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Floer homology and splittings of manifolds

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Dedicated to Professor Akio Hattori on his sixtieth birthday

1. Introduction

In [F], Floer defined a mod 8 graded homology group $I_*(M)$ for an oriented integral homology 3-sphere M. It is an invariant of the differentiable structure of M. Roughly speaking $I_*(M)$ is a homology group coming from the Morse theory of the Chern-Simons functional f on the infinite-dimensional manifold of all the gauge equivalence classes of irreducible connections on the principal SU(2)-bundle $M \times \mathrm{SU}(2)$. The critical point set of f is the set of the gauge equivalence classes of irreducible flat connections. In general, these critical sets may be degenerate and f not be a genuine Morse functional; a suitable perturbation of f is needed to define $I_*(M)$. In this paper we consider only non-degenerate critical points of f. This is partly because we want to avoid some inessential technical complications and partly because it is sufficient for our computations of $I_*(N_k)$ for homology 3-spheres N_k obtained by Dehn surgery along the figure eight knot in S^3.

For a smooth connection A on $M \times \mathrm{SU}(2)$, a self-adjoint Fredholm operator D_A is defined (Section 2). The gauge equivalence class $[A]$ of an irreducible flat connection A is a non-degenerate critical point of f if and only if $\operatorname{Ker} D_A = 0$. In this case $[A]$ determines a generator of the mod 8 graded chain group of $I_*(M)$. The mod 8 degree $d([A])$ is related to the spectral flow invariant (Section 2) as follows. Let A_0 and A_1 be two irreducible flat connections on $M \times \mathrm{SU}(2)$ such that $\operatorname{Ker} D_{A_0} = 0 = \operatorname{Ker} D_{A_1}$. Let $\{A_t\}$ be a smooth path of smooth connections on $M \times \mathrm{SU}(2)$ connecting A_0 and A_1. Then $d([A_1]) - d([A_0])$ is the mod 8 reduction of the spectral flow of the path of the self-adjoint Fredholm operators $\{D_{A_t}\}$.

The object of this paper is to give a practical method of calculation of the above spectral flow when M is split as $M = M_1 \cup M_2$, where $M_1 \cap M_2 = \partial M_1 = \partial M_2 = \Sigma$ and Σ is an orientable surface of genus g (≥ 2). Such a decomposition in 3-dimensional gauge theory was first considered by C. H.

Taubes in [T], where he proved that the Casson invariant of an oriented homology 3-sphere is equal to half of the Euler number of its Floer homology group. In this paper, we consider essentially the same differential operators on M_i ($i = 1, 2$) as in [T] but with different boundary conditions.

In Sections 3 and 4, we formulate some boundary value problems on 3-dimensional compact Riemannian manifolds with boundary. In Section 5, we give simple perturbations of metrics and connections which are necessary to make the boundary problems work well for our purpose. Using these data, in Section 6, we define an invariant $\gamma(\{A_t\})$ for a generic smooth path of smooth connections on $M \times \mathrm{SU}(2)$, here the terminology generic will be clarified in Sections 5 and 6 (Definitions 5.1 and 6.1). $\gamma(\{A_t\})$ is a homotopy invariant of a path derived from $\{A_t\}$ of the space of all the Lagrangian pairs in a $(6g - 6)$-dimensional symplectic vector space.

The following is our main theorem. In the theorem, M is not necessarily a homology 3-sphere.

THEOREM 1.1. *Let M be an oriented connected closed 3-manifold. Let M_1 and M_2 be codimension-0 submanifolds of M such that $M = M_1 \cup M_2$ and $M_1 \cap M_2 = \partial M_1 = \partial M_2 = \Sigma$ is a connected closed surface of genus g (≥ 2) oriented as the boundary of M_1. Let A_0 and A_1 be smooth irreducible flat connections on $M \times \mathrm{SU}(2)$ with $\operatorname{Ker} D_{A_0} = 0 = \operatorname{Ker} D_{A_1}$ such that A_i restricts to an irreducible flat connection B_i on $\Sigma \times \mathrm{SU}(2)$ ($i = 1, 2$).*

