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# Localization for the $K$-Theory of Noncommutative Rings

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ABSTRACT. If $S$ is denominator set in a ring $A$, we describe the third term in the long exact sequence relating the $K$-theory of $A$ and $S^{-1}A$. It is the Waldhausen $K$-theory of a category Perf$(A, S)$. If $A \to B$ is an analytic isomorphism along $S$, this third term satisfies excision, yielding a long exact Mayer-Vietoris sequence in $K$-theory.

The recent work **[TT]** of Thomason and Trobaugh establishes a localization theorem for the $K$-theory of commutative rings and quasi-compact quasi-separated schemes. This paper is partly an attempt to give a simple exposition of their proof in the important case $A \to S^{-1}A$, and partly an extension of their proof to the noncommutative case. When $S$ consists of nonzerodivisiors, we recover the calculations of **[GQ]** and **[Gr]**, since in that case our Perf$(A, S)$ has the same $K$-theory as the exact category $\mathcal{H}_S(A)$. We include an excision result which is new even in the commutative case.

To understand the statement of our localization theorem. we introduce some terms. Let $A$ be a ring with unit. A *strictly perfect* complex $P = P^\bullet$ is a bounded chain complex of finitely generated projective left $A$-modules. A chain complex $E = E^\bullet$ of left $A$-modules is a *perfect complex* if there is a strictly perfect complex $P^\bullet$ and a quasi-isomorphism $P \to E$. The category Perf$(A)$ of perfect complexes forms a "Waldhausen" category, i.e., a category with cofibrations (degreewise split monics) and weak equivalences (quasi-isomorphisms). The $K$-theory of $A$ is the same as the Waldhausen $K$-theory of Perf$(A)$. (See 1.1 below.)

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We are interested in localizing $A$ at a multiplicatively closed subset $S$ to form a *left quotient ring* $S^{-1}A$ whose elements have the form $s^{-1}a$ ($s \in S, a \in A$). This exists iff $S$ is a *left denominator* set, i.e., it satisfies the following conditions:

(i) ("Øre condition") $(\forall s \in S, a \in A)(\exists t \in S, b \in A) ta = bs$
(ii) (Annihilators) $(\forall s \in S, a \in A)$ if $as = 0$ then $(\exists t \in S) ta = 0$.

See **[F]**, 16.9. This hypothesis is sufficient to make $S^{-1}A$ flat as a right $A$-module, so that $M \mapsto S^{-1}M = S^{-1}A \otimes_A M$ is an exact functor from $A$-**mod** to $S^{-1}A$-**mod**. We remark that any *central* multiplicatively closed $S$ will automatically be left denominator set.

Suppose that $S$ is a left denominator set in $A$. The category $\mathrm{Perf}(A,S)$ of perfect $A$-module complexes $E$ such that $S^{-1}E$ is exact forms a Waldhausen subcategory of $\mathrm{Perf}(A)$, so it makes sense to talk about its algebraic $K$-theory. We will see below that it has the same $K$-theory as the subcategories $\mathrm{Perf}^{st}(A,S)$, $\mathrm{Perf}^b(A,S)$ and $\mathrm{Perf}^-(A,S)$ of strictly perfect, bounded, and bounded above complexes, respectively.

**Localization Theorem.** Let $S$ be a left denominator set in a ring $A$, for example a central multiplicatively closed set. Then there is a long exact sequence

$$\cdots \to K_{n+1}(S^{-1}A) \xrightarrow{\partial} K_n\mathrm{Perf}(A,S) \to K_n(A) \to K_n(S^{-1}A) \xrightarrow{\partial} \cdots$$

valid for all integers.

**Formula.** Here is an explicit formula for the boundary map $\partial : K_1(S^{-1}A) \to K_0\mathrm{Perf}(A,S)$. If $s \in S$, then $s$ is a unit of $S^{-1}A$ and $\partial(s)$ is represented by the complex

$$0 \to A \xrightarrow{s} A \to 0$$

concentrated in degrees 0 and 1. More generally, every matrix $\beta \in GL_n(S^{-1}A)$ is of the form $s^{-1}\alpha$ for some $s \in S$ and some $\alpha \in M_n(A)$. Then $\partial(\beta) = \partial(\alpha) - n\partial(s)$, where $\partial(\alpha)$ is represented by

$$0 \to A^n \xrightarrow{\alpha} A^n \to 0.$$

**Remark.** We will only construct a sequence ending in $K_0(A) \to K_0(S^{-1}A)$. However, the argument of **[C]** shows that we can define *negative* $K$-groups for $\mathrm{Perf}(A,S)$ and continue the above sequence to negative values of $n$. On a spectrum level, if $\mathcal{K}(A)$ denotes the nonconnective spectrum for the $K$-theory of $A$, the fiber $\mathcal{F}$ of $\mathcal{K}(A) \to \mathcal{K}(S^{-1}A)$ is a nonconnective delooping of the usual connective $K$-theory spectrum for $\mathrm{Perf}(A,S)$.

