we get an acyclic mapping cone Z:

$$\hat{R}j_{\tau *} j_\tau^* E^\cdot \xrightarrow{\hat{R}j_{\tau *}(f)} \hat{R}j_{\tau *} C_r(M^\cdot, d^r, P^\cdot) \xrightarrow{(0,\mathrm{id})} Z^\cdot = \mathrm{cone}(\hat{R}j_{\tau *}(f)).$$

Let $(0, \varphi)$ be the composite of

$$P^\cdot \to j_\tau^* C_r(M^\cdot, d^r, P^\cdot) \to \hat{R}j_{\tau*}C_r(M^\cdot, d^r, P^\cdot) \to Z^\cdot .$$

Since $P^\cdot$ is a bounded complex of small projective objects in $\mathcal{A}$ and $Z^\cdot$ is acyclic, we have $(0, \varphi) \overset{h}{=} 0$. Let $w = (u, v)$ be a homotopy for $(0, \varphi) \overset{h}{=} 0$. Then a standard computation shows that

$$u : P^\cdot \to \hat{R}j_{\tau*}j_\tau^* E^\cdot$$

is a chain map, $\varphi \overset{h}{=} \hat{R}j_{\tau*}(f) \circ u$, and

$$v : P^\cdot \to \hat{R}j_{\tau*}C_r(M^\cdot, d^r, P^\cdot)[-1]$$

is a homotopy for $\varphi \overset{h}{=} \hat{R}j_{\tau*}(f) \circ u$. Because

$$\begin{aligned}
\hat{R}j_{\tau*}j_\tau^* E^\cdot &= \operatorname{Tot}\Big(\bigoplus_{j=1}^m t_j^{-1}E^\cdot \to \bigoplus_{i<j} t_i^{-1}t_j^{-1}E^\cdot \to \cdots\Big)\\
&= \operatorname{Tot}\Big(\bigoplus_{j=1}^m (\varinjlim (E^\cdot \to G_jE^\cdot \to \cdots))\\
&\qquad \to \bigoplus_{i<j} (\varinjlim (E^\cdot \to G_iG_jE^\cdot \to \cdots)) \to \cdots\Big)\\
&= \varinjlim \Big(\operatorname{Tot}\Big(\bigoplus_{j=1}^m E^\cdot \to \bigoplus_{i<j} E^\cdot \to \cdots\Big)\\
&\qquad \to \operatorname{Tot}\Big(\bigoplus_{j=1}^m G_jE^\cdot \to \bigoplus_{i<j} G_iG_jE^\cdot \to \cdots\Big) \to \cdots\Big)\\
&= \varinjlim (K^\cdot(t_1^0, \ldots, t_m^0)E^\cdot \to K^\cdot(t_1, \ldots, t_m)E^\cdot \to \cdots)\\
&= \varinjlim_q K^\cdot(t_1^q, \ldots, t_m^q)E^\cdot
\end{aligned}$$

and $P^\cdot$ is a bounded complex of small projective objects, u can be factorized as $u = \psi_p \circ u'$ for some integer $p > 0$, where

$$u' : P^\cdot \to K^\cdot(t_1^p, \ldots, t_m^p)E^\cdot$$

is a chain map and

$$\psi_p : K^\cdot(t_1^p, \ldots, t_m^p)E^\cdot \to \varinjlim_q K^\cdot(t_1^q, \ldots, t_m^q)E^\cdot$$

is the structure map of the colimit. Since each $j_\tau^* E^\cdot \to j_\tau^* K^\cdot(t_1^q, \ldots, t_m^q)E^\cdot$ is a quasi-isomorphism by Lemma 1.4.4, so is each $j_\tau^* K^\cdot(t_1^q, \ldots, t_m^q)E^\cdot \to j_\tau^* K^\cdot(t_1^{q+1}, \ldots, t_m^{q+1})E^\cdot$, and so is

$$j_\tau^* \psi_p : j_\tau^* K^\cdot(t_1^p, \ldots, t_m^p)E^\cdot \to j_\tau^* \varinjlim_q K^\cdot(t_1^q, \ldots, t_m^q)E^\cdot .$$

Since $\varphi \overset{h}{=} \hat{R}j_{\tau*}(f) \circ u$, $\hat{R}j_{\tau*}(f)$ is a quasi-isomorphism, and $j_\tau^* \varphi$ is an r-quasi-isomorphism because φ is the composite

$$P^\cdot \to j_{\tau*}j_\tau^* P^\cdot \to j_\tau^* C_r(M^\cdot, d^r, P^\cdot) \to \hat{R}j_{\tau*}C_r(M^\cdot, d^r, P^\cdot),$$

$j_\tau^*(u)$ is also an r-quasi-isomorphism. Since $j_\tau^*(u) = j_\tau^*(\psi_p) \circ j_\tau^*(u')$ and $j_\tau^*(\psi_p)$ is a quasi-isomorphism, $j_\tau^*(u')$ is also an r-quasi-isomorphism. Since $E^i = 0$ when $i < n$ and $f : j_\tau^* E^\cdot \to C_r(M^\cdot, d^r, P^\cdot)$ is a quasi-isomorphism, we have $H^i(C_r(M^\cdot, d^r, P^\cdot)) = 0$ when $i < n$. Since $r \ll n$, we see that

$$j_\tau^*(u') : j_\tau^* P^\cdot \to j_\tau^* K^\cdot(t_1^p, \ldots, t_m^p)E^\cdot$$

can be extended to a quasi-isomorphism

$$g : C_r(M^\cdot, d^r, P^\cdot) \to j_\tau^* K^\cdot(t_1^p, \ldots, t_m^p)E^\cdot$$

by setting $g_k = 0$ in degree $k < r$. □

Now we can prove a special case of 3.3.2 and 3.3.1.

3.4.4. Proposition. *Let $\mathcal{A}$ be an* Ab5 *category with a set of small projective generators, $\{(t_j, G_j), j = 1, \ldots, m\}$ and $\{(u_k, H_k), k = 1, \ldots, r\}$ be two finite sets of compatible divisors on $\mathcal{A}$, $\tau = \bigcap_{j=1}^m t_j$, and $\mu = \bigcap_{k=1}^r u_k$; then:*

(a) For any two perfect complexes $E_1^\cdot, E_2^\cdot$ in $\mathcal{A}$, and a morphism $f : j_\tau^* E_1^\cdot \to j_\tau^* E_2^\cdot$ in $D(\mathcal{A}_\tau)$, there is a third perfect complex $E'^\cdot$ in $\mathcal{A}$ and morphisms $b : E_1^\cdot \to E'^\cdot$ and $c : E_2^\cdot \to E'^\cdot$ in $D(\mathcal{A})$ such that $j_\tau^*(b) = j_\tau^*(c) \circ f$ in $D(\mathcal{A}_\tau)$ and $j_\tau^*(c)$ is an isomorphism in $D(\mathcal{A}_\tau)$.

(b) In (a) if we further assume that $j_\mu^*(E_1^\cdot)$ and $j_\mu^*(E_2^\cdot)$ are acyclic in $\mathcal{A}_\mu$, then we can have $E'^\cdot$ with $j_\mu^*(E'^*)$ acyclic in $\mathcal{A}_\mu$.

Proof. Since $\mathcal{A}$ has a set of small projective generators, we can assume $E_1^\cdot$ and $E_2^\cdot$ are both strictly perfect and $E_1^i = 0 = E_2^i$ when $i < n$ for some n. Let f be represented as

$$j_\tau^* E_1^\cdot \xleftarrow{\sim} G^\cdot \to j_\tau^* E_2^\cdot .$$

Then $G^\cdot$ is a perfect complex in $\mathcal{A}_\tau$. By Lemma 3.4.3(b), there is a bounded above complex $F^\cdot$ with all $F^i \in j_\tau^*(\mathbf{P}(\mathcal{A}))$ and a quasi-isomorphism $F^\cdot \xrightarrow{\sim} G^\cdot$, so f

can be represented as

$$j_\tau^* E_1^\cdot \xleftarrow{\sim} F^\cdot \to j_\tau^* E_2^\cdot .$$

Let $H^\cdot = j_\tau^* E_1^\cdot \cup_{F^\cdot} j_\tau^* E_2^\cdot$ be the homotopy pushout. Then f can be represented as

$$j_\tau^* E_1^\cdot \to H^\cdot \xleftarrow{\sim} j_\tau^* E_2^\cdot ,$$

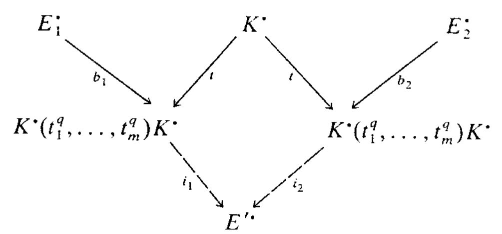
and obviously $H^\cdot$ is also bounded above and all $H^i \in j_\tau^*(\mathbf{P}(\mathscr{A}))$. By Lemma 3.4.3(c), choose $r \ll n$ such that we can find a complex $C_r(M^\cdot, d', P^\cdot)$ and a quasi-isomorphism $H^\cdot \to C_r(M^\cdot, d', P^\cdot)$, then f can be represented as

$$j_\tau^* E_1^\cdot \to C_r(M^\cdot, d', P^\cdot) \xleftarrow{\sim} j_\tau^* E_2^\cdot .$$

Now applying Lemma 3.4.3(d) to $j_\tau^* E_2^\cdot \xrightarrow{\sim} C_r(M^\cdot, d', P^\cdot)$, we get a quasi-isomorphism $C_r(M^\cdot, d', P^\cdot) \to j_\tau^* K^\cdot(t_1^p, \ldots, t_m^p)E_2^\cdot$ for some $p > 0$. Then f can be represented as

$$j_\tau^* E_1^\cdot \xrightarrow{f_1} j_\tau^* K^\cdot(t_1^p, \ldots, t_m^p)E_2^\cdot \xleftarrow[\sim]{f_2} j_\tau^* E_2^\cdot .$$

Put $K^\cdot = K^\cdot(t_1^p, \ldots, t_m^p)E_2^\cdot$; then $K^\cdot$ is a strictly perfect complex in $\mathscr{A}$ because all $G_{j_1}^p \cdots G_{j_u}^p E_2^\cdot$ are strictly perfect. Applying Lemma 3.4.2 to f_1, f_2, there are chain maps b_1, b_2 and $q > 0$ as in the diagram



such that $j_\tau^*(b_1) = j_\tau^*(t) \circ f_1$ and $j_\tau^*(b_2) = j_\tau^*(t) \circ f_2$. Let $E'^\cdot$ be the homotopy pushout, $b = i_i \circ B_1$ and $c = i_2 \circ B_2$; then f can be represented as

$$j_\tau^* E_1^\cdot \xrightarrow{j_\tau^*(b)} j_\tau^* E'^\cdot \xleftarrow{j_\tau^*(c)} j_\tau^* E_2^\cdot ,$$

i.e., $j_\tau^*(b) = j_\tau^*(c) \circ f$, and clearly $j_\tau^*(c)$ is an isomorphism in $D(\mathscr{A}_\tau)$ because f_2 is.

(b) In the proof of (a) above, if $j_\mu^* E_2^\cdot$ is acyclic in $\mathscr{A}_\mu$, then $j_\mu^*(K^\cdot) = j_\mu^*(K^\cdot(t_1^p, \ldots, t_m^p)E_2^\cdot)$ is acyclic in $\mathscr{A}_\mu$ because each $j_\mu^* G_{j_1}^p \cdots G_{j_u}^p E_2^\cdot$ is acyclic in $\mathscr{A}_\mu$. Then so is $j_\mu^*(K^\cdot(t_1^q, \ldots, t_m^q)K^\cdot)$, and so is $j_\mu^* E'^\cdot$ □

3.4.5. Proposition. *With the same assumptions and notations as in Proposition 3.4.4, we have:*

(a) *For a perfect complex $E^\cdot$ in $\mathscr{A}_\tau$, its class $[E^\cdot] \in K_0(\mathscr{A}_\tau)$ is in the image of $K_0(\mathscr{A}) \to K_0(\mathscr{A}_\tau)$ iff there is a perfect complex $F^\cdot$ in $\mathscr{A}$ such that $j_\tau^* F^\cdot$ is isomorphic to $E^\cdot$ in $D(\mathscr{A}_\tau)$.*

(b) *For a perfect complex $E^\cdot$ in $\mathscr{A}_\tau$ with $j_{\bar\mu}^*(E^\cdot)$ acyclic in $\mathscr{A}_{\tau\cup\mu}$, its class $[E^\cdot] \in K_0(\mathscr{A}_\tau \text{ off } \mathscr{A}_{\tau\cup\mu})$ is in the image of $K_0(\mathscr{A} \text{ off } \mathscr{A}_\mu) \to K_0(\mathscr{A}_\tau \text{ off } \mathscr{A}_{\tau\cup\mu})$ iff there is a perfect complex $F^\cdot$ in $\mathscr{A}$ with $j_\mu^*(F^\cdot)$ acyclic in $\mathscr{A}_\mu$ such that $j_\tau^* F^\cdot$ is isomorphic to $E^\cdot$ in $D(\mathscr{A}_\tau)$.*

Proof. (a) First, applying Lemma 2.2.1(d) to $\hat{R}j_{\tau *}E^\cdot$, we have a directed inductive system $\{E_\alpha^\cdot\}$ of strictly perfect complexes in $\mathscr{A}$ and a quasi-isomorphism $\varinjlim_\alpha E_\alpha^\cdot \to \hat{R}j_{\tau *}E^\cdot$. Let f be the morphism in $D(\mathscr{A}_\tau)$ represented by

$$E^\cdot \to j_\tau^* \hat{R}j_{\tau *}E^\cdot \xleftarrow{\sim} j_\tau^* \varinjlim_\alpha E_\alpha^\cdot ;$$

then f is an isomorphism in $D(\mathscr{A}_\tau)$, since $E^\cdot \to j_\tau^* \hat{R}j_{\tau *}E^\cdot$ is a quasi-isomorphism by Lemma 2.3.2(b). Because $E^\cdot$ is perfect,

$$\mathrm{Hom}_{D(\mathscr{A}_\tau)}(E^\cdot, j_\tau^* \varinjlim_\alpha E_\alpha^\cdot) \cong \varinjlim_\alpha \mathrm{Hom}_{D(\mathscr{A}_\tau)}(E^\cdot, j_\tau^* E_\alpha^\cdot) ,$$

hence f can be factorized in $D(\mathscr{A}_\tau)$ as

$$E^\cdot \to j_\tau^* E_\alpha^\cdot \to j_\tau^*(\varinjlim_\alpha E_\alpha^\cdot)$$

for some α. Thus $E^\cdot$ is a summand of $j_\tau^* E_\alpha^\cdot$ in $D(\mathscr{A}_\tau)$, i.e., there is another complex $E'^\cdot$ in $\mathscr{A}_\tau$ such that $E^\cdot \oplus E'^\cdot$ is isomorphic to $j_\tau^* E_\alpha^\cdot$ in $D(\mathscr{A}_\tau)$.

Next we will use Grayson's cofinality trick to finish the proof of (a) (cf. [6, Section 1] or [14, Proposition 5.5.4]).

Let π be the abelian monoid with the generators of all quasi-isomorphism classes $\langle E^\cdot \rangle$ of perfect complexes in $\mathscr{A}_\tau$, modulo the relations

(i) $\langle E_1^\cdot \rangle + \langle E_2^\cdot \rangle = \langle E_1^\cdot \oplus E_2^\cdot \rangle$,

(ii) $\langle E^\cdot \rangle = 0$ if $E^\cdot$ is isomorphic in $D(\mathscr{A}_\tau)$ to $j_\tau^* F$ for some perfect complex $F^\cdot$ in $\mathscr{A}$.

As is proved above, for any perfect complex $E^\cdot$ in $\mathscr{A}_\tau$, there is another complex $E'^\cdot$ in $\mathscr{A}_\tau$ and a perfect complex $F^\cdot$ in $\mathscr{A}$ such that $E^\cdot \oplus E'^\cdot$ is isomorphic to $j_\tau^* E_\alpha^\cdot$ in $D(\mathscr{A}_\tau)$, then $E'^\cdot$ is also perfect and $\langle E^\cdot \rangle + \langle E'^\cdot \rangle = \langle E^\cdot \oplus E'^\cdot \rangle = 0$. So actually π is already a group. Suppose $\langle G^\cdot \rangle = 0$ in π; this means that there are perfect complexes $H^\cdot$ in $\mathscr{A}_\tau$ and $K^\cdot, L^\cdot$ in $\mathscr{A}$ such that $G^\cdot \oplus H^\cdot \oplus j_\tau^* K^\cdot$ is isomorphic in $D(\mathscr{A}_\tau)$ to $H^\cdot \oplus j_\tau^* L^\cdot$. Let $H'^\cdot$ be such that $H^\cdot \oplus H'^\cdot$ is isomorphic in $D(\mathscr{A}_\tau)$ to $j_\tau^* F^\cdot$ for some perfect complex $F^\cdot$ in $\mathscr{A}$; then $G^\cdot \oplus j_\tau^*(F^\cdot \oplus K^\cdot)$ is isomorphic in $D(\mathscr{A}_\tau)$ to $j_\tau^*(F^\cdot \oplus L^\cdot)$, so $G^\cdot$ is isomorphic in $D(\mathscr{A}_\tau)$ to the cone of $j_\tau^*(F^\cdot \oplus K^\cdot) \to j_\tau^*(F^\cdot \oplus L^\cdot)$. By Proposition 3.4.4(a), the morphism $j_\tau^*(F^\cdot \oplus K^\cdot) \to j_\tau^*(F^\cdot \oplus L^\cdot)$ can be extended to a morphism in $D(\mathscr{A})$, so $\text{cone}(j_\tau^*(F^\cdot \oplus K^\cdot) \to j_\tau^*(F^\cdot \oplus L^\cdot))$ is isomorphic in $D(\mathscr{A}_\tau)$ to $j_\tau^*(\text{cone}(F^\cdot \oplus K^\cdot \to F^\cdot \oplus L^\cdot))$. Thus $\langle G^\cdot \rangle = 0$ in π iff $G^\cdot$ is isomorphic in $D(\mathscr{A}_\tau)$ to $j_\tau^* F^\cdot$ for some perfect complex $F^\cdot$ in $\mathscr{A}$.