Then there are a Riemannian metric on M and a smooth generic path of smooth connections, $\{A_t\}_{0 \leq t \leq 1}$, on $M \times \mathrm{SU}(2)$ connecting A_0 and A_1 such that (1) for $0 \leq t \leq 1$, A_t restricts to a product $B_t \times 1$ in a neighborhood of Σ for an irreducible flat connection B_t on $\Sigma \times \mathrm{SU}(2)$ and the trivial connection 1 in the normal direction of Σ and (2) the invariant $\gamma(\{A_t\})$ can be defined and

$$\mathrm{SF}(M, \{A_t\}) = \gamma(\{A_t\})$$

where $\mathrm{SF}(M, \{A_t\})$ denotes the spectral flow of $\{D_{A_t}\}_{0 \leq t \leq 1}$.

Theorem 1.1 will be proved in Section 7.

We expect that there may be simple and practical methods to calculate $\gamma(\{A_t\})$ in various cases. In the case of Heegaard splittings, a generic path $\{A_t\}_{0 \leq t \leq 1}$ in Theorem 1.1 can be found using the representation space of the surface group into SU(2) and $\gamma(\{A_t\})$ is the Maslov index associated with the Lagrangian submanifolds of the representation space corresponding to the handlebodies M_1 and M_2. In this case, the result of Theorem 1.1 seems to be closely related to the result of [F2].

As an application of Theorem 1.1, we obtain the following theorem:

THEOREM 1.2. *Let N_k be the homology 3-sphere obtained by the $(1/k)$-Dehn surgery along the figure eight knot in S^3 for an integer k. Let $I_*(N_k)$ be the Floer homology group of N_k. Then for k an integer < 0, $I_{\text{odd}}(N_k) = 0$ and $I_{\text{even}}(N_k)$ is a free abelian group whose rank is given by*

$$\begin{array}{ll}
(2m, \quad 2m, \quad 2m, \quad 2m) & \text{for } k = -4m \\
(2m-1, \quad 2m, \quad 2m-1, \quad 2m) & \text{for } k = -4m+1 \\
(2m-1, \quad 2m-1, \quad 2m-1, \quad 2m-1) & \text{for } k = -4m+2 \\
(2m-2, \quad 2m-1, \quad 2m-2, \quad 2m-1) & \text{for } k = -4m+3.
\end{array}$$

Here the $(j+1)$-th coordinate of the 4-vectors represents the rank of $I_{2j}(N_k)$ $(j = 0, 1, 2, 3)$. Since N_{-k} is orientation-preserving diffeomorphic to $-N_k$ (N_k with the opposite orientation), $I_*(N_{-k}) = I_{3-*}(N_k)$ for k an integer < 0.

The proof of this theorem will be given in Section 8.

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2. Gauge theory in 3 dimensions

Let M be an oriented closed 3-manifold with the integral homology of S^3. Every principal SU(2)-bundle over M is isomorphic to the trivial bundle, $P = M \times \mathrm{SU}(2)$. It is understood that a trivialization of P is fixed and the associated product connection is denoted by θ.

The space of smooth connections on P, $\mathscr{A} = \mathscr{A}(P)$, is an affine space; the choice of θ gives an isomorphism of $\mathscr{A}$ with $\Omega^1(M) \otimes \mathrm{su}(2)$. Here su(2) is the Lie algebra of SU(2) and $\Omega^p(M)$ $(p = 0, 1, 2, 3)$ is the space of smooth p-forms on M. Use the L_1^2-inner product on $\Omega^1(M) \otimes \mathrm{su}(2)$ to define $\mathscr{A}$ as a smooth manifold modelled on a (pre-)Hilbert space ([Pa]).