**Remark.** The proof of the Localization Theorem is much easier if we assume that $S$ is central. The difficulty with the general case is that clearing denominators in commutative diagrams is delicate. (See 3.1.)

**Remark.** It seems probable that the Localization Theorem remains valid if $A \to S^{-1}A$ is replaced by a flat epimorphism $A \to B$. However, the techniques of this paper do not immediately extend to that case. Several variants of the Localization Theorem may be found in the second author's thesis **[Yao]**.



In order to compute with the Localization Theorem, we provide an excision result for analytic isomorphisms along $S$.

**Definition.** (Cf. **[TT]** and **[We]**) Let $S$ be a left denominator set in $A$. We say that a ring map $f : A \to B$ is a (left) *analytic isomorphism* along $S$ if

a) $f(S)$ is a left denominator set in $B$
b) $A/As \xrightarrow{\cong} B/Bs$ for all $s \in S$
c) $\mathrm{Tor}_p^A(B, A/I) = 0$ for $p \neq 0$ and all left ideals $I$ of $A$ meeting $S$.

**Remark.** Condition b) implies that $A/I \cong B/BI$ for every ideal $I$ meeting $S$. If $A$ is commutative, then c) is implied by the condition of **[TT]** that $B_P$ is flat over $A_P$ for all primes $P$ of $A$ meeting $S$. If $S$ consists of central nonzerodivisors in $A$, we will show in 5.5(b) below that c) is equivalent to the assertion that $f(S)$ consists of right (hence left) nonzerodivisors in $B$. Thus our notion includes the notion of analytic isomorphism used in **[K]** and **[We]**. The term "analytic isomorphism" comes from the fact that the $S$-adic completions $\hat{A} = \varprojlim A/As$ and $\hat{B} = \varprojlim B/Bs$ are isomorphic.

**Excision Theorem.** Let $A \to B$ be an analytic isomorphism along $S$. Then the total tensor product map

$$B \otimes_A^{\mathbf{L}} - : K\,\mathrm{Perf}^-(A,S) \to K\,\mathrm{Perf}^-(B,S)$$

is a homotopy equivalence of spectra. Consequently, there is a long exact Mayer-Vietoris sequence (for all integers $n$):

$$\cdots \to K_{n+1}(S^{-1}B) \xrightarrow{\partial} K_n(A) \to K_n(B) \oplus K_n(S^{-1}A) \to K_n(S^{-1}B) \xrightarrow{\partial} \cdots.$$

**Remark.** Our proof follows the proof of **[TT, 3.19]**. If $S$ is a central set of nonzerodivisors on $A$ and $B$, this result was proven by Karoubi **[K]** by showing that $\mathcal{H}_S(A) \approx \mathcal{H}_S(B)$. (See **[We, 1.1]**).

## §1. The proof of the Localization Theorem.

The $K$-theory of $A$ is the $K$-theory of the category $\mathcal{P}(A)$ of fin. gen. projective left modules; either Quillen $K$-theory or Waldhausen $K$-theory may be used by **[Wa, 1.9]**. In order to compare the $K$-theory of $A$ to that of $\mathrm{Perf}(A,S)$, we invoke the following result.

LEMMA 1.1. *(Waldhausen) The following subcategories of* $\mathrm{Perf}(A)$ *have the same K-theory:*

a) $\mathcal{P}(A)$, *the K-theory of A*
b) $\mathrm{Perf}^{st}(A)$, *the strictly perfect complexes*
c) $\mathrm{Perf}^b(A)$, *the bounded perfect complexes*
d) $\mathrm{Perf}^-(A)$, *the bounded above perfect complexes*
e) $\mathrm{Perf}(A)$, *all perfect complexes.*

*In all cases, cofibrations are degreewise split monics, and the weak equivalences* **w** *are the quasi-isomorphisms.*

PROOF. By **[Gi, 6.2]**, the categories a) and b) have the same $K$-theory. The inclusion of strictly perfect complexes in either bounded or bounded above perfect complexes satisfies the approximation property (App) of **[Wa, 1.6.7]** by a standard exercise (see **[SGA6, I.2.7.1]** or **[TT, 1.9.5]**). The inclusion of category d) in category e) satisfies the dual approximation property ($\mathrm{App}^{op}$) since we can truncate complexes. By the Approximation Theorem **[Wa, 1.6.7]**, these categories have the same $K$-theory. □

PORISM 1.2. *It follows from the proof of 1.1 that the following subcategories of* $\mathrm{Perf}(A,S)$ *have the same* $K$*-theory:*

a) $\mathrm{Perf}^{st}(A,S) = \mathrm{Perf}^{st}(A) \cap \mathrm{Perf}(A,S)$
b) $\mathrm{Perf}^{b}(A,S) = \mathrm{Perf}^{b}(A) \cap \mathrm{Perf}(A,S)$
c) $\mathrm{Perf}^{-}(A,S) = \mathrm{Perf}^{-}(A) \cap \mathrm{Perf}(A,S)$
d) $\mathrm{Perf}(A,S)$

*In all cases, cofibrations are degreewise split monics, and the weak equivalences* **w** *are the quasi-isomorhisms.*