Hence it remains only to show that π is isomorphic to $K_0(\mathscr{A}_\tau)/\text{Im}\, K_0(\mathscr{A})$. Comparing the presentation of π and $K_0(\mathscr{A}_\tau)/\text{Im}\, K_0(\mathscr{A})$ in terms of generators and relations, we see that we only need to show that if $E_1^\cdot \to E_2^\cdot \to E_3^\cdot$ is a homotopy fibre sequence of perfect complexes in $\mathscr{A}_\tau$, then $\langle E_2^\cdot \rangle = \langle E_1^\cdot \rangle + \langle E_3^\cdot \rangle$. Let $E_1'^\cdot, E_3'^\cdot$ be such that $E_1^\cdot \oplus E_1'^\cdot$ and $E_3^\cdot \oplus E_3'^\cdot$ are isomorphic in $D(\mathscr{A}_\tau)$ to $j_\tau^* F_1^\cdot$ and $j_\tau^* F_2^\cdot$ respectively for some perfect complexes $F_1^\cdot$ and $F_2^\cdot$ in $\mathscr{A}$. By adding $E_1'^\cdot \to E_1'^\cdot \to 0$ and $0 \to E_3'^\cdot \to E_3'^\cdot$ to $E_1^\cdot \to E_2^\cdot \to E_3^\cdot$, we get a homotopy fibre sequence

$$E_1^\cdot \oplus E_1'^\cdot \to E_2^\cdot \oplus E_1'^\cdot \oplus E_3'^\cdot \to E_3^\cdot \oplus E_3'^\cdot ;$$

then

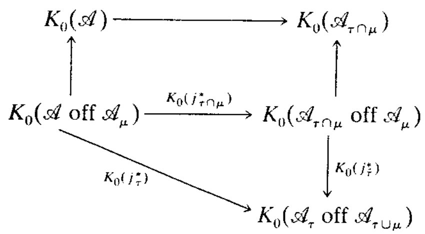
$$\begin{aligned} E_2^\cdot \oplus E_1'^\cdot \oplus E_3'^\cdot &\cong \text{cone}(E_3^\cdot \oplus E_3'^\cdot[-1] \to E_1^\cdot \oplus E_1'^\cdot) \\ &\cong \text{cone}(j_\tau^* F_3^\cdot[-1] \to j_\tau^* F_1^\cdot) \\ &\cong j_\tau^*(\text{cone}(F_3^\cdot[-1] \to F_1^\cdot)) \end{aligned}$$

are isomorphic in $D(\mathscr{A}_\tau)$, so

$$0 = \langle E_2^\cdot \rangle + \langle E_1'^\cdot \rangle + \langle E_3'^\cdot \rangle = \langle E_2^\cdot \rangle - \langle E_1^\cdot \rangle - \langle E_3^\cdot \rangle ,$$

as required.

(b) Look at the following commutative diagram



From the excision Theorem 3.2, $K_0(j_{\bar\tau}^*)$ is an isomorphism with the inverse $K_0(\hat{R}j_{\bar\tau}^*)$. If $E^\cdot \in \mathscr{P}(\mathscr{A}_\tau \text{ off } \mathscr{A}_{\tau\cup\mu})$ is such that $[E^\cdot]$ is in the image of $K_0(j_\tau^*)$, then $[\hat{R}j_{\bar\tau}^* E^\cdot]$ is in the image of $K_0(j_{\tau\cap\mu}^*)$, so $[\hat{R}j_{\bar\tau}^* E^\cdot]$ is in the image of $K_0(\mathscr{A}) \to K_0(\mathscr{A}_{\tau\cap\mu})$. From (a) above, there is a perfect complex $F^\cdot$ in $\mathscr{A}$ such that

$j_{\tau\cap\mu}^* F^\cdot$ is isomorphic to $\hat{R}j_{\bar\tau}^* E^\cdot$ in $D(\mathscr{A}_{\tau\cap\mu})$. So

$$j_\tau^* F^\cdot = j_{\bar\tau}^*(j_{\tau\cap\mu}^* F^\cdot) \cong j_{\bar\tau}^*(\hat{R}j_{\bar\tau}^* E^\cdot) \cong E^\cdot$$

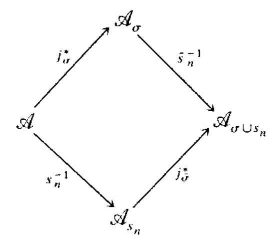
are isomorphic in $D(\mathscr{A}_\tau)$, and

$$j_\mu^* F^\cdot = j_{\bar\mu}^*(j_{\tau\cap\mu}^* F^\cdot) \cong j_{\bar\mu}^*(\hat{R}j_{\bar\tau}^* E^\cdot)$$

are isomorphic in $D(\mathscr{A}_\mu)$. Because $j_{\bar\mu}^*(\hat{R}j_{\bar\tau}^* E^\cdot)$ is acyclic in $\mathscr{A}_\mu$ by Lemma 2.5.1, $j_\mu^* F^\cdot$ is acyclic in $\mathscr{A}_\mu$. □

With $\mathscr{A}$ an Ab5 category having a set of small projective generators, Proposition 3.4.5 is 3.3.1 and Proposition 3.4.4 is 3.3.2, so we have proved Theorem 3.3 for this special case which serves as the first step of the induction we are going to use to prove Theorem 3.3 for general admissible abelian categories.

3.5.1. Lemma. *Let $\{\mathscr{A}, (s_i, F_i), i = 1, \ldots, n\}$ be an admissible abelian category, $\{(u_k, H_k), k = 1, \ldots, r\}$ be a finite set of divisors on $\mathscr{A}$ such that $\{(s_i, F_i), (u_k, H_k)\}$ are compatible, $\sigma = \bigcap_{i=1}^{n-1} s_i$, and $\mu = \bigcap_{k=1}^{r} u_k$; notice that we have the following commutative diagram:*



If $F^\cdot \in \mathscr{P}(\mathscr{A}_\sigma \text{ off } \mathscr{A}_{\sigma\cup\mu})$ is such that the class $[\bar{s}_n^{-1} F^\cdot]$ is in the image of

$$K_0(\mathscr{A}_{s_n} \text{ off } \mathscr{A}_{s_n\cup\mu}) \to K_0(\mathscr{A}_{\sigma\cup s_n} \text{ off } \mathscr{A}_{\sigma\cup s_n\cup\mu}),$$

then there is an $E^\cdot \in \mathscr{P}(\mathscr{A} \text{ off } \mathscr{A}_\mu)$ such that $j_\sigma^* E^\cdot$ is isomorphic to $F^\cdot$ in $D(\mathscr{A}_\sigma)$.

Proof. Since

$$[\bar{s}_n^{-1} F^\cdot] \in \text{Im}(K_0(\mathscr{A}_{s_n} \text{ off } \mathscr{A}_{s_n\cup\mu}) \to K_0(\mathscr{A}_{\sigma\cup s_n} \text{ off } \mathscr{A}_{\sigma\cup s_n\cup\mu})) ,$$

by Proposition 3.4.5(b), we have a $G^\cdot \in \mathscr{P}(\mathscr{A}_{s_n} \text{ off } \mathscr{A}_{s_n\cup\mu})$ and an isomorphism in $D(\mathscr{A}_{\sigma\cup s_n})$ between $\bar{s}_n^{-1} F^\cdot$ and $j_{\bar\sigma}^* G^\cdot$. Let this isomorphism be represented as

$$\bar{s}_n^{-1}F^{\cdot} \xrightarrow{\sim} H^{\cdot} \xleftarrow{\sim} j_{\bar{\sigma}}^{*}G^{\cdot}\ ;$$

then $H^{\cdot} \in \mathcal{P}(\mathcal{A}_{\sigma\cup s_n} \text{ off } \mathcal{A}_{\sigma\cup s_n\cup\mu})$. Through the adjointness of $(\bar{s}_n^{-1} = j_{\bar{s}_n}^{*}, j_{\bar{s}_n*})$ and $(j_{\bar{\sigma}}^{*}, j_{\bar{\sigma}*})$, we have chain maps $F^{\cdot} \to j_{\bar{s}_n*}H^{\cdot}$ in $C(\mathcal{A}_{\sigma})$ and $G^{\cdot} \to j_{\bar{\sigma}*}H^{\cdot} \to \hat{R}j_{\bar{\sigma}*}H^{\cdot}$ in $C(\mathcal{A}_{s_n})$, so we have chain maps in $C(\mathcal{A})$:

$$\hat{R}j_{\sigma*}F^{\cdot} \to \hat{R}j_{\sigma*}(j_{\bar{s}_n*}H^{\cdot}) = \hat{R}j_{\sigma\cup s_n*}H^{\cdot}$$

and

$$j_{s_n*}G^{\cdot} \to j_{s_n*}\hat{R}j_{\bar{\sigma}*}H^{\cdot} = \hat{R}j_{\sigma\cup s_n*}H^{\cdot}\ .$$

Let $E^{\cdot}$ be the homotopy pullback

$$\begin{array}{ccc} & E^{\cdot} & \\ \swarrow & & \searrow \\ \hat{R}j_{\sigma*}F^{\cdot} & & j_{s_n*}G^{\cdot} \\ \searrow & & \swarrow \\ & \hat{R}j_{\sigma\cup s_n*}H^{\cdot} & \end{array} \qquad (1)$$

We want to show that $E^{\cdot} \in \mathcal{P}(\mathcal{A} \text{ off } \mathcal{A}_{\mu})$.

Applying j_{σ}^{*} to (1), we get a homotopy pullback in $C(\mathcal{A}_{\sigma})$

$$\begin{array}{ccc} & j_{\sigma}^{*}E^{\cdot} & \\ \swarrow & & \searrow \\ j_{\sigma}^{*}\hat{R}j_{\sigma*}F^{\cdot} & & j_{\sigma}^{*}j_{s_n*}G^{\cdot} \\ \searrow & & \swarrow \\ & j_{\sigma}^{*}\hat{R}j_{\sigma\cup s_n*}H^{\cdot} & \end{array} \qquad (2)$$

Because $H^{\cdot} \to j_{\sigma\cup s_n}^{*}\hat{R}j_{\sigma\cup s_n*}H^{\cdot}$ is a quasi-isomorphism by Lemma 2.3.2(b), and $j_{\bar{s}_n*}$ is exact, we have a quasi-isomorphism

$$j_{\bar{s}_n*}H^{\cdot} \to j_{\bar{s}_n*}j_{\sigma\cup s_n}^{*}\hat{R}j_{\sigma\cup s_n*}H^{\cdot} = j_{\sigma}^{*}\hat{R}j_{\sigma\cup s_n*}H^{\cdot}\ .$$

Because $j_{\bar{\sigma}}^{*}G^{\cdot} \to H^{\cdot}$ is a quasi-isomorphism as assumed, we get another quasi-isomorphism

$$j_{\sigma}^{*}j_{s_n*}G^{\cdot} = j_{\bar{s}_n*}j_{\bar{\sigma}}^{*}G^{\cdot} \to j_{\bar{s}_n*}H^{\cdot}\ .$$

So the composite

$$j_{\sigma}^{*}j_{s_n*}G^{\cdot} \to j_{\bar{s}_n*}H^{\cdot} \to j_{\sigma}^{*}\hat{R}j_{\sigma\cup s_n*}H^{\cdot}$$

is also a quasi-isomorphism, which is the right lower side of the homotopy pull back (2). Thus the left upper side of (2), $j_{\sigma}^{*}E^{\cdot} \to j_{\sigma}^{*}\hat{R}j_{\sigma*}F^{\cdot}$, is also a quasi-isomorphism. But $F^{\cdot} \to j_{\sigma}^{*}\hat{R}j_{\sigma*}F^{\cdot}$ is a quasi-isomorphism and $F^{\cdot}$ is perfect in $\mathcal{A}_{\sigma}$, so $j_{\sigma}^{*}\hat{R}j_{\sigma*}F^{\cdot}$ is also perfect, and so is $j_{\sigma}^{*}E^{\cdot}$.

Applying s_n^{-1} to (1), we get a homotopy pullback in $C(\mathcal{A}_{s_n})$

$$\begin{array}{ccc} & s_n^{-1}E^{\cdot} & \\ \swarrow & & \searrow \\ s_n^{-1}\hat{R}j_{\sigma*}F^{\cdot} & & s_n^{-1}j_{s_n*}G^{\cdot} \\ \searrow & & \swarrow \\ & s_n^{-1}\hat{R}j_{\sigma\cup s_n*}H^{\cdot} & \end{array} \qquad (3)$$

Because $s_n^{-1}F^{\cdot} \to H^{\cdot}$ is a quasi-isomorphism as assumed and $\hat{R}j_{\bar{\sigma}*}$ preserves quasi-isomorphisms, we have a quasi-isomorphism:

$$s_n^{-1}\hat{R}j_{\sigma*}F^{\cdot} \cong \hat{R}j_{\bar{\sigma}*}\bar{s}_n^{-1}F^{\cdot} \to \hat{R}j_{\bar{\sigma}*} \cong s_n^{-1}\hat{R}j_{\sigma\cup s_n*}H^{\cdot}\ ,$$

which is the left lower side of the homotopy pullback (3). So the right upper side of (3), $s_n^{-1}E^{\cdot} \to s_n^{-1}j_{s_n*}G^{\cdot} = G^{\cdot}$, is also a quasi-isomorphism, and $s_n^{-1}E^{\cdot}$ is perfect in $\mathcal{A}_{s_n}$. Now since $j_{\sigma}^{*}E^{\cdot}$ and $s_n^{-1}E^{\cdot}$ are perfect, by Proposition 1.5.1, $E^{\cdot}$ is perfect in $\mathcal{A}$. Finally we need to check that $j_{\mu}^{*}E^{\cdot}$ is acyclic in $\mathcal{A}_{\mu}$. By Lemma 2.5.1, $j_{\mu}^{*}\hat{R}j_{\sigma*}F^{\cdot}$, $j_{\mu}^{*}\hat{R}j_{\sigma\cup s_n*}H^{\cdot}$ and $j_{\mu}^{*}j_{s_n*}G^{\cdot}$ are all acyclic in $\mathcal{A}_{\mu}$, thus so is $j_{\mu}^{*}E^{\cdot}$.

The isomorphism between $F^{\cdot}$ and $j_{\sigma}^{*}E^{\cdot}$ in $D(\mathcal{A}_{\sigma})$ is represented by

$$j_{\sigma}^{*}E^{\cdot} \xrightarrow{\sim} j_{\sigma}^{*}\hat{R}j_{\sigma*}F^{\cdot} \xleftarrow{\sim} F^{\cdot}\ . \qquad \square$$

3.5.2. Lemma. *Let* $\{\mathcal{A}, (s_i, F_i), i = 1, \ldots, n\}$ *be an admissible abelian category,* $\{(u_k, H_k),\ k = 1, \ldots, r\}$ *be a finite set of divisors on* $\mathcal{A}$ *such that* $\{(s_i, F_i), (u_k, H_k)\}$ *are compatible,* $\sigma = \bigcap_{i=1}^{n-1} s_i$, $u = \bigcap_{k=1}^{r} u_k$. *If* $E_1^{\cdot}, E_2^{\cdot} \in \mathcal{P}(\mathcal{A} \text{ off } \mathcal{A}_{\mu})$, $f : j_{\sigma}^{*}E_1^{\cdot} \to j_{\sigma}^{*}E_2^{\cdot}$ *is a morphism in* $D(\mathcal{A}_{\sigma})$, *then there is a third perfect complex* $E^{\prime\cdot} \in \mathcal{P}(\mathcal{A} \text{ off } \mathcal{A}_{\mu})$ *and two morphisms* $b : E_1^{\cdot} \to E^{\prime\cdot}$ *and* $c : E_2^{\cdot} \to E^{\prime\cdot}$ *in* $D(\mathcal{A})$ *such that* $j_{\sigma}^{*}(b) = j_{\sigma}^{*}(c) \circ f$ *in* $D(\mathcal{A}_{\sigma})$ *and* $j_{\sigma}^{*}(c)$ *is an isomorphism in* $D(\mathcal{A}_{\sigma})$.

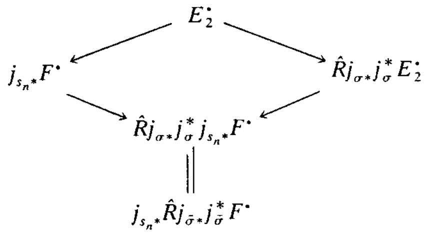
Moreover, we can choose $E^{\cdot}$ *such that* $[E^{\cdot}] = [E_2^{\cdot}]$ *in* $K_0(\mathcal{A} \text{ off } \mathcal{A}_{\mu})$.

Proof. In the following somewhat long and cumbersome proof, we want to find complexes $E^{\cdot}, H^{\cdot} \in \mathcal{P}(\mathcal{A} \text{ off } \mathcal{A}_{\mu})$, a morphism $b \in \operatorname{Hom}_{D(\mathcal{A})}(E_1^{\cdot}, E^{\cdot})$ represented by

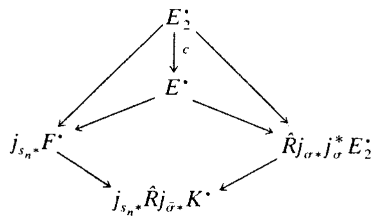
$$E_1^{\cdot} \xleftarrow{\sim} H^{\cdot} \to E^{\cdot}\ ,$$

and a chain map $c : E_2^{\cdot} \to E^{\cdot}$, such that $j_{\sigma}^{*}(B) = j_{\sigma}^{*}(c) \circ f$ in $D(\mathcal{A}_{\sigma})$ and $j_{\sigma}^{*}(c)$ is a quasi-isomorphism. When we do so, then we have proved the lemma except for the last sentence.

diagram:



Then we get the outside square of the following homotopy commutative diagram:



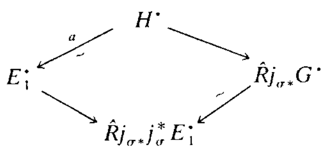
(8)

Because the lower square of (8) is the homotopy pullback (7), we get a chain map $c: E_2^{\cdot} \to E^{\cdot}$. In the proof of the perfectness of $E^{\cdot}$ above, we have shown that $j_\sigma^* E^{\cdot} \to j_\sigma^* \hat{R}j_{\sigma*} j_\sigma^* E_2^{\cdot}$ is a quasi-isomorphism, but the composite

$$j_\sigma^* E_2^{\cdot} \xrightarrow{j_\sigma^*(c)} j_\sigma^* E^{\cdot} \to j_\sigma^* \hat{R}j_{\sigma*} j_\sigma^* E_2^{\cdot}$$

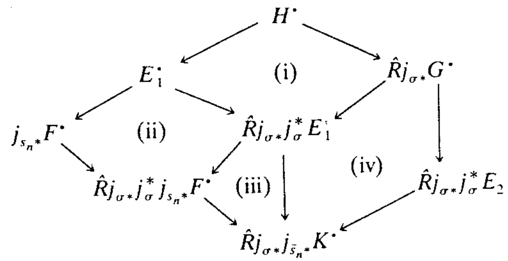
is a quasi-isomorphism by Lemma 2.3.2(b), so $j_\sigma^*(c)$ is a quasi-isomorphism. This is the c we wanted to get, as explained at the beginning of the proof.