In order to prove the Localization Theorem, we introduce a new notion of weak equivalence on the category with cofibrations $\mathrm{Perf}(A)$. We let **v** denote the class of all maps $E \to F$ such that $S^{-1}E \to S^{-1}F$ is a quasi-isomorphism. The subcategory of perfect complexes **v**-equivalent to zero is $\mathrm{Perf}(A,S)$, so Waldhausen's Fibration Theorem **[Wa, 1.6.4]** states that there is a homotopy fibration of spectra (yielding a long exact sequence on $K$-groups):

(1.3) $$K(\mathrm{Perf}(A,S)) \to K(\mathrm{Perf}(A),\mathbf{w}) \to K(\mathrm{Perf}(A),\mathbf{v}).$$

The middle term gives the $K$-theory of $A$, so it suffices to compare the right term to the $K$-theory of $\mathcal{B}$.

Since $S^{-1}A$ is flat over $A$, localization provides an exact functor from $\mathrm{Perf}(A)$ to $\mathrm{Perf}(S^{-1}A)$. This functor not only factors through the change in weak equivalence (from **w** to **v**), but it also factors through a category $\mathcal{B}$, which we now define.

**Definition 1.4.** Let $\mathcal{B}$ denote the full subcategory of $\mathrm{Perf}(S^{-1}A)$ consisting of those perfect $S^{-1}A$-module complexes $E^\bullet$ such that the class $[E^\bullet]$ in $K_0(S^{-1}A)$ is in the image of the map $K_0(A) \to K_0(S^{-1}A)$. We make $\mathcal{B}$ into a category with cofibrations (degreewise split monics) and weak equivalences (quasi-isomorphisms).

Thomason's version of the Cofinality Theorem for Waldhausen $K$-theory **[TT, 1.10.1]** applies to the inclusion of $\mathcal{B}$ in $\mathrm{Perf}(S^{-1}A)$, proving that $K_n(\mathcal{B}) \to K_n(S^{-1}A)$ is an isomorphism for $n \geq 1$, and that $K_0(\mathcal{B})$ is the image of $K_0(A) \to K_0(S^{-1}A)$. Therefore in order to prove the Localization Theorem it suffices to



establish the following assertion:

(1.5) $$K(\mathrm{Perf}(A),\mathbf{v}) \to K(\mathcal{B}) \text{ is a homotopy equivalence.}$$

This will be a consequence of the Thomason-Trobaugh Approximation Theorem **[TT, 1.9.8]**, once we prove (in section 3 below) that the map of derived categories

$$T : \mathbf{v}^{-1}\,\mathrm{Perf}(A) \to \mathbf{w}^{-1}\mathcal{B}$$

is an equivalence. The first step is to show that every complex $E^\bullet$ in $\mathcal{B}$ is quasi-isomorphic to $S^{-1}P^\bullet$ for some perfect $A$-module complex $P^\bullet$, i.e., that every object of $\mathbf{w}^{-1}\mathcal{B}$ is isomorphic to $T(P^\bullet)$ for some $P^\bullet$ in $\mathbf{v}^{-1}\,\mathrm{Perf}(A)$. This is the subject of the next section.

## §2. An Extension Criterion.

In this section we shall assume that $S$ is either central or a left denominator set in $A$. The following Exercise is trivial when $S$ is central. When $S$ is a left denominator set, it uses the fact that any finite subset $\{b_i\}$ of $S^{-1}A$ has a common denominator, i.e., is of the form $\{t^{-1}a_i\}$.

**Exercise 2.1.** Fix a left denominator set $S$ in $A$, and let $E^\bullet$ be a bounded chain complex of fin. gen. free left $S^{-1}A$-modules. Then there is a bounded complex $F^\bullet$ of fin. gen. free $A$-modules and an isomorphism $f : S^{-1}F^\bullet \to E^\bullet$ of $S^{-1}A$-module complexes. Moreover, if we use a choice of basis to represent the $f^i$ by matrices and assume that $E^i = 0$ for $i > n$, then $f^n = 1$ and every other $f^i$ is right multiplication by an element of $S$.

COROLLARY 2.2. *If* $P$ *is a fin. gen. projective* $A$*-module and*

$$0 \to E^m \to E^{m+1} \to \cdots \to E^{n-1} \to E^n \to 0$$

*is an* $S^{-1}A$*-module complex with* $E^i$ *fin. gen. free for* $i \neq n$ *and* $E^n \cong S^{-1}P$*, then there is a bounded chain complex* $P^\bullet$ *of fin. gen. projective* $A$*-modules with* $P^n = P$ *and an isomorphism* $f : S^{-1}P^\bullet \to E^\bullet$ *of* $S^{-1}A$*-module complexes.*

PROOF. Choose $Q$ so that $P \oplus Q$ is fin. gen. free and apply 2.1 to $E^\bullet \oplus (S^{-1}Q(n))$ to get a free complex $F^\bullet$ with $F^n = P \oplus Q$ and an isomorphism $S^{-1}F^\bullet \cong E^\bullet \oplus (S^{-1}Q(n))$ in which

$$S^{-1}F^n \cong (S^{-1}P) \oplus (S^{-1}Q) \cong E^n \oplus (S^{-1}Q)$$

is the canonical map. Now set $P^\bullet = F^\bullet/Q(n)$. □

COROLLARY 2.3. *If* $E^\bullet$ *is a strictly perfect* $S^{-1}A$*-module complex, then there is a bounded complex* $F^\bullet$ *of fin. gen. free* $A$*-modules, an* $S^{-1}A$*-module complex* $D^\bullet$ *and an isomorphism* $S^{-1}F^\bullet \cong D^\bullet \oplus E^\bullet$ *of* $S^{-1}A$*-module complexes.*

PROOF. Each $E^i$ is a fin. gen. projective $S^{-1}A$-module, so there are $S^{-1}A$-modules $D^i$ with $D^i \oplus E^i$ fin. gen. free. Assemble the $D^i$ into a complex (e.g., by 0 maps) and apply 2.1. □

**Remark 2.3.1.** This is the elementary analogue of [**TT, 5.5.1**]. Thomason and Trobaugh need to work harder, invoking the derived category of $S^{-1}A$, because of their geometric context. In *loc. cit.*, they state that "despite the flagrant triviality of the proof, this result is the key point in [**TT**]."

**Extension Criterion 2.4.** The following assertions are equivalent for every perfect $S^{-1}A$-module complex $E^\bullet$:

(i) $E^\bullet$ is quasi-isomorphic to $S^{-1}P^\bullet$ for some perfect $A$-module complex $P^\bullet$

(ii) The class $[E^\bullet] \in K_0(S^{-1}A)$ is in the image of $K_0(A) \to K_0(S^{-1}A)$.

PROOF. That (i) implies (ii) is clear. For the converse, we may suppose that $E^\bullet$ is strictly perfect, so $[E^\bullet] = \sum(-1)^i[E^i]$. By adding short complexes of the form $0 \to D^i = D^{i+1} \to 0$, we may assume every $E^i$ is free except $E^n$, that $E^i = 0$ for $i > n$, and that $[E^n] = [S^{-1}P]$ for some projective $A$-module $P$. Hence $E^n$ and $S^{-1}P$ are stably isomorphic $S^{-1}A$-modules, i.e., $E^n \oplus (S^{-1}A)^r \cong S^{-1}(P \oplus A^r)$ for some $r$. Adding $(S^{-1}A)^r$ in dimensions $n-1$ and $n$, we may assume that in fact $E^n \cong S^{-1}P$. Now apply 2.2 to obtain (i). □

## §3. Equivalence of Derived Categories.

If $\mathbf{w}$ is a class of maps in a skeletally small additive category $\mathcal{C}$, there is an additive category $\mathbf{w}^{-1}\mathcal{C}$ and a functor $Q : \mathcal{C} \to \mathbf{w}^{-1}\mathcal{C}$ sending $\mathbf{w}$ to isomorphisms which is universal in this respect. If $\mathbf{w}$ is a multiplicative system [**H, I.3**], this is an especially nice construction, since $\mathbf{w}^{-1}\mathcal{C}$ has the same objects as $\mathcal{C}$ and every morphism is represented by a diagram in $\mathcal{C}$ of the form

$$E \xleftarrow{w} E' \xrightarrow{\alpha} F.$$

This follows from the calculus of fractions [**V, 2.3.2**] [**H, 3.1**]

THEOREM 3.1. *Let $\mathcal{B} \subseteq$ Perf$(S^{-1}A)$ be as in (1.4), with $\mathbf{w}$ being the quasi-isomorphisms. Let $\mathbf{v}$ be the class of maps $v : E \to F$ in* Perf$(A)$ *such that $S^{-1}v : S^{-1}E \to S^{-1}F$ is a quasi-isomorphism. Then*

$$T : \mathbf{v}^{-1}\,\mathrm{Perf}(A) \to \mathbf{w}^{-1}\mathcal{B}$$

*is an equivalence of categories.*

**Reduction.** The Extension Criterion 2.4 shows that every object of $\mathcal{B}$, hence of $\mathbf{w}^{-1}\mathcal{B}$, comes from an object of Perf$(A)$. Therefore, it is enough to show that the functor $T$ is full and faithful. The following argument, copied from [**TT, 5.2.6**], shows that it is enough to prove that $T$ is full, for this implies that $T$ is also faithful. Since $\mathbf{v}$ is a multiplicative system, every map in $\mathbf{v}^{-1}\,$Perf$(A)$ is represented as $E \xleftarrow{v'} E' \xrightarrow{\alpha} F$ with $v'$ in $\mathbf{v}$. Suppose that $T$ sends this map (or equivalently, $\alpha$) to zero in $\mathbf{w}^{-1}\mathcal{B} \subseteq \mathbf{w}^{-1}\,$Perf$(S^{-1}A) \subseteq D(S^{-1}A)$. Let $C$ be the mapping cone of $\alpha$, so that

$$C(1) \xrightarrow{\pi} E' \xrightarrow{\alpha} F \xrightarrow{\delta} C$$



forms a distinguished triangle of perfect $A$-module complexes. Since Hom$(E, -)$ is a cohomological functor [**V, 1.2**] [**H, 1.1**] we have a diagram of abelian groups with exact rows:

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathbf{v}}(E, C(1)) & \xrightarrow{\pi} & \mathrm{Hom}_{\mathbf{v}}(E, E') & \xrightarrow{\alpha} & \mathrm{Hom}_{\mathbf{v}}(E, F) \\
\downarrow T & & \downarrow T & & \downarrow \\
\mathrm{Hom}_{\mathbf{w}}(S^{-1}E, S^{-1}C(1)) & \xrightarrow{\pi} & \mathrm{Hom}_{\mathbf{w}}(S^{-1}E, S^{-1}E') & \xrightarrow{0} & \mathrm{Hom}_{\mathbf{w}}(S^{-1}E, S^{-1}F)
\end{array}$$

For clarity, we have written Hom$_{\mathbf{v}}$ (resp. Hom$_{\mathbf{w}}$) for Hom in the triangulated category $\mathbf{v}^{-1}\,$Perf$(A)$ (resp. $\mathbf{w}^{-1}\mathcal{B}$). Since we have assumed that $T$ is full, the vertical maps are onto. Hence there is a map $E \xleftarrow{v} E'' \xrightarrow{\sigma} C(1)$ in $\mathbf{v}^{-1}\,$Perf$(A)$ such that $T(v^{-1}\sigma\pi)$ is the isomorphism $v'^{-1}$ in $\mathbf{w}^{-1}\mathcal{B}$. But then $S^{-1}(\sigma\pi)$ is a quasi-isomorphism. By definition of $\mathbf{v}$, $\sigma\pi$ is an isomorphism in $\mathbf{v}^{-1}\,$Perf$(A)$. Since $\sigma\pi\alpha = 0$, this forces $\alpha$ to be zero in $\mathbf{v}^{-1}\,$Perf$(A)$, proving that $T$ is faithful.

PROOF THAT $T$ IS FULL. Note that $T$ is an additive functor between additive categories. As every strictly perfect complex is a direct summand of a bounded f.g. free complex, we are reduced to showing that if $E$ and $F$ are bounded complexes of fin. gen. free $A$-modules, then

$$T : \mathrm{Hom}_{\mathbf{v}^{-1}\mathrm{Perf}(A)}(E, F) \to \mathrm{Hom}_{\mathbf{w}^{-1}\mathcal{B}}(S^{-1}E, S^{-1}F)$$

is onto. By [**SGA6, I.2.7**] and [**V, I.2.4.2**], every map in $\mathbf{w}^{-1}\mathcal{B}$ from $S^{-1}E$ to $S^{-1}F$ is represented by a chain map $\beta : S^{-1}E \to S^{-1}F$. Clearing denominators, we can choose $A$-module maps $\alpha^n : E^n \to F^n$ and $s \in S$ so that $\beta^n = s^{-1}\alpha^n$ for all $n$. As a warmup we consider the easy case first.

**Easy case: $S$ is central.** Because over $S^{-1}A$ we have

$$(\alpha^{n-1}d_F - d_E\alpha^n) = s(\beta^{n-1}d_F - d_E\beta^n) = 0,$$

some $t \in S$ annihilates $\alpha^{n-1}d_F - d_E\alpha^n$. Replacing $s$ by $ts$ and $\alpha^n$ by $t\alpha^n$, we have arranged that the $\{\alpha^n\}$ assemble to form a chain map $\alpha : E \to F$. Multiplication by $s$ is a chain map $E \to E$ lying in $\mathbf{v}$, and evidently the map

$$E \xleftarrow{s} E \xrightarrow{\alpha} F$$

in $\mathbf{v}^{-1}\,$Perf$(A)$ maps to $\beta$. We are done in this case.

**General case.** When $S$ is not central, multiplication by $s$ may not be a chain map. Since $E^n = 0$ for $n > N$, we may use the following lemma, together with descending induction on $n$, to see that by changing our choice of the $\alpha^n$ (and $s_n \in S$ so that $\beta^n = s_n^{-1}\alpha^n$) we can find a new chain complex

$$E' : \cdots \to E^{n-1} \xrightarrow{e^{n-1}} E^n \xrightarrow{e^n} \cdots \to E^N \to 0$$

and a diagram of chain maps

$$E \xleftarrow{\{s_n\}} E' \xrightarrow{\{\alpha^n\}} F.$$

Since $\{s_n\}$ is in $\mathbf{v}$, this represents a map in $\mathbf{v}^{-1}\operatorname{Perf}(A)$ lifting $\beta$. This will finish the proof.