Let $H^{\cdot}$ be the homotopy pullback



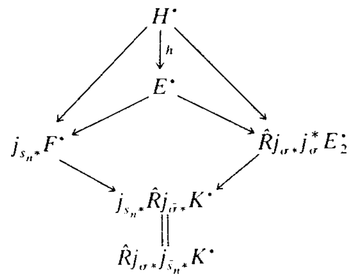
(9)

In order to define a chain map $H^{\cdot} \to E^{\cdot}$, we consider the following diagram:



(10)

which we want to be homotopy commutative. The square (i) is (9), so is homotopy commutative; the square (ii) is homotopy commutative by the naturality of the functors; the square (iii) is homotopy commutative by the homotopy commutativity of the left lower triangle of (5); and square (iv) is homotopy commutative by the homotopy commutativity of (6) and the naturality of the adjunction maps of adjoint pairs. Thus (10) is homotopy commutative. Compare the outside square of (10) with the homotopy pullback (7):



(11)

Then we have a chain map $h: H^{\cdot} \to E^{\cdot}$ which makes (11) homotopy commutative. Let $b \in \mathrm{Hom}_{D(\mathcal{A})}(E_1^{\cdot}, E^{\cdot})$ be represented by

$$E_1^{\cdot} \overset{a}{\underset{\sim}{\longleftarrow}} H^{\cdot} \xrightarrow{h} E^{\cdot},$$

where a is as in (9). Then a straightforward check shows that $j_\sigma^*(b) = j_\sigma^*(c) \circ f$ in $D(\mathcal{A}_\sigma)$.

To prove the last statement of the lemma, i.e., to choose $E^{\cdot}$ such that $[E^{\cdot}] = [E_2^{\cdot}]$ in $K_0(\mathcal{A} \text{ off } \mathcal{A}_\mu)$, we just simply let $\mathrm{cone}(c) \oplus E^{\cdot}$ replace the old $E^{\cdot}$. □

3.5.3. Corollary. *Let* $\{\mathcal{A}, (s_i, F_i), i = 1, \ldots, n\}$ *be an admissible abelian category,* $\{(u_k, H_k),\ k = 1, \ldots, r\}$ *be a finite set of divisors on* $\mathcal{A}$ *such that* $\{(s_i, F_i),$

$(u_k, H_k)\}$ are compatible, $\sigma = \bigcap_{i=1}^{n-1} s_i$, and $\mu = \bigcap_{k=1}^{r} u_k$; then we have a homotopy fibre sequence of spectra

$$K(\mathcal{A} \text{ off } \mathcal{A}_{\sigma\cup\mu}) \to K(\mathcal{A} \text{ off } \mathcal{A}_{\mu}) \to K(\mathcal{A}_\sigma \text{ off } \mathcal{A}_{\sigma\cup\mu})^{\sim} ,$$

and thus we have a long exact sequence

$$\cdots \to K_1(\mathcal{A}_\sigma \text{ off } \mathcal{A}_{\sigma\cup\mu}) \to K_0(\mathcal{A} \text{ off } \mathcal{A}_{\sigma\cap\mu})$$
$$\to K_0(\mathcal{A} \text{ off } \mathcal{A}_{\mu}) \to K_0(\mathcal{A}_\sigma \text{ off } \mathcal{A}_{\sigma\cup\mu}) .$$

Proof. This corollary is a special case of Theorem 3.3(b). In this special case, Lemma 3.5.1 is 3.3.1(b) and Lemma 3.5.2 is 3.3.2(b). □

3.5.4. Proposition. *Let $\{\mathcal{A}, (s_i, F_i), i = 1, \dots, n\}$ be an admissible abelian category, $\{(u_k, H_k), k = 1, \dots, r\}$ be a finite set of divisors on $\mathcal{A}$ such that $\{(s_i, F_i), (u_k, H_k)\}$ are compatible, $\sigma_p = \bigcap_{i=1}^{p} s_i$, and $\mu = \bigcap_{k=1}^{r} u_k$; then:*

(a) *For a perfect complex $E^\cdot \in \mathcal{P}(\mathcal{A}_{\sigma_p} \text{ off } \mathcal{A}_{\sigma_p\cup\mu})$, its class $[E^\cdot] \in K_\sigma(\mathcal{A}_{\sigma_p} \text{ off } \mathcal{A}_{\sigma_p\cup\mu})$ is in the image of $K_0(\mathcal{A} \text{ off } \mathcal{A}_\mu) \to K_0(\mathcal{A}_{\sigma_p} \text{ off } \mathcal{A}_{\sigma_p\cup\mu})$ iff there is a complex $F^\cdot \in \mathcal{P}(\mathcal{A} \text{ off } \mathcal{A}_\mu)$ such that $j_{\sigma_p}^* F^\cdot$ is isomorphic to $E^\cdot$ in $D(\mathcal{A}_{\sigma_p})$.*

(b) *For any two complexes $E_1^\cdot, E_2^\cdot \in \mathcal{P}(\mathcal{A} \text{ off } \mathcal{A}_\mu)$, and a morphism $f : j_{\sigma_p}^* E_1^\cdot \to j_{\sigma_p}^* E_2^\cdot$ in $D(\mathcal{A}_{\sigma_p})$, there is a third complex $E''^\cdot \in \mathcal{P}(\mathcal{A} \text{ off } \mathcal{A}_\mu)$ and two morphisms $b : E_1^\cdot \to E''^\cdot$ and $c : E_2^\cdot \to E''^\cdot$ in $D(\mathcal{A})$ such that $j_{\sigma_p}^*(b) = j_{\sigma_p}^*(c) \circ f$ in $D(\mathcal{A}_{\sigma_p})$ and $j_{\sigma_p}^*(c)$ is an isomorphism in $D(\mathcal{A}_{\sigma_p})$, and $[E''^\cdot] = [E_2^\cdot]$ in $K_0(\mathcal{A} \text{ off } \mathcal{A}_\mu)$.*

(c) *We have a homotopy fibre sequence of spectra*

$$K(\mathcal{A} \text{ off } \mathcal{A}_{\sigma_p\cap\mu}) \to K(\mathcal{A} \text{ off } \mathcal{A}_{\mu}) \to K(\mathcal{A}_{\sigma_p} \text{ off } \mathcal{A}_{\sigma_p\cup\mu})^{\sim} ,$$

and thus we have a long exact sequence of groups

$$\cdots \to K_1(\mathcal{A}_{\sigma_p} \text{ off } \mathcal{A}_{\sigma_p\cup\mu}) \to K_0(\mathcal{A} \text{ off } \mathcal{A}_{\sigma_p\cap\mu})$$
$$\to K_0(\mathcal{A} \text{ off } \mathcal{A}_{\mu}) \to K_0(\mathcal{A}_{\sigma_p} \text{ off } \mathcal{A}_{\sigma_p\cup\mu}) .$$

Proof. We use induction for $n - p = r$.

When $r = 1$, i.e., $p = n - 1$, the proposition is the combination of Lemma 3.5.1, Lemma 3.5.2 and Corollary 3.5.3.

Assume the proposition is true for $r - 1$, i.e., for $p + 1$.

To do the inductive step for (a), since

$$[E^\cdot] \in \mathrm{Im}(K_0(\mathcal{A} \text{ off } \mathcal{A}_\mu) \to K_0(\mathcal{A}_{\sigma_p} \text{ off } \mathcal{A}_{\sigma_p\cup\cup\mu})) ,$$

then

$$[E^\cdot] \in \mathrm{Im}(K_0(\mathcal{A}_{\sigma_{n-1}} \text{ off } \mathcal{A}_{\sigma_{n-1}\cup\mu}) \to K_0(\mathcal{A}_{\sigma_p} \text{ off } \mathcal{A}_{\sigma_p\cup\mu})) .$$

By the induction hypothesis, we have a complex $\tilde{E}^\cdot \in \mathcal{P}(\mathcal{A}_{\sigma_{n-1}} \text{ off } \mathcal{A}_{\sigma_{n-1}\cup\mu})$ such that $j_{\tilde{\sigma}_p}^* \tilde{E}^\cdot$ is isomorphic to $E^\cdot$ in $D(\mathcal{A}_{\sigma_p})$. Also since

$$[E^\cdot] \in \mathrm{Im}(K_0(\mathcal{A} \text{ off } \mathcal{A}_\mu) \to K_0(\mathcal{A}_{\sigma_p} \text{ off } \mathcal{A}_{\sigma_p\cup\mu})) ,$$

it means that there is a complex $H^\cdot \in \mathcal{P}(\mathcal{A} \text{ off } \mathcal{A}_\mu)$ such that $[j_{\sigma_p}^* H^\cdot] = [E^\cdot]$ in $K_0(\mathcal{A}_{\sigma_p} \text{ off } \mathcal{A}_{\sigma_p\cup\mu})$. Then

$$K_0(j_{\tilde{\sigma}_p}^*)([j_{\sigma_{n-1}}^* H^\cdot] - [\tilde{E}^\cdot]) = [j_{\tilde{\sigma}_p}^* j_{\sigma_{n-1}}^* H^\cdot] - [j_{\tilde{\sigma}_p}^* \tilde{E}^\cdot]$$
$$= [E^\cdot] - [E^\cdot] = 0$$

in $K_0(\mathcal{A}_{\sigma_p} \text{ off } \mathcal{A}_{\sigma_p\cup\mu})$. By the induction hypothesis, we have the long exact sequence

$$\cdots \to K_0(\mathcal{A}_{\sigma_{n-1}} \text{ off } \mathcal{A}_{(\sigma_{n-1}\cup\mu)\cap\sigma_p}) \to K_0(\mathcal{A}_{\sigma_{n-1}} \text{ off } \mathcal{A}_{\sigma_{n-1}\cup\mu})$$
$$\xrightarrow{K_0(j_{\tilde{\sigma}_p}^*)} K_0(\mathcal{A}_{\sigma_p} \text{ off } \mathcal{A}_{\sigma_p\cup\mu}) .$$

Since $[j_{\sigma_{n-1}}^* H^\cdot] - [\tilde{E}^\cdot] \in \mathrm{Ker}(K_0(j_{\tilde{\sigma}_p}^*))$, there is a complex $G^\cdot \in \mathcal{P}(\mathcal{A}_{\sigma_{n-1}} \text{ off } \mathcal{A}_{(\sigma_{n-1}\cup\mu)\cap\sigma_p})$ such that $[j_{\sigma_{n-1}}^* H^\cdot] - [\tilde{E}^\cdot] = [G^\cdot]$, i.e.,

$$[j_{\sigma_{n-1}}^* H^\cdot] = [\tilde{E}^\cdot] + [G^\cdot] = [\tilde{E}^\cdot \oplus G^\cdot]$$

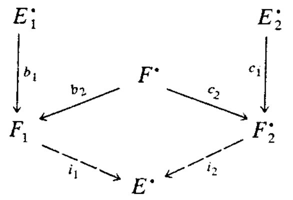
in $K_0(\mathcal{A}_{\sigma_{n-1}} \text{ off } \mathcal{A}_{\sigma_{n-1}\cup\mu})$. Notice that $j_{\tilde{\sigma}_p}^*(\tilde{E}^\cdot \oplus G^\cdot) = j_{\tilde{\sigma}_p}^* \tilde{E}^\cdot \oplus j_{\tilde{\sigma}_p}^* G^\cdot$ is still isomorphic to $E^\cdot$ in $D(\mathcal{A}_{\sigma_p})$ because $j_{\tilde{\sigma}_p}^* G^\cdot$ is acyclic. Now $[\tilde{E}^\cdot \oplus G^\cdot]$ is in the image of $K_0(\mathcal{A} \text{ off } \mathcal{A}_\mu) \to K_0(\mathcal{A}_{\sigma_{n-1}} \text{ off } \mathcal{A}_{\sigma_{n-1}\cup\mu})$, by Lemma 3.5.1, there is a complex $F^\cdot \in \mathcal{P}(\mathcal{A} \text{ off } \mathcal{A}_\mu)$ such that $j_{\sigma_{n-1}}^* F^\cdot$ is isomorphic to $\tilde{E}^\cdot \oplus G^\cdot$, so

$$j_{\sigma_p}^* F^\cdot = j_{\tilde{\sigma}_p}^*(j_{\sigma_{n-1}}^* F^\cdot) \cong j_{\tilde{\sigma}_p}^*(\tilde{E}^\cdot \oplus G^\cdot) \cong E^\cdot$$

in $D(\mathcal{A}_{\sigma_p})$. This finishes the "only if" part of (a), while the "if" part is obvious.

To prove (b), by the induction hypothesis, there is a complex $\tilde{E}^\cdot \in \mathcal{P}(\mathcal{A}_{\sigma_{n-1}} \text{ off } \mathcal{A}_{\sigma_{n-1}\cup\mu})$ and $\tilde{b} : j_{\sigma_{n-1}}^* E_1^\cdot \to \tilde{E}^\cdot$, $\tilde{c} : j_{\sigma_{n-1}}^* E_2^\cdot \to \tilde{E}^\cdot$ in $D(\mathcal{A}_{\sigma_{n-1}})$ such that $j_{\tilde{\sigma}_p}^*(\tilde{b}) = j_{\tilde{\sigma}_p}^*(\tilde{c}) \circ f$ and $j_{\tilde{\sigma}_p}^*(\tilde{c})$ is an isomorphism in $D(\mathcal{A}_{\sigma_p})$, and $[j_{\sigma_{n-1}}^* E_2^\cdot] = [\tilde{E}^\cdot]$ in $K_0(\mathcal{A}_{\sigma_{n-1}} \text{ off } \mathcal{A}_{\sigma_{n-1}\cup\mu})$. From Lemma 3.5.1, the equality $[j_{\sigma_{n-1}}^* E_2^\cdot] = [\tilde{E}^\cdot]$ implies that we can let $\tilde{E}^\cdot = j_{\sigma_{n-1}}^* F^\cdot$ for some $F^\cdot \in \mathcal{P}(\mathcal{A} \text{ off } \mathcal{A}_\mu)$. Applying Lemma 3.5.2 to $\tilde{b}$ and $\tilde{c}$ and noticing Remark 3.3.4, there are $F_1^\cdot, F_2^\cdot \in$

$\mathcal{P}(\mathcal{A} \text{ off } \mathcal{A}_\mu)$ and chain maps $b_1 : E_1^\cdot \to F_1^\cdot$, $b_2 : F^\cdot \to F_1^\cdot$, $c_1 : E_2^\cdot \to F_2^\cdot$ and $c_2 : F^\cdot \to F_2^\cdot$ such that $j_{\sigma_{n-1}}^*(b_1) = j_{\sigma_{n-1}}^*(b_2) \circ \tilde{b}$, $j_{\sigma_{n-1}}^*(c_1) = j_{\sigma_{n-1}}^*(c_2) \circ \tilde{c}$, and $j_{\sigma_{n-1}}^*(b_2), j_{\sigma_{n-1}}^*(c_2)$ are isomorphisms in $D(\mathcal{A}_{\sigma_{n-1}})$. Let $E^\cdot$ be the homotopy pushout as in the lower square of the following diagram:



Let $b = i_1 \circ b_1$, $c = i_2 \circ c_1$; then

$$j_{\sigma_p}^*(E_1^\cdot) \xrightarrow{j_{\sigma_p}^*(b)} j_{\sigma_p}^*(E^\cdot) \xleftarrow{j_{\sigma_p}^*(c)} j_{\sigma_p}^*(E_2^\cdot)$$

represents f, i.e., $j_{\sigma_p}^*(b) = j_{\sigma_p}^*(c) \circ f$ in $D(\mathcal{A}_{\sigma_p})$. Because $F_1^\cdot, F_2^\cdot, F^\cdot \in \mathcal{P}(\mathcal{A} \text{ off } \mathcal{A}_\mu)$, so $E^\cdot \in \mathcal{P}(\mathcal{A} \text{ off } \mathcal{A}_\mu)$. In order to meet the requirement that $[E^\cdot] = [E_2^\cdot]$ in $K_0(\mathcal{A} \text{ off } \mathcal{A}_\mu)$, we simply let cone$(c) \oplus E^\cdot$ replace the old $E^\cdot$.

(c) Obvious, since (a) above is 3.3.1 and (b) above is 3.3.2. □

Proof of Theorem 3.3. Actually Proposition 3.5.4 already implies Theorem 3.3 after a small modification. Let $\{\mathcal{A}(s_i, F_i), i = 1, \ldots, n\}$ be an admissible abelian category, $\{(t_j, G_j), j = 1, \ldots, m\}$ and $\{(u_k, H_k), k = 1, \ldots, r\}$ be another two finite sets of divisors on $\mathcal{A}$ such that $\{(s_i, F_i), (t_j, G_j), (u_k, H_k), i, j, k\}$ are compatible. Then $\{\mathcal{A}, (s_i, F_i), (s_i t_j, F_i G_j), i, j\}$ is also admissible, and $\{(s_i t_j, F_i G_j), i, j\}$ becomes a part of $\{(s_i, F_i), (s_i t_j, F_i G_j), i, j\}$ and all divisors are still compatible by Lemma 1.3.4. Let $\tau = \bigcap_{j=1}^m t_j$ and $\tau' = \bigcap_{i,j} s_i t_j$. Because $\bigcap_{i=1}^n (s_i\text{-Tor}) = 0$, we have $\tau = \tau'$. Since being perfect is independent of the choice of the structure divisors as pointed out in Proposition 2.4, Theorem 3.3 is Proposition 3.5.4 for the structure divisors $\{(s_i, F_i), (s_i t_j, F_i G_j), i, j\}$ □

4. Projective line

4.0. In this section we will generalize the construction of the polynomial ring $R[T]$ and projective line P_R^1 (cf. [11]) for a given ring R to the construction of $\mathcal{A}[T]$ and $P_{\mathcal{A}}^1$ for a given category $\mathcal{A}$. If $\mathcal{A}$ is an admissible abelian category, so are $\mathcal{A}[T]$ and $P_{\mathcal{A}}^1$. Mainly we will prove the following theorem which is a generalization of [11, Section 8.3]:

4.0.1. Theorem. *Let $\{\mathcal{A}, (s_i, F_i), i = 1, \ldots, n\}$ be an admissible abelian category, $\{(u_k, H_k), k = 1, \ldots, r\}$ be divisors on $\mathcal{A}$ such that $\{(s_i, F_i), (u_k, H_k)\}$ are compatible, $\mu = \bigcap_{k=1}^r u_k$; then there are homotopy equivalences*

(a) (absolute form) $K(\mathcal{A}) \times K(\mathcal{A}) \to K(P_{\mathcal{A}}^1)$,

(b) (relative form) $K(\mathcal{A} \text{ off } \mathcal{A}_\mu) \times K(\mathcal{A} \text{ off } \mathcal{A}_\mu) \to K(P_{\mathcal{A}}^1 \text{ off } P_{\mathcal{A}_\mu}^1)$.

This theorem will be used to define the negative K-groups, therefore the nonconnected K-theory spectra.

4.1.0. Definition. Let $\mathcal{N}$ denote the free monoid with one generator T, i.e., $\mathcal{N} = \{1 = T^0, T^2, \ldots\}$ with composition $T^i \circ T^j = T^{i+j}$. Let $\mathcal{A}$ be an Ab5 category, A an object in $\mathcal{A}$. Recall that an $\mathcal{N}$-action on A is a monoid morphism $\lambda : \mathcal{N} \to \mathrm{Hom}_{\mathcal{A}}(A, A)$. Obviously there is an 1–1 correspondence between $\mathcal{N}$-actions and $\mathrm{Hom}_{\mathcal{A}}(A, A)$ by sending an action λ to $\lambda(T) \in \mathrm{Hom}_{\mathcal{A}}(A, A)$. An object $A \in \mathcal{A}$ together with an action λ on it is called an $\mathcal{N}$-object in $\mathcal{A}$, denoted by (A, φ) with $\varphi = \lambda(T) \in \mathrm{Hom}_{\mathcal{A}}(A, A)$, or simply A if no confusion would arise. When we say 'the morphism $T^i : A \to A$', we actually mean the morphism $\lambda(T^i) = \varphi^i : A \to A$.