LEMMA. *Suppose we are given $s_n \in S$ and $\alpha^n : E^n \to F$ so that $\beta^n = s_n^{-1}\alpha^n$, and a map $e^n : E^n \to E^{n+1}$ such that $s_n d_E = e^n s_{n+1}$. Then there is a map $e : E^{n-1} \to E^n$, an $s_{n-1} \in S$ and an $\alpha : E^{n-1} \to F^{n-1}$ so that $ee^n = 0$, $\beta^{n-1} = s_{n-1}^{-1}\alpha$ and such that the following diagram commutes:*

$$\begin{array}{ccccc} E^{n-1} & \xleftarrow{s_{n-1}} & E^{n-1} & \xrightarrow{\alpha} & F^{n-1} \\ {\scriptstyle d_E}\downarrow & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle d_F} \\ E^n & \xleftarrow{s_n} & E^n & \xrightarrow{\alpha^n} & F^n. \end{array}$$

PROOF. Recall that $\beta^{n-1} = s^{-1}\alpha^{n-1}$ is given. Choose $s' \in S$ and $e' : E^{n-1} \to E^n$ so that $e's_n = s'd_E$. Then choose $t \in S$ and $a \in A$ so that $as' = ts$. Set $e'' = ae'$ and $\alpha'' = t\alpha^{n-1}$, so that

$$(e''e^n)s_{n+1} = (ae')(s_n d_E) = a(s'd_E)d_E = 0$$

and over $S^{-1}A$ we have

$$\begin{aligned}(\alpha'' d_F - e''\alpha^n) &= t(s\beta^{n-1})d_F - (ae')(s_n\beta^n) \\ &= as'(\beta^{n-1}d_F - d_E\beta^n) \\ &= 0.\end{aligned}$$

Therefore there is an $s'' \in S$ so that $s''e''e^n = 0$ and

$$s''(\alpha'' d_F - e''\alpha^n) = 0.$$

Set $e = s''e''$, $s_{n-1} = s''as' = s''ts$ and $\alpha = s''\alpha''$. □

## §4. Criteria for perfectness and pseudocoherence.

The following two results are straightforward modifications of results in [**TT, 2.4**]. We need them for the excision result in the next section.

Recall from [**SGA6, I.2**] that an $A$-module chain complex $P^\bullet$ is said to be *strictly pseudo-coherent* if it is a bounded above complex of fin. gen. projective $A$-modules. A complex $E^\bullet$ is said to be *pseudo-coherent* if there is a quasi-isomorphism $P^\bullet \to E^\bullet$ with $P^\bullet$ strictly pseudo-coherent. Recall also that $\tau^n E$ is the good truncation

$$\cdots \to 0 \to d(E^{n-1}) \to E^n \to E^{n+1} \to \cdots.$$

THEOREM 4.1. ([**TT, 2.4.2**]) *Let $E$ be an $A$-module chain complex. The following are equivalent:*

a) *$E$ is pseudo-coherent*
b) *For all integers $n$ and $k$, and all directed systems $\{F_\alpha\}$ of $A$-module complexes, the canonical map (4.1.1) is an isomorphism.*

$$(4.1.1)\qquad \varinjlim_\alpha H^k(\mathbf{R}\operatorname{Hom}(E, \tau^n F_\alpha)) \xrightarrow{\cong} H^k(\mathbf{R}\operatorname{Hom}(E, \varinjlim_\alpha \tau^n F_\alpha))$$



c) *Same as b) except we require the $F_\alpha$ to be strictly perfect*
d) *Same as c) except we require the $F_\alpha$ to be uniformly bounded above, and we require $E$ to be cohomologically bounded above.*
e) *For all integers $n$, and all directed systems $\{F_\alpha\}$ of $A$-module complexes, the canonical map (4.1.2) is an isomorphism.*

$$(4.1.2)\qquad \varinjlim_\alpha \operatorname{Hom}_{D(A)}(E, \tau^n F_\alpha) \xrightarrow{\cong} \operatorname{Hom}_{D(A)}(E, \varinjlim_\alpha \tau^n F_\alpha)$$

f) *Same as e) except we require the $F_\alpha$ to be strictly perfect*
g) *Same as f) except we require the $F_\alpha$ to be uniformly bounded above, and we require $E$ to be cohomologically bounded above.*

PROOF. We merely note the changes that are needed to modify the proof of [**TT, 2.4.2**]. Note that the meaning of "perfect" is slightly different in *op. cit.* In the proof that b) ⇒ e) we cite [**H, I.6.4**] instead of [**TT, 2.4.1**] to see that

$$H^\circ \mathbf{R}\operatorname{Hom}(E, \tau^n F) = \operatorname{Hom}_{D(A)}(E, \tau^n F).$$

In the proof that g) ⇒ a) we cite [**SGA6, I.2.12 and I.2.7**] instead of [**TT, 2.2.13**] to see that if $E \oplus E'$ is $n$-pseudo-coherent then so is $E$, and if $E$ is $n$-pseudo-coherent for all $n$ then $E$ is pseudo-coherent. □