Let $\mathcal{A}[T]$ denote the category of all $\mathcal{N}$-objects in $\mathcal{A}$ where morphisms are $\mathcal{N}$-action preserving morphisms. If we regard $\mathcal{N}$ as a category with a single object $*$ and $\mathrm{Hom}(*, *) = \mathcal{N}$, then $\mathcal{A}[T] = \mathrm{Cat}(\mathcal{N}, \mathcal{A})$, the category of all functors from $\mathcal{N}$ to $\mathcal{A}$ and natural transformations. So $\mathcal{A}[T]$ is also an Ab5 category. If $\{(A_\alpha, \varphi_\alpha), \Phi_\beta^\alpha\}$ is an inductive system in $\mathcal{A}[T]$, then

$$\varinjlim \{(\mathcal{A}_\alpha, \varphi_\alpha), \Phi_\beta^\alpha\} = (\varinjlim \{A_\alpha, \Phi_\beta^\alpha\}, \varinjlim \varphi_\alpha).$$

For any $A \in \mathcal{A}$, let $A[T] = (\bigoplus_{i=0}^\infty A_i, T) \in \mathcal{A}[T]$, where all $A_i = A$ (i actually indicates the 'degree') and T is the shift to the right by degree 1: $(a_0, a_1, \ldots) \to (0, a_0, a_1, \ldots)$. Then $(\ )[T] : \mathcal{A} \to \mathcal{A}[T]$ is an exact and exactness-reflecting functor. We also have the forgetful functor from $\mathcal{A}[T]$ to $\mathcal{A}$ sending (A, φ) to A, which is also an exact and exactness-reflecting functor.

4.1.1. Lemma. *Let $\mathcal{A}$ be an* Ab5 *category, then*:

(a) *The functor $(\ )[T] : \mathcal{A} \to \mathcal{A}[T]$ is left adjoint to the forgetful functor*; *more explicitly, for any $A \in \mathcal{A}$, $(B, \xi) \in \mathcal{A}[T]$, we have a natural isomorphism*

$$\Theta : \mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{A}[T]}\Big(\Big(\bigoplus_{i=0}^\infty A_i, T\Big), (B, \xi)\Big),$$

$$f \to \Theta(f) = (\xi^i \circ f).$$

(b) $(\ )[T]$ *preserves small projective objects and generators.*

(c) *If (s, F) is a divisor on $\mathcal{A}$, then (s, F) naturally induces a divisor on $\mathcal{A}[T]$*

which will be still denoted as (s, F) *with*

$$F((A, \varphi)) = (FA, F\varphi)\,, \quad s_{(A,\varphi)} = s_A\,, \qquad \forall (A, \varphi) \in \mathscr{A}[T]\,.$$

Moreover, $\mathscr{A}_s[T] = \mathscr{A}[T]_s$, *and the diagram*

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{(\ )[T]} & \mathscr{A}[T] \\ \downarrow{\scriptstyle s^{-1}} & & \downarrow{\scriptstyle s^{-1}} \\ \mathscr{A}_s & \xrightarrow{(\ )[T]} & \mathscr{A}_s[T] = \mathscr{A}[T]_s \end{array}$$

commutes.

(d) *If* $\{\mathscr{A}, (s_i, F_i), i = 1, \ldots, n\}$ *is an admissible abelian category, then so is* $\{\mathscr{A}[T], (s_i, F_i), i = 1, \ldots, n\}$. *If* $\{\mathscr{A}, (s_i, F_i), i = 1, \ldots, n\}$ *is strongly admissible, then so is* $\{\mathscr{A}[T], (s_i, F_i), i = 1, \ldots, n\}$.

Proof. Easy. Omitted. □

4.1.2. Let $\mathscr{N}^- = \{1, T^{-1}, T^{-2}, \ldots\}$ be the free monoid generated by one element T^{-1}. Replacing $\mathscr{N}$ by $\mathscr{N}^-$, we have the category $\mathscr{A}[T^{-1}]$. Of course $\mathscr{A}[T]$ and $\mathscr{A}[T^{-1}]$ are the same except that we use different symbols for the generator of the free monoid. We do so for later convenience.

Let $\mathscr{Z} = \{\ldots, T^{-1}, T^0 = 1, T, \ldots\}$ be the free group generated by T. Replacing $\mathscr{N}$ by $\mathscr{Z}$ in Definition 4.1.0, we can define $\mathscr{Z}$-action, $\mathscr{Z}$-objects, the category $\mathscr{A}[T, T^{-1}]$ of all $\mathscr{Z}$-objects and $\mathscr{Z}$-action preserving morphisms, and the functor $(\ )[T, T^{-1}] : \mathscr{A} \to \mathscr{A}[T, T^{-1}]$ sending A to $(\bigoplus_{i=-\infty}^{\infty} A, T)$, etc. Obviously we also have Lemma 4.1.1 for $\mathscr{A}[T, T^{-1}]$ and $(\ )[T, T^{-1}]$.

Since $\mathscr{N} \subset \mathscr{Z}$, $\mathscr{N}^- \subset \mathscr{Z}$, every $\mathscr{Z}$-action naturally induces an $\mathscr{N}$-action and $\mathscr{N}^-$-action by restriction, so $\mathscr{A}[T, T^{-1}]$ is naturally embedded in $\mathscr{A}[T]$ and $\mathscr{A}[T^{-1}]$ as a full subcategory. Moreover, $\mathscr{A}[T, T^{-1}]$ is also a localization of $\mathscr{A}[T]$ or $\mathscr{A}[T^{-1}]$ through a canonical divisor defined as follows: Let T denote the natural transformation $\mathrm{Id}_{\mathscr{A}[T]} \to \mathrm{Id}_{\mathscr{A}[T]}$ with $T_{(\mathscr{A},\varphi)} = \varphi$ for any $(A, \varphi) \in \mathscr{A}[T]$; then obviously (T, Id) becomes a divisor on $\mathscr{A}[T]$, and

$$\begin{aligned} T^{-1}(A, \varphi) &= \varinjlim\, ((A, \varphi) \xrightarrow{\varphi} (A, \varphi) \xrightarrow{\varphi} \cdots) \\ &= (\varinjlim\, (A \xrightarrow{\varphi} A \xrightarrow{\varphi} \cdots), \varinjlim\, \varphi) \in \mathscr{A}[T, T^{-1}] \end{aligned}$$

because $\varinjlim \varphi \in \mathrm{Aut}(A)$. So we have $\mathscr{A}[T]_T = \mathscr{A}[T, T^{-1}]$. In the same way we have the divisor $(T^{-1}, \mathrm{Id}_{\mathscr{A}[T^{-1}]})$ on $\mathscr{A}[T^{-1}]$ and $\mathscr{A}[T^{-1}]_{T^{-1}} = \mathscr{A}[T, T^{-1}]$. If (s, F) is a divisor on $\mathscr{A}$, then (s, F) and $(T, \mathrm{Id}_{\mathscr{A}[T]})$ are always compatible on $\mathscr{A}[T]$.

4.1.3. Definition. *Let* $\mathscr{A}$ be an Ab5 category. The projective line over $\mathscr{A}$ is the category $P^1_{\mathscr{A}}$ of all triples $M = (A^+, \theta, A^-)$, where $A^+ \in \mathscr{A}[T]$, $A^- \in \mathscr{A}[T^{-1}]$ and $\theta : T^{-1}(A^+) \to T(A^-)$ is an isomorphism in $\mathscr{A}[T, T^{-1}]$; a morphism in $P^1_{\mathscr{A}}$ is a pair

$$f = (f_1, f_2) : M = (A^+, \theta, A^-) \to N = (B^+, \eta, B^-)\,,$$

where $f^+ \in \mathrm{Hom}_{\mathscr{A}[T]}(A^+, B^+)$, $f^- \in \mathrm{Hom}_{\mathscr{A}[T^{-1}]}(A^-, B^-)$ such that $\eta \circ T^{-1}(f_1) = T(f_2) \circ \theta$. $P^1_{\mathscr{A}}$ is also an Ab5 category.

Let (s, F) be a divisor on $\mathscr{A}$, then we can naturally extend it to a divisor on $P^1_{\mathscr{A}}$, still denoted as (s, F), where $F : P^1_{\mathscr{A}} \to P^1_{\mathscr{A}}$ sends $M = (A^+, \theta, A^-)$ to $FM = (FA^+, F\theta, FA^-)$ and $s : \mathrm{Id}_{P^1_{\mathscr{A}}} \to F$ is the natural transformation with $s_M = (s_{A^+}, s_{A^-})$. We have $(P^1_{\mathscr{A}})_s = P^1_{\mathscr{A}_s}$.

There are two canonical divisors on $P^1_{\mathscr{A}}$, which we define as follows. For an integer n, let $(\ )(n) : P^1_{\mathscr{A}} \to P^1_{\mathscr{A}}$ be the functor sending $M = (A^+, \theta, A^-)$ to $M(n) = (A^+, T^{-n} \circ \theta, A^-)$. Let $t_1, t_2 : \mathrm{Id} \to (\ )(1)$ be the natural transformations with $t_{1M} = (1, T^{-1})$ and $t_{2M} = (T, 1)$ for any $M \in P^1_{\mathscr{A}}$. Clearly $(t_1, (\ )(1))$ and $(t_2, (\ )(1))$ are two compatible divisors on $P^1_{\mathscr{A}}$, and

$$\begin{aligned} t_1^{-1}M &= \varinjlim\, ((A^+, \theta, A^-) \xrightarrow{(1, T^{-1})} (A^-, T^{-1} \circ \theta, A^-) \xrightarrow{(1, T^{-1})} \cdots) \\ &= (A^+, \theta, T(A^-))\,, \end{aligned}$$

$$\begin{aligned} t_2^{-1}M &= \varinjlim\, ((A^+, \theta, A^-) \xrightarrow{(T, 1)} (A^+, T^{-1} \circ \theta, A^-) \xrightarrow{(T, 1)} \cdots) \\ &= (T^{-1}(A^+), \theta, A^-)\,. \end{aligned}$$

So the embedding functor $j_{t_1 *} : \mathscr{A}[T] \to P^1_{\mathscr{A}}$ sending A to $j_{t_1 *}(A) = (A, 1, T^{-1}A)$ induces a category equivalence between $\mathscr{A}[T]$ and $(P^1_{\mathscr{A}})_{t_1}$, and $j_{t_1 *}$ has an exact adjoint $j_{t_1}^* : P^1_{\mathscr{A}} \to \mathscr{A}[T]$ sending $M = (A^+, \theta, A^-)$ to $j_{t_1}^*(M) = A^+$. Thus $(j_{t_1}^*, j_{t_1 *})$ becomes a localizing adjoint pair of functors for the divisor $(t_1, (\ )(1))$. Similarly we have $(j_{t_2}^*, j_{t_2 *})$ a localizing adjoint pair of functors between $P^1_{\mathscr{A}}$ and $\mathscr{A}[T^{-1}]$ for the divisor $(t_2, (\ )(1))$. If (s, F) is a divisor on $\mathscr{A}$, then the extended divisor (s, F) on $P^1_{\mathscr{A}}$ is always compatible with $(t_1, (\ )(1))$ and $(t_2, (\ )(1))$.

4.1.4. Lemma. (a) *If* $\{\mathscr{A}, (s_i, F_i), i = 1, \ldots, n\}$ *is an admissible abelian category, then so is* $\{P^1_{\mathscr{A}}, (s_i t_j, F_i(\ )(1)), i = 1, \ldots, n, j = 1, 2\}$.

(b) *If* $\{\mathscr{A}, (s_i, F_i), i = 1, \ldots, n\}$ *is a strongly admissible abelian category, then so is* $\{P^1_{\mathscr{A}}, (s_i t_j, F_i(\ )(1)), i = 1, \ldots, n, j = 1, 2\}$.

Proof. Only (b) needs a little proof.

Since $\mathscr{A}$ is strongly admissible, i.e., $\mathbf{P}(\mathscr{A})$ generates $\mathscr{A}$, we claim that

$$\{(P[T], T^n, P[T^{-1}]) \mid P \in \mathbf{P}(\mathscr{A}), n = 0, \pm 1, \ldots\} \subset \mathbf{P}(P^1_{\mathscr{A}})$$

generates $P^1_{\mathscr{A}}$, thus $P^1_{\mathscr{A}}$ is strongly admissible. In fact, let

$$f=(f^+, f^-): M=(A^+, \theta, A^-)\to N=(B^+, \eta, B^-)$$

be a nonzero morphism in $P^1_{\mathscr{A}}$; then f^+ and f^- are not both zero. Let us assume $f^+\neq 0$ (a similar proof for $f^-\neq 0$). Because $\mathbf{P}(\mathscr{A})[T]=\{P[T]\mid P\in\mathbf{P}(\mathscr{A})\}$ generates $\mathscr{A}[T]$ by Lemma 4.1.1(b), we have a $P\in\mathbf{P}(\mathscr{A})$ and $g^+: P[T]\to A^+$ such that $f^+\circ g^+\neq 0$. Because $P[T^{-1}]$ is also a small projective object in $\mathscr{A}[T^{-1}]$ by Lemma 4.1.1(b), $\mathrm{Hom}_{\mathscr{A}[T^{-1}]}(P[T^{-1}], -)$ commutes with colimits, so the composite

$$P[T^{-1}]\to P[T, T^{-1}]=T^{-1}(P[T])\xrightarrow{T^{-1}(g^+)} T^{-1}(A^+)$$
$$\xrightarrow{\theta} T(A^-)=\varinjlim (A^-\xrightarrow{T^{-1}} A^-\xrightarrow{T^{-1}}\cdots)$$

can be factorized as

$$P[T^{-1}]\xrightarrow{g^-} A^-\xrightarrow{\varphi_n}\varinjlim (A^-\xrightarrow{T^{-1}} A^-\xrightarrow{T^{-1}}\cdots)$$

for some n, where φ_n is the structure morphism of the colimit. Then

$$g=(g^+, g^-): (P[T], T^n, P[T^{-1}])\to (A^+, \theta, A^-)$$

is a morphism in $P^1_{\mathscr{A}}$ and $f\circ g\neq 0$, so

$$\{(P[T], T^n, P[T^{-1}])\mid P\in\mathbf{P}(\mathscr{A}), n=0, \pm 1, \ldots\}\subset \mathbf{P}(P^1_{\mathscr{A}})$$

generates $P^1_{\mathscr{A}}$. □

From now on in this section, we will always assume $\mathscr{A}$ to be an admissible abelian category.

4.2.1. Definition. Define the functor

$$\pi^*: \mathscr{A}\to P^1_{\mathscr{A}}, \qquad \pi^*(A)=(A[T], 1, A[T^{-1}]), \quad \forall A\in\mathscr{A};$$

then π^* is an exact and exactness reflecting functor.

Define the functors π_* and $R^1\pi_*$:

$$\pi_*: P^1_{\mathscr{A}}\to A, \qquad \pi_*(M)=\ker(d^0), \quad \forall M=(A^+, \theta, A^-)\in P^1_{\mathscr{A}},$$

$$R^1\pi_*: P^1_{\mathscr{A}}\to\mathscr{A}, \quad R^1\pi_*(M)=\mathrm{coker}(d^0), \quad \forall M=(A^+, \theta, A^-)\in P^1_{\mathscr{A}},$$

where $d^0=(\theta\circ\varphi_0^+, -\varphi_0^-): A^+\oplus A^-\to T(A^-)$, and

$$\varphi_0^+: A^+\to T^{-1}(A^+)=\varinjlim (A^+\xrightarrow{T} A^+\xrightarrow{T}\cdots)$$
$$\varphi_0^-: A^-\to T(A^-)=\varinjlim (A^-\xrightarrow{T^{-1}} A^-\xrightarrow{T^{-1}}\cdots)$$

are the structure morphisms of the colimits. Notice that when we take kernel and cokernel we forget $\mathscr{N}$-, $\mathscr{N}^-$- and $\mathscr{Z}$-actions.

4.2.2. Lemma. (a) *If $0\to M'\to M\to M''\to 0$ is an exact sequence in $P^1_{\mathscr{A}}$, then we have the exact sequence in $\mathscr{A}$:*

$$0\to\pi_*(M')\to\pi_*(M)\to\pi_*(M'')$$
$$\to R^1\pi_*(M')\to R^1\pi_*(M)\to R^1\pi_*(M'')\to 0.$$

(b) *Let $A\in\mathscr{A}$; then*

$$\pi_*(\pi^*(A)(n))=\begin{cases}\bigoplus_{i=1}^{n+1} A & n\geq 0,\\ 0 & n<0,\end{cases}$$

$$R^1\pi_*(\pi^*(A)(n))=\begin{cases}\bigoplus_{i=1}^{-n-1} A & n\leq -2,\\ 0 & n>-2.\end{cases}$$

(c) *If $M\in P^1_{\mathscr{A}}$ is locally f.g., then there is an integer n_0 such that when $n\geq n_0$, $R^1\pi_*(M(n))=0$.*

Proof. (a) Obvious by definition and the snake lemma.

(b) Obvious by definition.

(c) First assume $\mathscr{A}$ has a set of small projective generators, then of course $\mathscr{A}$ is strongly admissible. From the proof of Lemma 4.1.4(b), $\{\pi^*P(n)\mid P\in\mathbf{P}(\mathscr{A}), n=0, \pm 1, \ldots)$ generates $P^1_{\mathscr{A}}$, so there is a surjection $\coprod_{\alpha\in I}\pi^*(P_\alpha)(n_\alpha)\to M$, where $P_\alpha\in\mathbf{P}(\mathscr{A})$. Then by Corollary 1.6.10, there is a finite subset $J\subset I$ such that $\coprod_{\alpha\in J}\pi^*(P_\alpha)(n_\alpha)\to M$ is a surjection. Choose n_0 such that $n_\alpha+n_0>-2$ for all $\alpha\in J$, then when $n\geq n_0$

$$R^1\pi_*\Big(\coprod_{\alpha\in J}\pi^*(P_\alpha)(n_\alpha)(n)\Big)=\coprod_{\alpha\in J} R^1\pi_*(\pi^*(P_\alpha)(n_\alpha+n))=0$$

by (b) above. But by (a) above, we have a surjection

$$R^1\pi_*\Big(\coprod_{\alpha\in J}\pi^*(P_\alpha)(n_\alpha)(n)\Big)\to R^1\pi_*(M(n)),$$

so $R^1\pi_*(M(n))=0$ when $n\geq n_0$.