THEOREM 4.2. ([**TT, 2.4.3**]) *Let $E$ be an $A$-module chain complex. The following are equivalent:*

a) *$E$ is perfect*
b) *$E$ is cohomologically bounded below, and for any directed system $\{F_\alpha\}$ of $A$-module complexes, the canonical map (4.2.1) is an isomorphism.*

$$(4.2.1)\qquad \varinjlim_\alpha \operatorname{Hom}_{D(A)}(E, F_\alpha) \xrightarrow{\cong} \operatorname{Hom}_{D(A)}(E, \varinjlim_\alpha F_\alpha)$$

c) *$E$ is cohomologically bounded, and (4.2.1) is an isomorphism for any directed system $\{F_\alpha\}$ of strictly perfect complexes which is uniformly cohomologically bounded above.*

PROOF. We merely note the changes needed for the proof of [**TT, 2.4.3**] to go through. For a) ⇒ b) we cite [**H, I.6.4**] instead of [**TT, 2.4.1**], as above. For c) ⇒ a), the proof in [**TT**] shows that some $E \oplus E'$ is isomorphic in $D(A)$ to a strictly perfect complex. As in the proof of 4.1 above, this implies that $E$ is pseudo-coherent. By [**SGA6, I.5.8.1**] $E$ is perfect. □

## §5. Excision.

In this section, we assume that $f : A \to B$ is an analytic isomorphism along a left denominator set $S$ in $A$. In order to compare $\operatorname{Perf}(A, S)$ and $\operatorname{Perf}(B, S)$, we first compare the derived categories of $A$ and $B$.

Recall the construction of the total tensor product

$$\mathbf{L}f^* = B \otimes_A^{\mathbf{L}} - : D^-(A) \to D^-(B).$$

Any bounded above $A$-module complex $E$ has a quasi-isomorphism $P \to E$ with a bounded above projective complex $P$, and $\mathbf{L}f^*(E)$ is $B \otimes_A P$. The choice of $P$ may be made functorial—use the total complex of the canonical free resolution—and therefore defines a functor from bounded above $A$-module complexes to bounded above $B$-module complexes. Restricting still further, but retaining the notation, we get functors

$$\mathbf{L}f^* : \mathrm{Perf}^-(A) \to \mathrm{Perf}^-(B)$$
$$\mathbf{L}f^* : \mathrm{Perf}^-(A,S) \to \mathrm{Perf}^-(B,S).$$

The former induces the map $f^* : K(A) \to K(B)$. The latter map is the focal point of this section: we shall prove that it induces an isomorphism on $K$-theory.

PROPOSITION 5.1. *Let $f : A \to B$ be an analytic isomorphism along $S$.*

a) *If $E$ is a bounded above complex of $A$-modules such that $S^{-1}E$ is exact, then the canonical map $E \to B \otimes_A^{\mathbf{L}} E$ is an isomorphism in the derived category $D^-(A)$.*
b) *If $F$ is a bounded above complex of $B$-modules such that $S^{-1}F$ is exact, then the canonical map (obtained by thinking of $F$ as an $A$-module complex)*
$$B \otimes_A^{\mathbf{L}} F \to F$$
*is an isomorphism in the derived category $D^-(B)$.*

PROOF. (Cf. [**TT, 2.6.3 (a,b)**]) For purposes of checking we may assume that $E$ and $F$ are also bounded below. The usual devissage argument (see *op. cit.*) now reduces to the case in which $E$ and $F$ are concentrated in one degree, i.e., $S$-torsion (left) modules. Since for every $A$-module $M$ we have

$$H_*(B \otimes_A^{\mathbf{L}} M) = \mathrm{Tor}_*^A(B,M),$$

we are done by the following lemma. □

LEMMA 5.2. *Let $A \to B$ be an analytic ismorphism along $S$. Then for every $S$-torsion left $A$-module $M$ we have $M \cong B \otimes_A M$ and $Tor_p^A(B,M) = 0$ for $p \neq 0$.*

PROOF. If $M \cong A/I$ then $S$ meets $I$ and we are done by the definition of analytic isomorphism. An induction on the number of generators of $M$ proves this result if $M$ is finitely generated. As every $M$ is the union of its fin. gen. submodules, and Tor commutes with filtered colimits, the result holds for infinitely generated $M$ as well. □

PROPOSITION 5.3. *If $E$ is a perfect $B$-module complex with $S^{-1}E$ exact, then $E$ is also perfect as an $A$-module complex.*

PROOF. (Cf. [**TT, 2.6.3 (d)**].) By truncating, we may assume that $E$ is bounded above. We appeal to criterion 4.2 (c) to see that $E$ is perfect, so let $\{F_\alpha\}$ be a uniformly bounded above directed system of strictly perfect $A$-module



complexes. Let $\Gamma_S F$ denote the mapping cone of $F \to S^{-1}F$, translated by $+1$, so that $S^{-1}(\Gamma_S F)$ will be exact, and the natural map

$$\mathrm{Hom}_{D(A)}(E, \Gamma_S F) \to \mathrm{Hom}_{D(A)}(E,F)$$

is an isomorphism. (To see this, use the long exact Hom sequence and note that $\mathrm{Hom}(E, S^{-1}F)$ is trivial because $S^{-1}E \cong 0$ in $D(A)$.) Using (5.1) and the adjointness property [**V, 2.3.3**], we see that if $E$ and $F$ are bounded above

$$\begin{aligned}\mathrm{Hom}_{D(A)}(E,\Gamma_S F) &\cong \mathrm{Hom}_{D(A)}(E, B \otimes_A^{\mathbf{L}} \Gamma_S F)\\ &\cong \mathrm{Hom}_{D(B)}(B \otimes_A^{\mathbf{L}} E, B \otimes_A^{\mathbf{L}} \Gamma_S F)\\ &\cong \mathrm{Hom}_{D(B)}(E, B \otimes_A^{\mathbf{L}} \Gamma_S F).\end{aligned}$$

Now set $F = \varinjlim_\alpha F_\alpha$, and note that $\varinjlim_\alpha(\Gamma_S F_\alpha) \cong \Gamma_S F$.