Now let $\{\mathscr{A}, (s_i, F_i)\}$ be an admissible abelian category. Then $s_i^{-1}(M)$ is a locally f.g. in $P^1_{\mathscr{A}_{s_i}}$ for all i. Since $P^1_{\mathscr{A}_{s_i}}$, has a set of small projective generators, there is an n_0 such that $R^1\pi_*(s_i^{-1}(M)(n)) = 0$ when $n \geq n_0$. Choose n_0 large enough so that this holds for all i. Then when $n \geq n_0$, because s_i^{-1} is exact, we have $s_i^{-1}(R^1\pi_*(M(n))) = R^1\pi_*(s_i^{-1}(M(n))) = 0$, so $R^1\pi_*(M(n)) = 0$. □

4.2.3. We have two canonical divisors $(t_1, (\ )(1))$ and $(t_2, (\ ))(1))$ on $P^1_{\mathscr{A}}$ as defined in Definition 4.1.3. Because $(t_1\text{-Tor}) \cap (t_2\text{-Tor}) = 0$, for any $M \in P^1_{\mathscr{A}}$, we have the following sequence which is exact in $P^1_{\mathscr{A}}$ for all integer n by Lemma 1.4.4:

$$0 \to M(n) \xrightarrow{(t_2, -t_1)} M(n+1) \oplus M(n+1) \xrightarrow{(t_1, t_2)} M(n+2) \to 0 . \tag{12}$$

4.2.4. Definition. $M \in P^1_{\mathscr{A}}$ is called *regular* if $R^1\pi_*(M(-1)) = 0$.

4.2.5. Lemma. (a) *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence in $P^1_{\mathscr{A}}$, then:*

(i) *If M', M'' are regular, so is M.*

(ii) *If M is regular, so is M''.*

(iii) *If M is regular, and $\pi_*(M(-1)) \to \pi_*(M''(-1))$ is surjective, then M' is regular.*

(b) *If M is regular, so is $M(n)$ for all $n \geq 0$.*

(c) *If $A \in \mathscr{A}$, then $\pi^*(A)$ is regular.*

Proof. Obvious. □

4.2.6. Lemma. *If $M \in P^1_{\mathscr{A}}$ is regular, then there is a surjection*

$$\nu_M : \pi^*(\pi_*(M)) \to M .$$

Proof. First notice that π^* and π_* are an adjoint pair of functors. In fact, for any $M = ((A^+, \eta), \theta, (A^-, \xi)) \in P^1_{\mathscr{A}}$ and any $B \in \mathscr{A}$, define

$$\Phi : \mathrm{Hom}_{P^1_{\mathscr{A}}}(\pi^*B, M) \to \mathrm{Hom}_{\mathscr{A}}(B, \pi_*M) ,$$
$$f \to \Phi(f) = \pi_*(f)$$

and

$$\Psi : \mathrm{Hom}_{\mathscr{A}}(B, \pi_*M) \to \mathrm{Hom}_{P^1_{\mathscr{A}}}(\pi^*B, M) ,$$
$$g \to \Psi(g) = (f_1, f_2) ,$$

where f_1 and f_2 are defined as follows: Since π_*M is a subobject of $A^+ \oplus A^-$, denote

$$p^+ \text{ (or } p^-) : \pi_*(M) \hookrightarrow A^+ \oplus A^- \to A^+ \text{ (or } A^-)$$

the restriction of the projection $A^+ \oplus A^- \to A^+$ (or A^-), then let

$$f_1 = \Theta(p^+ \circ g) : B[T] \to (A^+, \eta) ,$$
$$f_2 = \Theta(p^- \circ g) : B[T^{-1}] \to (A^-, \xi) ,$$

where Θ is defined as in Lemma 4.1.1(a). Then it is easy to check that Φ and Ψ are well defined and $\Phi \circ \Psi = 1$ and $\Psi \circ \Phi = 1$. So we take the morphism $\nu_M : \pi^*(\pi_*(M)) \to M$ to be the adjunction map of the adjoint pair. To prove ν_M is surjective, we need to prove that

$$\Theta(p^+) = (\eta^i \circ p^+)_{i=0}^{\infty} : \pi_*(M)[T] = \bigoplus_{i=0}^{\infty} \pi_*(M) \to (A^+, \eta)$$

and

$$\Theta(p^-) = (\xi^{-i} \circ p^-)_{i=0}^{-\infty} : \pi_*(M)[T^{-1}] = \bigoplus_{i=0}^{\infty} \pi_*(M) \to (A^-, \xi)$$

are both surjective.

Let

$$p_r^+ : \pi_*(M(r)) \hookrightarrow A^+ \oplus A^- \to A^+$$

denote the restriction of the projection $A^+ \oplus A^- \to A^+$ (or A^-). First we prove that $\bigcup_r \mathrm{Im}(p_r^+) = A^+$. Since $\mathscr{A}$ is an admissible abelian category, by Proposition 1.6.9, $\mathscr{A}$ is generated by locally f.g. objects, so we have $A^+ = \bigcup A_\alpha^-$, where $\{A_\alpha^+\}$ is the set of all locally f.g. subobjects of A^+. We want to prove that each A_α^+ is contained in an $\mathrm{Im}(p_r^+)$ for some r.

Consider the composite

$$h_\alpha : A_\alpha^+ \hookrightarrow A^- \to T^{-1}(A^+) \xrightarrow{\theta} T(A^-) = \varinjlim (A^{-1} \xrightarrow{\xi} A^- \to \cdots) .$$

Since A_α^+ is locally f.g., $\mathrm{Hom}_{\mathscr{A}}(A_\alpha^+, -)$ commutes with directed colimits, h_α can be factorized as

$$A_\alpha^+ \to A^- \xrightarrow{\varphi_r^-} \varinjlim (A^- \xrightarrow{\xi} A^- \to \cdots) ,$$

where φ_r^- is the structure map of the colimit ('r' here indicates that the map sends A^- to the 'rth' A^- in the colimit), so $\mathrm{Im}\, h_\alpha \subset \mathrm{Im}\, \varphi_r^-$. Let A_r' be the pullback of the following diagram:

$$\begin{array}{ccc} A'_r & \longrightarrow & A^- \\ \downarrow & & \downarrow{\scriptstyle \varphi_r^-} \\ A_\alpha^+ & \longrightarrow & \operatorname{Im}(\varphi_r^-) \subset \varinjlim (A^- \xrightarrow{\xi} A^- \longrightarrow \cdots) \end{array}$$

then we have a surjection $A'_r \to A_\alpha^+$. But the definition $\pi_*(M(r)) = \ker(d^0)$ means that $\pi_*(M(r))$ is the pullback of the following diagram:

$$\begin{array}{ccc} A'_r & \longrightarrow & A^- \\ \downarrow{\scriptstyle p_r^+} & & \downarrow{\scriptstyle \varphi_r^- = T^r \circ \varphi_0^-} \\ A_\alpha^+ & \xrightarrow[\theta \circ \varphi_0^-]{} & \operatorname{Im}(\varphi_r^-) \subset \varinjlim (A^- \xrightarrow{\xi} A^- \to \cdots) \end{array}$$

so A'_r is in fact a subobject of $\pi_*(M(r))$; then $A_\alpha^+ \subset \operatorname{Im}(p_r^+)$. Thus we have proved that $\bigcup_r \operatorname{Im}(p_r^+) = A^+$.

Next we want to prove that for any $r \geq 0$,

$$\operatorname{Im}\left(\bigoplus_{i=0}^{r} \pi_*(M) \xrightarrow{(\eta^i \circ p^+)} A^+\right) = \operatorname{Im}\left(\bigoplus_{i=0}^{r} \pi_*(M(i)) \xrightarrow{(p_i^+)} A^+\right),$$

i.e., $\sum_{i=0}^{r} \operatorname{Im}(\eta^i \circ p^+) = \sum_{i=0}^{r} \operatorname{Im}(p_i^+)$.

We use induction on r. When $r = 0$, it is obvious.

Assume the equality holds for $r-1$, $r \geq 1$.

To prove the inductive step, consider the exact sequence from (12):

$$0 \to M(r-2) \xrightarrow{(t_2, -t_1)} M(r-1) \oplus M(r-1) \xrightarrow{(t_1, t_2)} M(r) \to 0.$$

Because $M(r-1)$ is regular, we have $R^1\pi_*(M(r-2)) = 0$; then by Lemma 4.2.2(a) we have a surjection

$$\pi_* M(r-1) \oplus \pi_* M(r-1) \xrightarrow{(t_1, t_2)} \pi_* M(r).$$

So we have $\operatorname{Im}(p_r^+) = \operatorname{Im}((t_1 + t_2) \circ p_{r-1}^+)$. Recall $t_1 = (1, T^{-1}) = (1, \xi)$, $t_2 = (T, 1) = (\eta, 1)$, so $t_1 \circ p_{r-1}^+ = p_{r-1}^+$, $t_2 \circ p_{r-1}^+ = \eta \circ p_{r-1}^+$, and $\operatorname{Im}(p_r^+) = \operatorname{Im}(p_{r-1}^+) + \operatorname{Im}(\eta \circ p_{r-1}^+)$. Then

$$\begin{aligned} \sum_{i=0}^{r} \operatorname{Im}(p_i^+) &= \sum_{i=0}^{r-1} \operatorname{Im}(p_i^+) + \operatorname{Im}(\eta \circ p_{r-1}^+) \\ &= \sum_{i=0}^{r-1} \operatorname{Im}(\eta^i \circ p^+) + \eta(\operatorname{Im}(p_{r-1}^+)) \\ &= \sum_{i=0}^{r-1} \operatorname{Im}(\eta^i \circ p^+) + \eta\left(\sum_{i=0}^{r-1} \operatorname{Im}(\eta^i \circ p^+)\right) \\ &= \sum_{i=0}^{r-1} \operatorname{Im}(\eta^i \circ p^+) + \sum_{i=0}^{r} \operatorname{Im}(\eta^i \circ p^+) \\ &= \sum_{i=0}^{r} \operatorname{Im}(\eta^i \circ p^+). \end{aligned}$$

Thus the induction proof is fulfilled, we have $\operatorname{Im}(\Theta(p^+)) = A^+$. Similarly we prove that $\operatorname{Im}(\Theta(p^-)) = A^-$, thus finishing the proof of the lemma. □

4.2.7. Definition. Let $\mathrm{r}P_{\mathscr{A}}^1$ be the full subcategory of $P_{\mathscr{A}}^1$ of all regular objects. Obviously we can regard $\mathrm{r}P_{\mathscr{A}}^1$ as an exact subcategory of $P_{\mathscr{A}}^1$ in Quillen's sense. If we restrict π_* to $\mathrm{r}P_{\mathscr{A}}^1$, then π_* is an exact functor.

Define a functor

$$Z : \mathrm{r}P_{\mathscr{A}}^1 \to P_{\mathscr{A}}^1, \qquad Z(F) = \ker(\pi^*\pi_* F \xrightarrow{\nu_F} F), \quad \forall F \in \mathrm{r}P_{\mathscr{A}}^1,$$

where ν_F is the adjunction map. Because π_* and π^* are both exact on $P_{\mathscr{A}}^1$, so is Z. Because

$$\pi_*(\pi^*\pi_* F) = \pi_* F \xrightarrow{\pi_*\nu_F} \pi_* F$$

is an isomorphism, by Lemma 4.2.2(a), $\pi_*(Z(F)) = 0$, and by Lemma 4.2.5(a)(iii), $Z(F)(1)$ is regular.

Define another functor

$$T_1 : \mathrm{r}P_{\mathscr{A}} \to P_{\mathscr{A}}^1, \qquad T_1(F) = \pi_*(Z(F)(1)), \quad \forall F \in \mathrm{r}P_{\mathscr{A}}^1.$$

Then T_1 is also exact because Z and π_* are both exact on $\mathrm{r}P_{\mathscr{A}}^1$.

4.2.8. Lemma. *For any $F \in \mathrm{r}P_{\mathscr{A}}^1$, we have an exact sequence in $P_{\mathscr{A}}^1$:*

$$0 \to \pi^*(T_1 F)(-1) \to \pi^*(\pi_* F) \to F \to 0.$$

Proof. Follow the idea of [11, p. 132]. □

We will use induction to prove Theorem 4.0.1. So we first consider a special case: $\mathscr{A}$ is an Ab5 category with a set of small projective generators.

4.3.1. Lemma. *Let $\mathscr{A}$ be an* Ab5 *category with a set of small projective generators; if $M=(P^+,\theta,P^-)\in \mathbf{P}(P^1_{\mathscr{A}})$, and $M(1)$ is regular, then $\pi_*(M)\in \mathbf{P}(\mathscr{A})$. If further M is regular, then $T_1(M)\in \mathbf{P}(\mathscr{A})$.*

Proof (cf. [11, 1.12 and 1.13]). $M=(P^+,\theta,P^-)\in \mathbf{P}(P^1_{\mathscr{A}})$ implies $P^+\in \mathbf{P}(\mathscr{A}[T])$ and $P^-\in \mathbf{P}(\mathscr{A}[T^{-1}])$. Since $\mathbf{P}(\mathscr{A})[T]$ generates $\mathscr{A}[T]$ by Lemma 4.1.1(b), we have a surjection $\coprod P_\alpha[T]\to P^+$, where $P_\alpha\in \mathbf{P}(\mathscr{A})$. But P^+ is small projective in $\mathscr{A}[T]$, this surjection splits, so P^+ is a summand of $\coprod P_\alpha[T]$. Thus P^+ is a projective object in $\mathscr{A}$ (forget the $\mathcal{N}$-action). Similarly, P^- and $T(P^-)$ are also projective objects in $\mathscr{A}$. By the definitions of π_* and $R^1\pi_*$ we have an exact sequence which now splits,

$$0\to \pi_*(M)\to P^+\oplus P^-\to T(P^-)\to R^1\pi_*(M)=0\to 0\,,$$

thus $\pi_*(M)$ is projective in $\mathscr{A}$.

It remains to prove $\pi_*(M)$ is small. According to the proof of Lemma 4.1.4(b), $\{\pi^*P(n)\mid P\in \mathbf{P}(\mathscr{A}),\ n=0,\pm1,\dots\}$ generates $P^1_{\mathscr{A}}$, so we have an exact sequence

$$0\to \ker\to \coprod_{\alpha\in I}\pi^*(P_\alpha)(n_\alpha)\to M\to 0\,,$$

where $P_\alpha\in \mathbf{P}(\mathscr{A})$, $\ker\in \mathbf{P}(P^1_{\mathscr{A}})$, and I can be a finite set because M is locally small projective. By Lemma 4.2.2(c), there is an $n_0>0$ such that when $n\ge n_0$, $\ker(n)$ is regular, so by Lemma 4.2.2(a), we have a surjection

$$\pi_*\Big(\coprod_{\alpha\in I}\pi^*(P_\alpha)(n_\alpha+n)\Big)\to \pi_*(M(n))\,.$$

Because $\pi_*(\coprod_{\alpha\in I}\pi^*(P_\alpha)(n_\alpha+n))$ is small projective by Lemma 4.2.2(a), $\pi_*(M(n))$ is small projective when $n\ge n_0$. Applying Lemma 4.2.2(a) to the exact sequence from (12):

$$0\to M(n-1)\to M(n)\oplus M(n)\to M(n+1)\to 0\,,$$

and because $M(1)$ is regular, when $n\ge 1$ we have a short exact sequence:

$$0\to \pi_*(M(n-1))\to \pi_*(M(n))\oplus \pi_*(M(n))\to \pi_*(M(n-1))\to 0$$

which splits since $\pi_*(M(n+1))$ is projective. So $\pi_*(M(n-1))$ is small since $\pi_*(M(n+1))$ is small. Then by induction over decreasing n, all $\pi_*(M(n))$ are small for $n\ge 0$, in particular, $\pi_*(M)$ is small.

If further M is regular, then $Z(M)$ is regular. Since in the exact sequence

$$0\to Z(M)\to \pi^*\pi_*(M)\to M\to 0$$

both $M,\pi^*\pi_*(M)\in \mathbf{P}(P^1_{\mathscr{A}})$, $Z(M)\in \mathbf{P}(P^1_{\mathscr{A}})$, thus $T_1(M)=\pi_*(Z(M)(1))\in \mathbf{P}(\mathscr{A})$.

4.3.2. Proposition. *Let $\mathscr{A}$ be an* Ab5 *category with a set of small projective generators, $\{(u_k,H_k),\ k=1,\dots,r\}$ be compatible divisors on $\mathscr{A}$, $\mu=\bigcap_{k=1}^r u_k$; then there are homotopy equivalences induced by the functor $(E^\cdot,F^\cdot)\to \pi^*E^\cdot\oplus\pi^*F^\cdot(-1)$:*

(a) (absolute form) $K(\mathscr{A})\times K(\mathscr{A})\to K(P^1_{\mathscr{A}})$,

(b) relative form) $K(\mathscr{A}\text{ off }\mathscr{A}_\mu)\times K(\mathscr{A}\text{ off }\mathscr{A}_\mu)\to K(P^1_{\mathscr{A}}\text{ off }P^1_{\mathscr{A}_\mu})$.

Proof. Follow the idea of [14, 4.9–4.12], only simpler. Since $\mathscr{A}$ and $P^1_{\mathscr{A}}$ are both strongly admissible, we can consider only strictly perfect complexes by Lemma 2.6.2 and the derived category theorem. Consider the following complicial biWaldhausen categories:

$\mathbf{A}_k$ = strictly perfect complexes $E^\cdot$ in $P^1_{\mathscr{A}}$ such that all $E^n(k)$ are regular,
$\mathbf{A}$ = strictly perfect complexes in $P^1_{\mathscr{A}}$,
$\mathbf{B}$ = strictly perfect complexes in $\mathscr{A}$.

Then the embeddings

$$\mathbf{A}_k\subset \mathbf{A}_{k+1}\cdots\subset \mathbf{A}\subset \mathscr{P}(P^1_{\mathscr{A}})\,,\qquad \mathbf{B}\subset\mathscr{P}(\mathscr{A})$$

induce homotopy equivalences $K^{\mathrm{W}}(\mathbf{A}_k)\cong K^{\mathrm{W}}(\mathbf{A}_{k+1})$, and therefore

$$K^{\mathrm{W}}(\mathbf{A}_k)\cong K^{\mathrm{W}}(\mathbf{A})\,,\quad \forall k=0,1,\dots.$$

Then the functor

$$\Phi=\pi^*(\ )\oplus\pi^*(\ )(-1):\mathbf{B}\times\mathbf{B}\to \mathbf{A}_1\subset\mathbf{A}$$

induces homotopy equivalence since the composite functors $\Phi\circ\Psi$ and $\psi'\circ\Phi$ both induce homotopy equivalences, where

$$\Psi=(\pi^*(\ ),\pi^*(\ )(1)):\mathbf{A}_1\to\mathbf{B}\times\mathbf{B}\,,$$

$$\Psi'=(\pi_*,T_1):\mathbf{A}_0\to\mathbf{B}\times\mathbf{B}\,.$$

(b) In the proof of (a) above, if we replace all $\mathbf{A}_k$, $\mathbf{A}$ and $\mathbf{B}$ by $\mathbf{A}_k\cap\mathscr{P}(P^1_{\mathscr{A}}\text{ off }P^1_{\mathscr{A}_\mu})$, $\mathbf{A}\cap\mathscr{P}(P^1_{\mathscr{A}}\text{ off }P^1_{\mathscr{A}_\mu})$ and $\mathbf{B}\cap\mathscr{P}(\mathscr{A}\text{ off }\mathscr{A}_\mu)$, the proof is still true, thus we get the required homotopy equivalence. □

4.3.3. Proposition. *Let $\{\mathscr{A},(s_i,F_i),\ i=1,\dots,n\}$ be an admissible abelian cate-*

gory, $\{(u_k, H_k),\ k = 1, \ldots, r\}$ be a finite set of divisors on $\mathcal{A}$ such that $\{(s_i, F_i), (u_k, H_k)\}$ are compatible, $\sigma = \bigcap_{i=1}^{n-1} s_i$, $\mu = \bigcap_{k=1}^{r} u_k$; then we have a natural long exact sequence

$$\begin{aligned} \cdots &\to K_1(\mathcal{A}_{\sigma\cup s_n} \text{ off } \mathcal{A}_{\sigma\cup s_n\cup\mu}) \to K_0(\mathcal{A} \text{ off } \mathcal{A}_\mu) \\ &\to K_0(\mathcal{A}_\sigma \text{ off } \mathcal{A}_{\sigma\cup\mu}) \oplus K_0(\mathcal{A}_{s_n} \text{ off } \mathcal{A}_{s_n\cup\mu}) \\ &\to K_0(\mathcal{A}_{\sigma\cup s_n} \text{ off } \mathcal{A}_{\sigma\cup s_n\cup\mu}) . \end{aligned}$$

Proof. From Theorem 3.3, we have the following homotopy commutative diagram of spectra where the rows are homotopy fibre sequences:

$$\begin{array}{ccccc} K(\mathcal{A} \text{ off } \mathcal{A}_{\sigma\cap\mu}) & \longrightarrow & K(\mathcal{A} \text{ off } \mathcal{A}_\mu) & \longrightarrow & K(\mathcal{A}_\sigma \text{ off } \mathcal{A}_{\sigma\cup\mu})^{\sim} \\ \downarrow & & \downarrow & & \downarrow \\ K(\mathcal{A}_{s_n} \text{ off } \mathcal{A}_{\sigma\cap(s_n\cup\mu)}) & \longrightarrow & K(\mathcal{A}_{s_n} \text{ off } \mathcal{A}_{s_n\cup\mu}) & \longrightarrow & K(\mathcal{A}_{\sigma\cup s_n} \text{ off } \mathcal{A}_{\sigma\cup s_n\cup\mu})^{\sim} \end{array}$$

So we have the following commutative diagram of K-groups with the rows exact:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & K_1(\mathcal{A}_\sigma \text{ off } \mathcal{A}_{\sigma\cup\mu}) & \longrightarrow & K_0(\mathcal{A} \text{ off } \mathcal{A}_{\sigma\cap\mu}) & \longrightarrow & K_0(\mathcal{A} \text{ off } \mathcal{A}_\mu) & \longrightarrow & K_0(\mathcal{A}_\sigma \text{ off } \mathcal{A}_{\sigma\cup\mu})^{\sim} \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots \longrightarrow & K_1(\mathcal{A}_{\sigma\cup s_n} \text{ off } \mathcal{A}_{\sigma\cup s_n\cup\mu}) & \longrightarrow & K_0(\mathcal{A}_{s_n} \text{ off } \mathcal{A}_{\sigma\cap(s_n\cup\mu)}) & \longrightarrow & K_0(\mathcal{A}_{s_n} \text{ off } \mathcal{A}_{s_n\cup\mu}) & \longrightarrow & K_0(\mathcal{A}_{\sigma\cup s_n} \text{ off } \mathcal{A}_{\sigma\cup s_n\cup\mu})^{\sim} \end{array}$$

By Theorem 3.2.2, we have isomorphisms

$$K_i(\mathcal{A} \text{ off } \mathcal{A}_{\sigma\cap\mu}) \to K_i(\mathcal{A}_{s_n} \text{ off } \mathcal{A}_{\sigma\cap(s_n\cup\mu)}) , \quad i \geq 0 ,$$

so we get the following long exact sequence:

$$\begin{aligned} \cdots &\to K_1(\mathcal{A}_{\sigma\cup s_n} \text{ off } \mathcal{A}_{\sigma\cup s_n\cup\mu}) \to K_0(\mathcal{A} \text{ off } \mathcal{A}_\mu) \\ &\to K_0(\mathcal{A}_\sigma \text{ off } \mathcal{A}_{\sigma\cup\mu})^{\sim} \oplus K_0(\mathcal{A}_{s_n} \text{ off } \mathcal{A}_{s_n\cup\mu}) \\ &\to K_0(\mathcal{A}_{\sigma\cup s_n} \text{ off } \mathcal{A}_{\sigma\cup s_n\cup\mu})^{\sim} . \end{aligned}$$

Comparing this long exact sequence with the one required in the proposition, we see that we need to get rid of '$\sim$' (recall from the proof of Theorem 3.3,

$$K_0(\mathcal{A}_\sigma \text{ off } \mathcal{A}_{\sigma\cup\mu})^{\sim} = \operatorname{Im}(K_0(\mathcal{A} \text{ off } \mathcal{A}_\mu) \to K_0(\mathcal{A}_\sigma \text{ off } \mathcal{A}_{\sigma\cup\mu})) ,$$

etc.), i.e., we need to prove the following sequence

$$\begin{aligned} K_0(\mathcal{A} \text{ off } \mathcal{A}_\mu) &\xrightarrow{(K_0(j_\sigma^*), K_0(j_{s_n}^*))} K_0(\mathcal{A}_\sigma \text{ off } \mathcal{A}_{\sigma\cup\mu}) \oplus K_0(\mathcal{A}_{s_n} \text{ off } \mathcal{A}_{s_n\cup\mu}) \\ &\xrightarrow{K_0(j_{\bar{s}_n}^*) - K_0(j_{\bar{\sigma}}^*)} K_0(\mathcal{A}_{\sigma\cup s_n} \text{ off } \mathcal{A}_{\sigma\cup s_n\cup\mu}) \end{aligned}$$

is exact at the middle. Clearly

$$(K_0(j_{\bar{s}_n}^*) - K_0(j_{\bar{\sigma}}^*)) \circ (K_0(j_\sigma^*), K_0(j_{s_n}^*)) = 0 .$$

Let $(a, b) \in \ker(K_0(j_{\bar{s}_n}^*) - K_0(j_{\bar{\sigma}}^*))$. By a fact pointed out in [14, 1.5.7] that every element of $K_0(\mathbf{A})$ is the class $[c]$ of some c in $\mathbf{A}$ when $\mathbf{A}$ is a Waldhausen category with a mapping cylinder functor satisfying the cylinder axiom, there are $E^{\cdot} \in \mathcal{P}(\mathcal{A}_\sigma \text{ off } \mathcal{A}_{\sigma\cup\mu})$ and $F^{\cdot} \in \mathcal{P}(\mathcal{A}_{s_n} \text{ off } \mathcal{A}_{s_n\cup\mu})$ such that $a = [E^{\cdot}]$ and $b = [F^{\cdot}]$. Then $[j_{\bar{s}_n}^* E^{\cdot}] = [j_{\bar{\sigma}}^* F^{\cdot}]$, so by Lemma 3.5.1, there is a $G^{\cdot} \in \mathcal{P}(\mathcal{A} \text{ off } \mathcal{A}_\mu)$ such that $j_\sigma^* G^{\cdot}$ is quasi-isomorphic to $E^{\cdot}$, thus $a = [E^{\cdot}] \in K_0(\mathcal{A}_\sigma \text{ off } \mathcal{A}_{\sigma\cup\mu})^{\sim}$. By the exactness of the sequence with '$\sim$', we have $(a, b) \in \operatorname{Im}((K_0(j_\sigma^*), K_0(j_{s_n}^*)))$. This finishes the proof of the proposition. □

Proof of Theorem 4.0.1. We use induction on the number n of the structure divisors $\{(s_1, F_1), \ldots, (s_n, F_n)\}$. Here, for the sake of simplicity, we only write down the proof for the absolute case, i.e., $(u_k, H_k) = (0, \mathrm{Id})$ for all k, then $\mathcal{A}_\mu = 0$. For the relative case, we just need replace ( )($\mathcal{A}$) by ( )($\mathcal{A}$ off $\mathcal{A}_\mu$), etc.

When $n = 1$, i.e., $\mathcal{A}$ has a set of small projective generators, then Theorem 4.0.1 for this case is just Proposition 4.3.2.

Assume the theorem is true for $n - 1$. To prove the inductive step, let $\sigma = \bigcap_{i=1}^{n-1} s_i$. Consider the following commutative diagram of K-groups:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \begin{pmatrix} K_1(\mathcal{A}_{\sigma\cup s_n}) \\ \oplus \\ K_1(\mathcal{A}_{\sigma\cup s_n}) \end{pmatrix} & \longrightarrow & \begin{pmatrix} K_0(\mathcal{A}) \\ \oplus \\ K_0(\mathcal{A}) \end{pmatrix} & \longrightarrow & \begin{pmatrix} K_0(\mathcal{A}_\sigma) \\ \oplus \\ K_0(\mathcal{A}_\sigma) \end{pmatrix} \oplus \begin{pmatrix} K_0(\mathcal{A}_{s_n}) \\ \oplus \\ K_0(\mathcal{A}_{s_n}) \end{pmatrix} & \longrightarrow & \begin{pmatrix} K_0(\mathcal{A}_{\sigma\cup s_n}) \\ \oplus \\ K_0(\mathcal{A}_{\sigma\cup s_n}) \end{pmatrix} \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots \longrightarrow & K_1(P^1_{\mathcal{A}_{\sigma\cup s_n}}) & \longrightarrow & K_0(P^1_{\mathcal{A}}) & \longrightarrow & K_0(P^1_{\mathcal{A}_\sigma}) \oplus K_0(P^1_{\mathcal{A}_{s_n}}) & \longrightarrow & K_0(P^1_{\mathcal{A}_{\sigma\cup s_n}}) \end{array}$$

By Proposition 4.3.3, the two rows are exact; then applying the induction hypothesis to $\mathcal{A}_{\sigma\cup s_n}$ which has $n - 1$ structure divisors $\{(s_i s_n, F_i F_n),\ i = 1, \ldots, n-1\}$, and applying the five-lemma to the above diagram, we have isomorphisms

$$K_i(\mathcal{A}) \oplus K_i(\mathcal{A}) \to K_i(P^1_{\mathcal{A}}) , \quad i \geq 0 ,$$

so we have a homotopy equivalence

$$K(\mathcal{A}) \times K(\mathcal{A}) \to K(P^1_{\mathcal{A}}) .$$

This finishes the proof of Theorem 4.0.1. □

5. Negative degree K-groups

5.0. Thanks to the results in Sections 3 and 4, we can follow [14] very closely to construct negative degree K-groups so that we can extend to negative degree K-groups the results obtained in Sections 3 and 4. In terms of spectra, we can construct nonconnected K-theory spectra so that we can get rid of covering spectra in the homotopy fibre sequences in Theorem 3.3 and get real homotopy fibre sequences. We first write down the modifications we need in our context.

5.1.1. Lemma. *Let $\{\mathcal{A}, (s_i, F_i)\}$ be an admissible abelian category, $(\ )[T] : \mathcal{A} \to \mathcal{A}[T]$ be the functor as defined in Definition* 4.1.0; *then*:

(a) *$K((\ )[T]) : K(\mathcal{A}) \to K(\mathcal{A}[T])$ is a split monomorphism up to homotopy.*

(b) *If further there is another finite set of divisors $\{(u_k, H_k)\}$ on $\mathcal{A}$ such that $\{(s_i, F_i), (u_k, H_k)\}$ are compatible, $\mu = \bigcap u_k$, then*

$$K((\ )[T]) : K(\mathcal{A} \text{ off } \mathcal{A}_\mu) \to K(\mathcal{A}[T] \text{ off } \mathcal{A}_\mu[T])$$

is a split monomorphism up to homotopy.

Proof. (a) Let $\mathbf{D}$ denote the image of $(\ )[T] : \mathcal{A} \to \mathcal{A}[T]$; then $\mathbf{D}$ generates $\mathcal{A}[T]$ because for any $(A, \varphi) \in \mathcal{A}[T]$, we have a surjection

$$(\varphi^i) : A[T] = \bigoplus_{i=0}^{\infty} A \to (A, \varphi)$$

in $\mathcal{A}[T]$. Let $\mathbf{A}$ be the Waldhausen subcategory of $\mathcal{P}(\mathcal{A}[T])$ of perfect complexes of objects in $\mathbf{D}$. Since $\mathbf{D}$ generates $\mathcal{A}[T]$, every perfect complex in $\mathcal{A}[T]$ is quasi-isomorphic to a perfect complex of objects in $\mathbf{D}$, so by Theorem 3.1.1 we have $K(\mathcal{A}[T]) = K^{W}(\mathbf{A})$. Obviously $(\ )[T] : \mathcal{P}(\mathcal{A}) \to \mathbf{A}$ is an exact functor of the Waldhausen categories.

Define a functor

$$C : \mathcal{A}[T] \to \mathcal{A}, \qquad C(A, \varphi) = \operatorname{coker}(\varphi), \quad \forall (A, \varphi) \in \mathcal{A}[T].$$

Then the restriction $C|_{\mathbf{D}} : \mathbf{D} \to \mathcal{A}$ is exact. Thus C will induce an exact functor of Waldhausen categories $\mathbf{A} \to \mathcal{P}(\mathcal{A})$ if we can prove that C sends perfect complexes $\in \mathbf{A}$ to perfect complexes $\in \mathcal{P}(\mathcal{A})$, or equivalently, for each i, $s_i(CE^\cdot) = C(s_i(E^\cdot))$ is perfect in $\mathcal{A}_{s_i}$ when $E^\cdot \in \mathbf{A}$. This can be easily obtained from the following sublemma:

Sublemma. *Assume $\mathcal{A}$ has a set of small projective generators, $E^\cdot$ is perfect in $\mathcal{A}[T]$; then $LC(E^\cdot)$ is perfect in $\mathcal{A}$, where LC is the left derived functor of C. In particular, if $E^\cdot \in \mathbf{A}$, because $C|_{\mathbf{D}}$ is exact, $LC|_{\mathbf{A}} = C|_{\mathbf{A}}$, so $C(E^\cdot)$ is perfect in $\mathcal{A}$.*

Proof. Since $\mathcal{A}[T]$ has enough projective objects,

$$LC : D^-(\mathcal{A}[T]) \to D^-(\mathcal{A})$$

exists.

Define a functor

$$N : \mathcal{A} \to \mathcal{A}[T], \qquad N(A) = (A, 0) \in \mathcal{A}[T], \quad \forall A \in \mathcal{A}.$$

Then obviously (C, N) is an adjoint pair of functors between $\mathcal{A}[T]$ and $\mathcal{A}$, and N is exact. We claim that (LC, N) is also an adjoint pair of functors between the derived categories. To see this, let $E^\cdot \in D^-(\mathcal{A}[T])$, $F^\cdot \to E^\cdot$ be a projective resolution of $E^\cdot$. By Lemma 4.1.1(b), we can choose all F^n to have the form $F^n = P^n[T]$, where P^n is a small projective object in $\mathcal{A}$; then $C(F^\cdot)$ is a complex of small projective objects in $\mathcal{A}$. Let $G^\cdot \in D^-(\mathcal{A})$; then

$$\begin{aligned}
&\operatorname{Hom}_{D(\mathcal{A})}(LC(E^\cdot), G^\cdot) \\
&\quad \cong \operatorname{Hom}_{D(\mathcal{A})}(C(F^\cdot), G^\cdot) = H^0(\operatorname{Hom}^\cdot(C(F^\cdot), G^\cdot)) \\
&\quad \cong H^0(\operatorname{Hom}(F^\cdot, N(G^\cdot))) = \operatorname{Hom}_{D(\mathcal{A}[T])}(F^\cdot, N(G^\cdot)) \\
&\quad \cong \operatorname{Hom}_{D(\mathcal{A}[T])}(E^\cdot, N(G^\cdot)).
\end{aligned}$$

So (LC, N) are adjoint for the derived categories. Now if $E^\cdot$ is perfect in $\mathcal{A}[T]$, $\{F^\cdot_\alpha\}$ is an arbitrary inductive system of complexes in $\mathcal{A}$, then

$$\begin{aligned}
&\operatorname{Hom}_{D(\mathcal{A})}(LC(E^\cdot), \varinjlim F^\cdot_\alpha) \\
&\quad \cong \operatorname{Hom}_{D(\mathcal{A}[T])}(F^\cdot, N(\varinjlim F^\cdot_\alpha)) \\
&\quad \cong \varinjlim \operatorname{Hom}_{D(\mathcal{A}[T])}(E^\cdot, N(F^\cdot_\alpha)) \\
&\quad \cong \varinjlim \operatorname{Hom}_{D(\mathcal{A})}(LC(E^\cdot), F^\cdot_\alpha).
\end{aligned}$$

So $LC(E^\cdot)$ is perfect in $\mathcal{A}$ by Proposition 2.4(a). This finishes the proof of the sublemma. □

Now $C \circ (\ )[T] = \mathrm{Id}$, so $K((\ )[T])$ is a split monomorphism up to homotopy.

(b) In the proof of (a) above, replace $\mathbf{A}$ by $\mathbf{A} \cap (\mathcal{P}(\mathcal{A}[T] \text{ off } \mathcal{A}_\mu[T])$ and replace $\mathcal{P}(\mathcal{A})$ by $\mathcal{P}(\mathcal{A} \text{ off } \mathcal{A}_\mu)$. □

5.1.2. Theorem (Bass fundamental theorem). *Let $\{\mathcal{A}, (s_i, F_i)\}$ be an admissible abelian category, and $\{(u_k, G_k)\}$ be another finite set of divisors on $\mathcal{A}$ such that $\{(s_i, F_i), (u_k, G_k)\}$ are compatible, $\mu = \bigcap u_k$; then:*

(a) *For $n \geq 1$, there are exact sequences*

$$0 \to K_n(\mathscr{A}) \to K_n(\mathscr{A}[T]) \oplus K_n(\mathscr{A}[T^{-1}])$$
$$\to K_n(\mathscr{A}[T, T^{-1}]) \xrightarrow{\partial_T} K_{n-1}(\mathscr{A}) \to 0\,,$$

$$0 \to K_n(\mathscr{A} \text{ off } \mathscr{A}_\mu) \to K_n(\mathscr{A}[T] \text{ off } \mathscr{A}_\mu[T]) \oplus K_n(\mathscr{A}[T^{-1}] \text{ off } \mathscr{A}_\mu[T^{-1}])$$
$$\to K_n(\mathscr{A}[T, T^{-1}] \text{ off } \mathscr{A}_\mu[T, T^{-1}]) \xrightarrow{\partial_T} K_{n-1}(\mathscr{A} \text{ off } \mathscr{A}_\mu) \to 0\,.$$

(b) *For* $n \geq 1$,

$$\partial_T : K_n(\mathscr{A}[T, T^{-1}]) \to K_{n-1}(\mathscr{A})\,,$$
$$\partial_T : K_n(\mathscr{A}[T, T^{-1}] \text{ off } \mathscr{A}_\mu[T, T^{-1}]) \to K_{n-1}(\mathscr{A} \text{ off } \mathscr{A}_\mu)\,,$$

split naturally, i.e., there are

$$h_T : K_{n-1}(\mathscr{A}) \to K_n(\mathscr{A}[T, T^{-1}])\,,$$
$$h_T : K_{n-1}(\mathscr{A} \text{ off } \mathscr{A}_\mu) \to K_n(\mathscr{A}[T, T^{-1}] \text{ off } \mathscr{A}_\mu[T, T^{-1}])\,,$$

such that $\partial_T \circ h_T = \text{id}$.