Using criterion 4.2 (c) in $D(B)$, we therefore have

$$\begin{aligned}\varinjlim_\alpha \mathrm{Hom}_{D(A)}(E,\Gamma_S F_\alpha) &\cong \varinjlim_\alpha \mathrm{Hom}_{D(B)}(E, B \otimes_A^{\mathbf{L}} \Gamma_S F_\alpha)\\ &\cong \mathrm{Hom}_{D(B)}(E, \varinjlim_\alpha B \otimes_A^{\mathbf{L}} \Gamma_S F_\alpha)\\ &\cong \mathrm{Hom}_{D(B)}(E, B \otimes_A^{\mathbf{L}} \Gamma_S F) \cong \mathrm{Hom}_{D(A)}(E, \Gamma_S F).\end{aligned}$$

Hence

$$\varinjlim_\alpha \mathrm{Hom}_{D(A)}(E, F_\alpha) \cong \mathrm{Hom}_{D(A)}(E,F).$$

Using 4.2 (c), this proves that $E$ is perfect in $D(A)$. □

COROLLARY 5.4. *The forgetful functor from $B$-modules to $A$-modules induces a functor $u : \mathrm{Perf}(B,S) \to \mathrm{Perf}(A,S)$ and an equivalence of derived categories*

$$\mathbf{w}^{-1}\,\mathrm{Perf}(B,S) \approx \mathbf{w}^{-1}\,\mathrm{Perf}(A,S),$$

*whose inverse is the total tensor product $B \otimes_A^{\mathbf{L}} -$.*

PROOF OF EXCISION THEOREM. By 5.1 and [**Wa, 1.3.1**] the compositions

$$\mathrm{Perf}^-(A,S) \xrightarrow{\mathbf{L}f^*} \mathrm{Perf}^-(B,S) \xrightarrow{u} \mathrm{Perf}^-(A,S)$$
$$\mathrm{Perf}^-(B,S) \xrightarrow{u} \mathrm{Perf}^-(A,S) \xrightarrow{\mathbf{L}f^*} \mathrm{Perf}^-(B,S)$$

induce maps on $K$-theory which are homotopy equivalent to the identity. The existence of the Mayer-Vietoris sequence is a formal consequence of the homotopy equivalence

$$K\,\mathrm{Perf}^-(A,S) \xrightarrow{\sim} K\,\mathrm{Perf}^-(B,S),$$

given the Localization Theorem. (See, e.g., [**We, 1.2**]). □

We conclude with the following promised result, that our notion of analytic isomorphism generalizes both the notion of "isomorphism infinitely near $Y$" of [**TT**] and the notion of analytic isomorphism used in [**K**] and [**We**].

LEMMA 5.5. *Let $S$ be central in $A$ and $f : A \to B$ a map such that $A/As \cong B/Bs$ for all $s \in S$, and $f(S)$ is a left denominator set in $B$.*

a) *$f$ is an analytic isomorphism iff $\mathrm{Tor}_p^A(B, A/As) = 0$ for $p \neq 0$ and all $s \in S$*

b) *If $S$ consists of nonzerodivisors in $A$, then $f$ is an analytic isomorphism iff $S$ consists of right nonzerodivisors in $B$.*

PROOF. If $s$ is a nonzerodivisor on $A$, then $\mathrm{Tor}_p^A(B, A/As) = 0$ for $p > 1$ and $\mathrm{Tor}_1^A(B, A/As) \cong \{b \in B : bs = 0\}$. Therefore a) implies b). To see a), let $I$ be a left ideal of $A$ containing $s \in S$ and set $J = As$. As $A/I \cong A/J \otimes_A A/I$ there is a spectral sequence

$$E^2_{pq} = \mathrm{Tor}_p^{A/J}(\mathrm{Tor}_q^A(B, A/J), A/I) \Rightarrow \mathrm{Tor}_{p+q}^A(B, A/I).$$

If $\mathrm{Tor}_q^A(B, A/As) = 0$ for $q \neq 0$ and $B/Bs \cong A/As$, the spectral sequence collapses to give

$$\mathrm{Tor}_p^A(B, A/I) \cong \mathrm{Tor}_p^{A/J}(A/J, A/I).$$

This vanishes for $p \neq 0$, proving (a). □

**Remark.** The proof goes through if, instead of assuming $S$ central, we assume that $As$ is a 2-sided ideal of $A$ for all $s \in S$.

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