Naturality here means the following: Let $\mathscr{A}\mathscr{A}$ *be the category of all admissible abelian categories* $\{\mathscr{A}, (s_i, F_i)\}$ *with another finite set of divisors* $\{(u_k, G_k)\}$ *on* $\mathscr{A}$ *such that all divisors involved are compatible. A morphism in* $\mathscr{A}\mathscr{A}$ *is a functor*

$$F : \{\mathscr{A}, (s_i, F_i), (u_k, G_k)\} \to \{\mathscr{A}', (s'_i, F'_i), (u'_k, G'_k)\}$$

which sends complexes $\in \mathscr{P}(\mathscr{A} \text{ off } \mathscr{A}_\mu)$ *to complexes* $\in \mathscr{P}(\mathscr{A}' \text{ off } \mathscr{A}'_{\mu'})$, *where* $\mu = \bigcap u_k$ *and* $\mu' = \bigcap u'_k$. *Then the following diagram commutes for any morphism* F *in* $\mathscr{A}\mathscr{A}$:

$$\begin{array}{ccccc}
K_{n-1}(\mathscr{A} \text{ off } \mathscr{A}_\mu) & \xrightarrow{h_T} & K_n(\mathscr{A}[T, T^{-1}] \text{ off } \mathscr{A}_\mu[T, T^{-1}]) & \xrightarrow{\partial_T} & K_{n-1}(\mathscr{A} \text{ off } \mathscr{A}_\mu) \\
\downarrow{\scriptstyle K_{n-1}(F)} & & \downarrow{\scriptstyle K_n(F)} & & \downarrow{\scriptstyle K_{n-1}(F)} \\
K_{n-1}(\mathscr{A}' \text{ off } \mathscr{A}'_{\mu'}) & \xrightarrow{h_T} & (\mathscr{A}'[T, T^{-1}] \text{ off } \mathscr{A}'_{\mu'}[T, T^{-1}]) & \xrightarrow{\partial_T} & K_{n-1}(\mathscr{A}' \text{ off } \mathscr{A}'_{\mu'})
\end{array}$$

(c) *For* $n = 0$, *there are exact sequences*

$$0 \to K_0(\mathscr{A}) \to K_0(\mathscr{A}[T]) \oplus K_0(\mathscr{A}[T^{-1}]) \to K_0(\mathscr{A}[T, T^{-1}])\,,$$

$$0 \to K_0(\mathscr{A} \text{ off } \mathscr{A}_\mu) \to K_0(\mathscr{A}[T] \text{ off } \mathscr{A}_\mu[T]) \oplus K_0(\mathscr{A}[T^{-1}] \text{ off } \mathscr{A}_\mu[T^{-1}])$$
$$\to K_0(\mathscr{A}[T, T^{-1}] \text{ off } \mathscr{A}_\mu[T, T^{-1}])\,.$$

Proof (cf. [14, 6.1]). Notice that in [14] $K(X)$ is a ring spectrum where X is a scheme. But in the present case, $K(\mathscr{A})$ is no longer a ring spectrum. So we must modify the proof in [14].

For simplicity of exposition, we only write down the proof for the absolute case. For a proof of the relative case, we just need to replace $\mathscr{P}(\mathscr{A})$ by $\mathscr{P}(\mathscr{A} \text{ off } \mathscr{A}_\mu)$, etc.

(a),(c) Consider the following homotopy commutative diagram of spectra:

$$\begin{array}{ccccc}
K(P^1_{\mathscr{A}} \text{ off } \mathscr{A}[T]) & \longrightarrow & K(P^1_{\mathscr{A}}) & \xrightarrow{K(\iota_1^{-1})} & K(\mathscr{A}[T])^{\sim} \\
\downarrow & & \downarrow{\scriptstyle K(\iota_2^{-1})} & & \downarrow{\scriptstyle K(T^{-1})} \\
K(\mathscr{A}[T^{-1}] \text{ off } \mathscr{A}[T, T^{-1}]) & \longrightarrow & K(\mathscr{A}[T^{-1}]) & \xrightarrow{K(T)} & K(\mathscr{A}[T, T^{-1}])^{\sim}
\end{array}$$

where the two rows are homotopy fibre sequences by Theorem 3.3 and the left vertical map is a homotopy equivalence by Theorem 3.2, so we get a homotopy cartesian square

$$\begin{array}{ccc}
K(P^1_{\mathscr{A}}) & \xrightarrow{K(\iota_1^{-1})} & K(\mathscr{A}[T])^{\sim} \\
\downarrow{\scriptstyle K(\iota_2^{-1})} & & \downarrow{\scriptstyle K(T^{-1})} \\
K(\mathscr{A}[T^{-1}]) & \xrightarrow[K(T)]{} & K(\mathscr{A}[T, T^{-1}])^{\sim}\,.
\end{array}$$

Then we have a map of spectra $\Omega(K(\mathscr{A}[T, T^{-1}])^{\sim}) \to K(P^1_{\mathscr{A}})$ or rather

$$\partial_T : K(\mathscr{A}[T, T^{-1}])^{\sim} \to K(P^1_{\mathscr{A}}) \wedge S^1$$

which induces the connecting map ∂_T in the following long exact Mayer–Vietoris sequence of K-groups (by Proposition 4.3.3, we can remove '$\sim$'):

$$\cdots \to K_{n+1}(\mathscr{A}[T, T^{-1}]) \xrightarrow{\partial_T} K_n(P^1_{\mathscr{A}})$$
$$\xrightarrow{(K_n(\iota_1^{-1}), K_n(\iota_2^{-1}))} K_n(\mathscr{A}[T]) \oplus K_n(\mathscr{A}[T^{-1}])$$
$$\xrightarrow{K_n(T^{-1}) - K_n(T)} K_n(\mathscr{A}[T, T^{-1}]) \to \cdots \to K_0(\mathscr{A}[T, T^{-1}])\,.$$

From Theorem 4.0.1, we have isomorphisms

$$K_n(\Phi) : K_n(\mathscr{A}) \oplus K_n(\mathscr{A}) \to K_n(P^1_{\mathscr{A}})\,, \quad \forall n \geq 0\,.$$

Let $l_n, r_n : K_n(\mathscr{A}) \to K_n(\mathscr{A}) \oplus K_n(\mathscr{A})$ send x to $(x, 0)$, $(0, x)$ respectively, and $i_n = l_n - r_n$. Then

$$K_n(t_1^{-1})\circ K_n(\Phi)\circ i_n = K_n((\ )[T]) - K_n((\ )[T]) = 0\,,$$

$$K_n(t_2^{-1})\circ K_n(\Phi)\circ i_n = K_n((\ )[T^{-1}]) - K_n((\ )[T^{-1}]) = 0\,.$$

But $(K_n(T_1^{-1})\circ K_n(\Phi),\ K_n(t_2^{-1})\circ K_n(\Phi))\circ l_n$ is an injection by Lemma 5.1.1, and $\mathrm{Im}(l_n)\oplus \mathrm{Im}(i_n) = K_n(\mathscr{A})\oplus K_n(\mathscr{A})$, so the above long exact sequence breaks into the following exact sequence for $n\geq 0$:

$$0\to K_n(\mathscr{A})\to K_n(\mathscr{A}[T])\oplus K_n(\mathscr{A}[T^{-1}])$$
$$\to K_n(\mathscr{A}[T,T^{-1}])\xrightarrow{\partial_T} K_{n-1}(\mathscr{A})\to 0\,,$$

which is (a). For $n=0$, we get

$$0\to K_0(\mathscr{A})\to K_0(\mathscr{A}[T])\oplus K_0(\mathscr{A}[T^{-1}])\to K_0(\mathscr{A}[T,T^{-1}])\,,$$

which is (c).

(b) The tensor product (in the obvious sense) induces the following homotopy commutative diagram of spectra (here Z is the ring of integers):

$$\begin{array}{ccc} K(\mathscr{A})\wedge K(Z[T,T^{-1}])^{\sim} & \xrightarrow{1\wedge\partial_T} & K(\mathscr{A})\wedge K(P^1_Z)\wedge S^1 \\ \downarrow{\scriptstyle\oplus} & & \downarrow{\scriptstyle\oplus\wedge 1} \\ K(\mathscr{A}[T,T^{-1}])^{\sim} & \xrightarrow{\partial_T} & K(P^1_{\mathscr{A}})\wedge S^1 \end{array}$$

By (a) above, $\mathrm{Im}(K_n(\partial_T)) = K_{n-1}(\mathscr{A})$, so we have the homotopy commutative diagram

$$\begin{array}{ccc} K(\mathscr{A})\wedge K(Z[T,T^{-1}])^{\sim} & \xrightarrow{1\wedge\partial_T} & K(\mathscr{A})\wedge K(Z)\wedge S^1 \\ \downarrow{\scriptstyle\oplus} & & \downarrow{\scriptstyle\oplus\wedge 1} \\ K(\mathscr{A}[T,T^{-1}])^{\sim} & \xrightarrow{\partial_T} & K(\mathscr{A})\wedge S^1 \end{array} \qquad (13)$$

On the other hand, let T denote the map of spectra $S^1\to K(Z[T,T^{-1}])^{\sim}$ which represents the element $T\in K_1(Z[T,T^{-1}])^{\sim} = K_1(Z[T,T^{-1}])$; then we have the following homotopy commutative diagram:

$$\begin{array}{ccc} K(\mathscr{A})\wedge K(Z)\wedge S^1 & \longrightarrow & K(\mathscr{A})\wedge K(Z[T,T^{-1}])^{\sim} \\ \downarrow & & \downarrow \\ K(\mathscr{A})\wedge S^1 & \longrightarrow & K(\mathscr{A}[T,T^{-1}])^{\sim} \end{array} \qquad (14)$$

Composing (14) with (13) horizontally, we have the following homotopy commutative diagram:

$$\begin{array}{ccccc} K(\mathscr{A})\wedge K(Z)\wedge S^1 & \longrightarrow & K(\mathscr{A})\wedge K(Z[T,T^{-1}])^{\sim} & \longrightarrow & K(\mathscr{A})\wedge K(Z)\wedge S^1 \\ \downarrow & & \downarrow & & \downarrow \\ K(\mathscr{A})\wedge S^1 & \longrightarrow & K(\mathscr{A}[T,T^{-1}])^{\sim} & \longrightarrow & K(\mathscr{A})\wedge S^1 \end{array} \qquad (15)$$

The composite of the top row of (15) is an automorphism [14, Theorem 6.1(b)], and because $K(\mathscr{A})\wedge K(Z)\xrightarrow{\otimes} K(\mathscr{A})$ induces a $K(Z)$-module structure on $K(\mathscr{A})$, so the composite of the bottom row of (15) is also an automorphism. Let h_T be the composite of the inverse of this automorphism and of the map $K(\mathscr{A})\wedge S^1\to K(\mathscr{A}[T,T^{-1}])^{\sim}$ at the bottom of (14); then $\partial_T\circ h_T = \mathrm{id}$, so ∂_T is split. Obviously h_T is natural. □

Let $\mathscr{A}\mathscr{A}$ be the category defined as in Theorem 5.1.2(b), then we can construct covariant functors K^{B}_n from $\mathscr{A}\mathscr{A}$ to the category of all abelian groups for all $n=0, \pm1, \pm2, \ldots$ and K^{B} from $\mathscr{A}\mathscr{A}$ to the category of spectra. We refer the reader to [14, Section 6] for detailed construction of K^{B}_n and K^{B}. Here we only write down the final results of the construction.

5.2. Theorem (Bass fundamental theorem). *Let $\{\mathscr{A}, (s_i, F_i), i=1,\ldots,n\}$ be an admissible abelian category, $\{(u_k, H_k), k=1,\ldots,r\}$ be another finite set of divisors on $\mathscr{A}$ such that $\{(s_i,F_i), (u_k,H_k)\}$ are compatible, $\mu=\bigcap_{k=1}^{r} u_k$; then:*

(a) *The natural map $K(\mathscr{A}\text{ off }\mathscr{A}_\mu)\to K^{\mathrm{B}}(\mathscr{A}\text{ off }\mathscr{A}_\mu)$ induces isomorphisms on π_n for $n\geq 0$:*

$$K_n(\mathscr{A}\text{ off }\mathscr{A}_\mu)\xrightarrow{\cong} K^{\mathrm{B}}_n(\mathscr{A}\text{ off }\mathscr{A}_\mu)\,.$$

(b) *For any integer n, there is a natural exact sequence.*

$$0\to K^{\mathrm{B}}_n(\mathscr{A}\text{ off }\mathscr{A}_\mu)\to K^{\mathrm{B}}_n(\mathscr{A}[T]\text{ off }\mathscr{A}_\mu[T])\oplus K^{\mathrm{B}}_n(\mathscr{A}[T^{-1}]\text{ off }\mathscr{A}_\mu[T^{-1}])$$
$$\to K^{\mathrm{B}}_n(\mathscr{A}[T,T^{-1}]\text{ off }\mathscr{A}_\mu[T,T^{-1}])\to K^{\mathrm{B}}_{n-1}(\mathscr{A}\text{ off }\mathscr{A}_\mu)\to 0\,.$$

(c) *There is a natural homotopy fibre sequence of spectra*

$$K(\mathscr{A}[T]\text{ off }\mathscr{A}_\mu[T])\ \overset{h}{\underset{K(\mathscr{A}\text{ off }\mathscr{A}_\mu)}{\bigcup}}\ K(\mathscr{A}[T^{-1}]\text{ off }\mathscr{A}_\mu[T^{-1}])$$
$$\xrightarrow{b} K(\mathscr{A}[T,T^{-1}]\text{ off }\mathscr{A}_\mu[T,T^{-1}])\to \Sigma K(\mathscr{A}\text{ off }\mathscr{A}_\mu)\,.$$

(d) *For any integer n and any positive integer $k\geq 1$, we have the natural isomorphism $\partial_{T_k}\circ\cdots\circ\partial_{T_1}\circ(\bigcup T_1)\circ\cdots\circ(\bigcup T_k) = \mathrm{id}$:*

$$\begin{array}{ccc} K_n^{\mathrm{B}}(\mathcal{A} \text{ off } \mathcal{A}_\mu) & \xrightarrow{\partial_{T_k}\circ\cdots\circ\partial_{T_1}} & K_{n+k}^{\mathrm{B}}(\mathcal{A}[T_1^{\pm1},\ldots,T_k^{\pm1}] \text{ off } \mathcal{A}_\mu[T_1^{\pm1},\ldots,T_k^{\pm1}]) \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle (\cup T_1)\circ\cdots\circ(\cup T_k)} \\ & & K_n^{\mathrm{B}}(\mathcal{A} \text{ off } \mathcal{A}_\mu) \end{array}$$

(e) *For any positive integer* $k \geq 1$, *the composite* $d_{T_k} \circ \cdots \circ d_{T_1} \circ (\bigcup T_1) \circ \cdots \circ (\bigcup T_k)$ *is a natural homotopy equivalence, i.e., the following diagram is homotopy commutative*:

$$\begin{array}{ccc} \Sigma^k K^{\mathrm{B}}(\mathcal{A} \text{ off } \mathcal{A}_\mu) & \xrightarrow{(\cup T_1)\circ\cdots\circ(\cup T_k)} & K^{\mathrm{B}}(\mathcal{A}[T_1^{\pm1},\ldots,T_k^{\pm}] \text{ off } \mathcal{A}_\mu[T_1^{\pm1},\ldots,T_k^{\pm1}]) \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle d_{T_k}\circ\cdots\circ d_{T_1}} \\ & & \Sigma^k K^{\mathrm{B}}(\mathcal{A} \text{ off } \mathcal{A}_\mu) \end{array}$$ □

From the construction of K_n^{B} and K^{B} and the degree shifting formula, Theorem 5.2(d), we can re-write the results in Sections 3 and 4 in terms of K_n^{B} and K^{B} easily.

5.3. Theorem (Excision). *Let* $\{\mathcal{A}, (s_i, F_i), i = 1, \ldots, n\}$ *be an admissible abelian category*, $\{(t_j, G_j), j = 1, \ldots, m\}$ *and* $\{(u_k, H_k), k = 1, \ldots, r\}$ *be other two finite sets of divisors on* $\mathcal{A}$ *such that* $\{(s_i, F_i), (t_j, G_j), (u_k, H_k)\}$ *are compatible*, $\tau = \bigcap_{j=1}^m t_j$, $\mu = \bigcap_{k=1}^r u_k$, *and* $(\tau\text{-Tor}) \cap (\mu\text{-Tor}) = 0$; *then we have induced by the localization functor* j_τ^*, *a natural homotopy equivalence*,

$$K^{\mathrm{B}}(j_\tau^*) : K^{\mathrm{B}}(\mathcal{A} \text{ off } \mathcal{A}_\mu) \xrightarrow{\cong} K^{\mathrm{B}}(\mathcal{A}_\tau \text{ off } \mathcal{A}_{\tau\cup\mu})$$

and isomorphisms for all integers n,

$$K_n^{\mathrm{B}}(j_\tau^*) : K_n^{\mathrm{B}}(\mathcal{A} \text{ off } \mathcal{A}_\mu) \xrightarrow{\cong} K_n^{\mathrm{B}}(\mathcal{A}_\tau \text{ off } \mathcal{A}_{\tau\cup\mu}) . \quad □$$

5.4. Theorem (Projective line bundle theorem). *Let* $\{\mathcal{A}, (s_i, F_i), i = 1, \ldots, n\}$ *be an admissible abelian category*, $\{(u_k, H_k), k = 1, \ldots, r\}$ *be another finite set of divisors on* $\mathcal{A}$ *such that* $\{(s_i, F_i), (u_k, H_k)\}$ *are compatible*, $\mu = \bigcap_{k=1}^r u_k$; *then we have natural homotopy equivalences*,

(i) $K^{\mathrm{B}}(\mathcal{A}) \times K^{\mathrm{B}}(\mathcal{A}) \xrightarrow{\cong} K^{\mathrm{B}}(P_{\mathcal{A}}^1)$,

(ii) $K^{\mathrm{B}}(\mathcal{A} \text{ off } \mathcal{A}_\mu) \times K^{\mathrm{B}}(\mathcal{A} \text{ off } \mathcal{A}_\mu) \xrightarrow{\cong} K^{\mathrm{B}}(P_{\mathcal{A}}^1 \text{ off } P_{\mathcal{A}_\mu}^1)$,

and isomorphisms for all integers n,

(i) $K_n^{\mathrm{B}}(\mathcal{A}) \times K_n^{\mathrm{B}}(\mathcal{A}) \xrightarrow{\cong} K_n^{\mathrm{B}}(P_{\mathcal{A}}^1)$,

(ii) $K_n^{\mathrm{B}}(\mathcal{A} \text{ off } \mathcal{A}_\mu) \times K_n^{\mathrm{B}}(\mathcal{A} \text{ off } \mathcal{A}_\mu) \xrightarrow{\cong} K_n^{\mathrm{B}}(P_{\mathcal{A}}^1 \text{ off } P_{\mathcal{A}_\mu}^1)$.

5.5. Theorem (Localization). *Let* $\{\mathcal{A}, (s_i, F_i), i = 1, \ldots, n\}$ *be an admissible abelian category*, $\{(t_j, G_j), j = 1, \ldots, m\}$ *and* $\{(u_k, H_k), k = 1, \ldots, r\}$ *be another two finite sets of divisors on* $\mathcal{A}$ *such that* $\{(s_i, F_i), (t_j, G_j), (u_k, H_k)\}$ *are compatible*, $\tau = \bigcap_{j=1}^m t_j$, $\mu = \bigcap_{k=1}^r u_k$; *then there are homotopy fibre sequences of spectra*,

(i) $K^{\mathrm{B}}(\mathcal{A} \text{ off } \mathcal{A}_\tau) \to K^{\mathrm{B}}(\mathcal{A}) \to K^{\mathrm{B}}(\mathcal{A}_\tau)$,

(ii) $K^{\mathrm{B}}(\mathcal{A} \text{ off } \mathcal{A}_{\tau\cap\mu}) \to K^{\mathrm{B}}(\mathcal{A} \text{ off } \mathcal{A}_\mu) \to K^{\mathrm{B}}(\mathcal{A}_\tau \text{ off } \mathcal{A}_{\tau\cup\mu})$,

and the long exact sequences of K-groups drawn from the above homotopy fibre sequence,

(i) $\cdots \to K_n^{\mathrm{B}}(\mathcal{A} \text{ off } \mathcal{A}_\tau) \to K_n^{\mathrm{B}}(\mathcal{A}) \to K_n^{\mathrm{B}}(\mathcal{A}_\tau) \to \cdots$,

(ii) $\cdots \to K_n^{\mathrm{B}}(\mathcal{A} \text{ off } \mathcal{A}_{\tau\cap\mu}) \to K_n^{\mathrm{B}}(\mathcal{A} \text{ off } \mathcal{A}_\mu) \to K_n^{\mathrm{B}}(\mathcal{A}_\tau \text{ off } \mathcal{A}_{\tau\cup\mu}) \to \cdots$.

Proof. See [14, Theorem 7.4]. □

5.6.1. Proposition (Mayer–Vietoris). *Let* $\{\mathcal{A}, (s_i, F_i)\}$ *be an admissible abelian category*, $\{(t_{j_1}, G_{j_1})\}$, $\{(t_{j_2}, G_{j_2})\}$ *and* $\{(u_k, H_k)\}$ *be another three finite sets of divisors on* $\mathcal{A}$ *such that all the divisors involved are compatible*, $\tau_1 = \bigcap_{j_1} t_{j_1}$, $\tau_2 = \bigcap_{j_2} t_{j_2}$, $\mu = \bigcap_k u_k$; *then we have homotopy cartesian squares*:

(i)
$$\begin{array}{ccc} K^{\mathrm{B}}(\mathcal{A}_{\tau_1\cap\tau_2}) & \longrightarrow & K^{\mathrm{B}}(\mathcal{A}_{\tau_1}) \\ \downarrow & & \downarrow \\ K^{\mathrm{B}}(\mathcal{A}_{\tau_2}) & \longrightarrow & K^{\mathrm{B}}(\mathcal{A}_{\tau_1\cup\tau_2}) \end{array}$$

(ii)
$$\begin{array}{ccc} K^{\mathrm{B}}(\mathcal{A}_{\tau_1\cap\tau_2} \text{ off } \mathcal{A}_{(\tau_1\cap\tau_2)\cup\mu}) & \longrightarrow & K^{\mathrm{B}}(\mathcal{A}_{\tau_1} \text{ off } \mathcal{A}_{\tau_1\cup\mu}) \\ \downarrow & & \downarrow \\ K^{\mathrm{B}}(\mathcal{A}_{\tau_2} \text{ off } \mathcal{A}_{\tau_2\cup\mu}) & \longrightarrow & K^{\mathrm{B}}(\mathcal{A}_{\tau_1\cup\tau_2} \text{ off } \mathcal{A}_{\tau_1\cup\tau_2\cup\mu}) \end{array}$$

Proof. Consider the following commutative diagram:

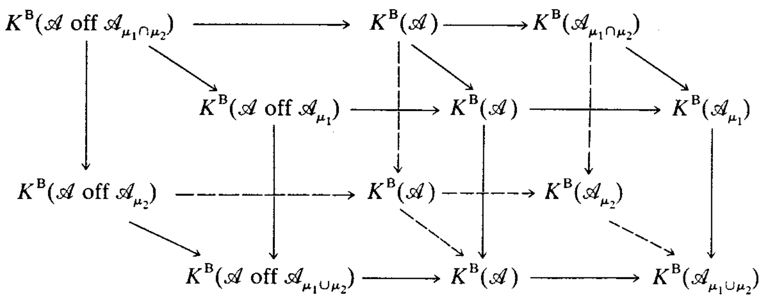
$$\begin{array}{ccccc} K^{\mathrm{B}}(\mathcal{A}_{\tau_1\cap\tau_2} \text{ off } \mathcal{A}_{\tau_1\cap(\tau_2\cup\mu)}) & \longrightarrow & K^{\mathrm{B}}(\mathcal{A}_{\tau_1\cap\tau_2} \text{ off } \mathcal{A}_{(\tau_1\cap\tau_2)\cup\mu}) & \longrightarrow & K^{\mathrm{B}}(\mathcal{A}_{\tau_1} \text{ off } \mathcal{A}_{\tau_1\cup\mu}) \\ \cong\downarrow & & \downarrow & & \downarrow \\ K^{\mathrm{B}}(\mathcal{A}_{\tau_2} \text{ off } \mathcal{A}_{\tau_2\cap(\tau_1\cup\mu)}) & \longrightarrow & K^{\mathrm{B}}(\mathcal{A}_{\tau_2} \text{ off } \mathcal{A}_{\tau_2\cup\mu}) & \longrightarrow & K^{\mathrm{B}}(\mathcal{A}_{\tau_1\cup\tau_2} \text{ off } \mathcal{A}_{\tau_1\cup\tau_2\cup\mu}) \end{array}$$

The rows are homotopy fibre sequences by Theorem 5.5, the left vertical map is a homotopy equivalence by Theorem 5.3, so the right square is a homotopy cartesian square. □

5.6.2. Proposition. *Let* $\{\mathcal{A}, (s_i, F_i)\}$ *be an admissible abelian category*, $\{(u_{k_1}, H_{k_1})\}$ *and* $\{(u_{k_2}, H_{k_2})\}$ *be another two finite sets of divisors on* $\mathcal{A}$ *such that all the divisors involved are compatible*, $\mu_1 = \bigcap_{k_1} u_{k_1}$, $\mu_2 = \bigcap_{k_2} u_{k_2}$; *then we have a homotopy cartesian square*:

$$\begin{array}{ccc} K^{B}(\mathcal{A} \text{ off } \mathcal{A}_{\mu_1\cap\mu_2}) & \longrightarrow & K^{B}(\mathcal{A} \text{ off } \mathcal{A}_{\mu_1}) \\ \downarrow & & \downarrow \\ K^{B}(\mathcal{A} \text{ off } \mathcal{A}_{\mu_2}) & \longrightarrow & K^{B}(\mathcal{A} \text{ off } \mathcal{A}_{\mu_1\cup\mu_2}) \end{array}$$

Proof. Consider the following commutative diagram:



The rows are homotopy fibre sequences by Theorem 6.4, the right vertical square is homotopy cartesian by Proposition 6.5.1, and the middle vertical square is homotopy cartesian obviously, so the left vertical square is homotopy cartesian also. □

5.6.3. We will use homotopy limits of systems of spectra to formulate the Mayer–Vietoris theorem for covers with more than two sets of divisors. For definition and basic properties of homotopy limits of systems of spectra, we refer to [13, Sections 1 and 5].

Let $\{\mathcal{A}, (s_i, F_i)\}$ be an admissible abelian category, $\{(t_{j_1}, G_{j_1})\}, \ldots, \{(t_{j_p}, G_{j_p})\}$ be p finite sets of divisors on $\mathcal{A}$ such that all the divisors involved are compatible, $\tau_1 = \bigcap_{j_1} t_{j_1}, \ldots, \tau_p = \bigcap_{j_p} t_{j_p}$. We call $\mathcal{U} = \{\tau_1, \ldots, \tau_p\}$ a cover of $\mathcal{A}$ if

$$(\tau_1\text{-Tor}) \cap \cdots \cap (\tau_p\text{-Tor}) = 0 .$$

If F is an arbitrary functor from $\mathcal{A}\mathcal{A}$ to the category of spectra, we denote by

$$\check{H}^{\cdot}(\mathcal{A}, \mathcal{U}; F) = \operatorname{holim}\Big(\prod_{l=1}^{p} F(\mathcal{A}_{\tau_l}) \rightrightarrows \prod_{l_0, l_1 = 1}^{p} F(\mathcal{A}_{\tau_{l_0}\cup\tau_{l_1}}) \substack{\leftarrow\\\rightarrow\\\rightarrow} \cdots\Big)$$

the homotopy limit over Δ of the cosimplicial spectrum where the coface maps and codegeneracy maps are defined in the standard way. There is a natural augmentation

$$\varepsilon : F(\mathcal{A}) \to \check{H}^{\cdot}(\mathcal{A}, \mathcal{U}; F)$$

which is induced by

$$(F(j_{\tau_l}^{*})) : F(\mathcal{A}) \to \prod_{l=1}^{p} F(\mathcal{A}_{\tau_l}) .$$

Let $\mathcal{V} = \{\sigma_1, \ldots, \sigma_q\}$ be another cover of $\mathcal{A}$. We call $\mathcal{V}$ a refinement of $\mathcal{U}$ if the all divisors involved are compatible, and if there is a map

$$\varphi : \{1, \ldots, q\} \to \{1, \ldots, p\}$$

such that $\sigma_k \geq \tau_{\varphi(k)}$ for $k = 1, \ldots, q$. Let $\mathcal{V}$ be a refinement of $\mathcal{U}$; there is an augmentation-preserving map $\check{H}^{\cdot}(\mathcal{A}, \mathcal{U}; F) \to \check{H}^{\cdot}(\mathcal{A}, \mathcal{V}; F)$ induced by $\{J_r\}$, where

$$J_r : \prod_{l_0, \ldots, l_r = 1}^{p} F(\mathcal{A}_{\tau_{l_0}\cup\cdots\cup\tau_{l_r}}) \to \prod_{k_0, \ldots, k_r = 1}^{q} F(\mathcal{A}_{\sigma_{k_0}\cup\cdots\cup\sigma_{k_r}})$$

is such that $J_r = ((J_r)_{k_0, \ldots, k_r})$ with $(J_r)_{k_0, \ldots, k_r}$ the composite

$$\prod_{l_0, \ldots, l_r = 1}^{p} F(\mathcal{A}_{\tau_{l_0}\cup\cdots\cup\tau_{l_r}}) \to F(\mathcal{A}_{\tau_{\varphi(k_0)}\cup\cdots\cup\tau_{\varphi(k_r)}}) \to F(\mathcal{A}_{\sigma_{k_0}\cup\cdots\cup\sigma_{k_r}}) .$$

By [13, Section 1.20] (or more accurately the same proof as the one given there), the induced map $\check{H}^{\cdot}(\mathcal{A}, \mathcal{U}; F) \to \check{H}^{\cdot}(\mathcal{A}, \mathcal{V}; F)$ is independent of the choice of the φ up to homotopy, so if $\mathcal{U}$ and $\mathcal{V}$ refine each other, then $\check{H}^{\cdot}(\mathcal{A}, \mathcal{U}; F)$ is homotopy equivalent to $\check{H}^{\cdot}(\mathcal{A}, \mathcal{V}; F)$.

Let $\mathcal{U} = \{\tau_1, \ldots, \tau_p\}$ be a cover of $\mathcal{A}$. If $\{(u_k, H_k)\}$ is another finite set of divisors on $\mathcal{A}$ which are compatible with all the other divisors involved, $\mu = \bigcap_k u_k$, then

$$\mu \cup \mathcal{U} = \{\overline{\tau_1 \cup \mu}, \cdots \overline{\tau_p \cup \mu}\}$$

is a cover of $\mathcal{A}_\mu$.

5.7. Theorem (Mayer–Vietoris). *Let $\{\mathcal{A}, (s_i, F_i)\}$ be an admissible abelian category, $\{(u_k, H_k)\}, \{(t_{j_1}, G_{j_1})\}, \ldots, \{(t_{j_p}, G_{j_p})\}$ be another $p + 1$ finite sets of divisors on $\mathcal{A}$ such that all the divisors involved are compatible, $\mu = \bigcap_k u_k$, $\tau_1 = \bigcap_{j_1} t_{j_1}, \ldots, \tau_p = \bigcap_{j_p} t_{j_p}$, $\mathcal{U} = \{\tau_1, \ldots, \tau_p\}$. If*

$$(\tau_1\text{-Tor}) \cap \cdots \cap (\tau_p\text{-Tor}) = 0 ,$$

then the augmentations are homotopy equivalences:

(i) $K^B(\mathcal{A}) \to \check{H}^{\cdot}(\mathcal{A}, \mathcal{U}; K^B)$

$$= \operatorname{holim}\Big(\prod_{l=1}^{p} K^B(\mathcal{A}_{\tau_l}) \rightrightarrows \prod_{l_0, l_1 = 1}^{p} K^B(\mathcal{A}_{\tau_{l_0}\cup\tau_{l_1}}) \substack{\leftarrow\\\rightarrow\\\rightarrow} \cdots\Big) ,$$

(ii) $$K^{\mathrm{B}}(\mathscr{A} \text{ off } \mathscr{A}_\mu) \to \check{H}^{\bullet}(\mathscr{A}, \mathscr{U}; K^{\mathrm{B}}(\mathscr{A}_{(\ )} \text{ off } \mathscr{A}_{(\ )\cup\mu}))$$
$$= \operatorname{holim}\left(\prod_{l=1}^{p} K^{\mathrm{B}}(\mathscr{A}_{\tau_l} \text{ off } \mathscr{A}_{\tau_l\cup\mu}) \leftrightarrows \cdots\right).$$

Therefore, we have the strongly convergent Mayer–Vietoris spectral sequences

$$E_2^{s,t} = \check{H}^s(\mathscr{A}, \mathscr{U}; K_t^{\mathrm{B}}) = H^s\left(\prod_{l=1}^{p} K_t^{\mathrm{B}}(\mathscr{A}_{\tau_l}) \to \prod_{l_0,l_1=1}^{p} K_t^{\mathrm{B}}(\mathscr{A}_{\tau_{l_0}\cup\tau_{l_1}}) \to \cdots\right)$$
$$\Rightarrow K_{t-s}^{\mathrm{B}}(\mathscr{A}),$$
$$E_2^{s,t} = \check{H}^s(\mathscr{A}, \mathscr{U}; K_t^{\mathrm{B}}(\mathscr{A}_{(\ )} \text{ off } \mathscr{A}_{(\ )\cup\mu}))$$
$$= H^s\left(\prod_{l=1}^{p} K_t^{\mathrm{B}}(\mathscr{A}_{\tau_l} \text{ off } \mathscr{A}_{\tau_l\cup\mu}) \to \cdots\right)$$
$$\Rightarrow K_{t-s}^{\mathrm{B}}(\mathscr{A} \text{ off } \mathscr{A}_\mu).$$

Proof. See [14, 8.2.5 and 8.3]. We need to replace a cover of a scheme by a cover of an admissible abelian category, and replace the intersection of open subschemes by the union of torsion theories, etc. □

6. Applications

6.1. Theorem. *Let R be an arbitrary ring, $t_1, \ldots, t_n$ be n elements in the center of R or if not, there are $\varphi_1, \ldots, \varphi_n \in \mathrm{Aut}(R)$ such that for any $a \in R$, $t_i a = \varphi_i(a) t_i$, $\varphi_j(t_i) = t_i$, $\varphi_i \circ \varphi_j = \varphi_j \circ \varphi_i$ for any $i, j = 1, \ldots, n$. If $Rt_1 + \cdots + Rt_n = R$, then we have a homotopy equivalence*:

$$K^{\mathrm{B}}(R) \to \operatorname{holim}\left(\prod_{j=1}^{n} K^{\mathrm{B}}(R[t_j^{-1}]) \leftrightarrows \prod_{i,j=1}^{n} K^{\mathrm{B}}(R[t_i^{-1}t_j^{-1}]) \mathrel{\substack{\leftarrow\\ \rightarrow\\ \rightarrow}} \cdots\right)$$

and therefore a strongly convergent spectral sequence

$$E_2^{p,q} = H^p\left(\prod_{j=1}^{n} K_q^{\mathrm{B}}(R[t_j^{-1}]) \to \prod_{i,j=1}^{n} K_q^{\mathrm{B}}(R[t_i^{-1}t_j^{-1}]) \to \cdots\right)$$
$$\Rightarrow K_{q-p}^{\mathrm{B}}(R).$$

Proof (cf. Example 1.2.3). $(t_1, \mathrm{Id}), \ldots, (t_n, \mathrm{Id})$ are compatible divisors on R-Mod, and $Rt_1 + \cdots + Rt_n = R$ implies $(t_1\text{-Tor}) \cap \cdots \cap (t_n\text{-Tor}) = 0$, so $\{t_1, \ldots, t_n\}$ becomes a cover of R-Mod, then the corollary follows Theorem 5.7. □

Proposition 6.2. *Let X be a smooth variety over a field k; then the embedding from the structure sheaf $\mathscr{O}_X$ of X to the sheaf $\mathscr{D}_X$ of germs of differential operators on X induces isomorphisms of K-groups of all integers n*:

$$K_n(X) \cong K_n(\mathscr{D}_X) \quad \textit{for all } n.$$

Proof. Since X is a variety over a field k, X has an ample family of line bundles, thus $X = \bigcup_{i=1}^{r} X_{s_i}$, where X_{s_i} is the locus of a global section s_i of some line bundle F_i on X and is affine. Then $(s_i, F_i), \ldots, (s_r, F_r)$ are divisors on the categories $\mathscr{A} = \mathrm{Qcoh}(X)$ and $\mathscr{A}' =$ the category of all sheaves of $\mathscr{D}_X$-modules in $\mathrm{Qcoh}(X)$, and form covers for $\mathscr{A}$ and $\mathscr{A}'$; and the localizations $\mathscr{A}_{s_i} = \Gamma(X_{s_i}, \mathscr{O}_X)$-Mod and $\mathscr{A}'_{s_i} = \Gamma(X_{s_i}, \mathscr{D}_X)$-Mod (cf. Examples 1.1.4 and 1.1.5). From [11, 7.2], we have isomorphisms for all $n \geq 0$, $K(\Gamma(X_{s_i}, \mathscr{O}_X)) \to K_n(\Gamma(X_{s_i}, \mathscr{D}_X))$. Then the proposition follows Theorem 5.7. □

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