With the product structure fixed, the group $\mathscr{G}$ of smooth automorphisms of P is identical to $C^\infty(M; \mathrm{SU}(2))$. It acts on $\mathscr{A}$ in the usual way (as $(g, \mathscr{A}) \to g\mathscr{A} = gdg^{-1} + gAg^{-1}$). Let $\mathscr{B} = \mathscr{A}/\mathscr{G}$ with the quotient topology. Let $\mathscr{R} \subset \mathscr{A}$ denote the space of reducible connections. The group $\mathscr{G}$ acts on $\mathscr{A}^* = \mathscr{A} - \mathscr{R}$ with stabilizer ± 1. Set $\mathscr{B}^* = (\mathscr{A}^*)/\mathscr{G}$. Think of $\mathscr{B}^*$ as an infinite-dimensional manifold which is modelled on a pre-Hilbert space by using L_1^2-theory in [Pa]. This manifold structure makes the projection from $\mathscr{A}^* \to \mathscr{B}^*$ a principal $\mathscr{G}$-bundle.

Give M a Riemannian metric, and let $*: \Omega^p(M) \to \Omega^{3-p}(M)$ denote the associated Hodge star operator. The metric defines an L^2 inner product on $\Omega^p(M) \otimes \mathrm{su}(2)$: the inner product of p-form a with p-form b is

$$(a, b)_{L^2} = -\int_M \mathrm{tr}(a \wedge *b).$$

In particular the above inner product on $\Omega^1(M) \otimes \mathrm{su}(2)$ defines a $\mathscr{G}$-equivariant Riemannian metric on $\mathscr{A}^*$ and also defines a Riemannian metric on $\mathscr{B}^*$.

The Chern-Simons functional f on $\mathscr{A}^*$ is defined by

$$f(A) = \int_M \mathrm{tr}(A \wedge dA + (2/3)A \wedge A \wedge A)$$

$(A \in \mathscr{A}^*)$. Then, for $g \in \mathscr{G}$, $f(gA) = f(A) + c\deg(g)$, where c is a constant and $\deg(g)$ is the mapping degree of $g: M \to \mathrm{SU}(2)$. Then f descends to a functional $f: \mathscr{B}^* \to \mathbf{R}/c\mathbf{Z}$. The curvature of a connection A is the su(2)-valued 2-form $F_A = dA + A \wedge A$. The assignment $A \to -*F_A$ defines a $\mathscr{G}$-equivariant vector field on $\mathscr{A}^*$ and its descendant to $\mathscr{B}^*$ is the gradient vector field of f. Thus the critical point set of f on $\mathscr{B}^*$ is precisely the set of the $\mathscr{G}$-orbit of the irreducible flat connections on $M \times \mathrm{SU}(2)$. This set is identical to the set of the conjugacy classes of the irreducible representations of $\pi_1(M)$ to SU(2),

$$\mathrm{Hom}(\pi_1(M), \mathrm{SU}(2))^*/\mathrm{ad}\,\mathrm{SU}(2).$$

Notation convention. From now on, we adopt the following abbreviation: we denote a direct sum

$$\Omega^p(\ \ ) \otimes \mathrm{su}(2) + \cdots + \Omega^{p'}(\ \ ) \otimes \mathrm{su}(2)$$

by

$$(\Omega^p + \cdots + \Omega^{p'})(\ \ ) \otimes \mathrm{su}(2).$$

Definition 2.1. For a smooth connection A on $M \times \mathrm{SU}(2)$, we define the operator

$$D_A: (\Omega^1 \oplus \Omega^0)(M) \otimes \mathrm{su}(2) \to (\Omega^1 \oplus \Omega^0)(M) \otimes \mathrm{su}(2)$$

by

$$D_A(a, b) = (*d_A a + d_A b, d_A^* a) \tag{2.1}$$

for $a \in \Omega^1(M) \otimes \mathrm{su}(2)$ and $b \in \Omega^0(M) \otimes \mathrm{su}(2)$, where d_A denotes the covariant derivative.

D_A defines a self-adjoint Fredholm operator from the L_1^2-completion of $(\Omega^1 \oplus \Omega^0)(M) \otimes \mathrm{su}(2)$ to the L^2-completion of $(\Omega^1 \oplus \Omega^0)(M) \otimes \mathrm{su}(2)$. A $\mathscr{G}$-orbit $[A]$ of an irreducible flat connection A on $M \times SU(2)$ is a non-degenerate critical point of f if and only if Ker $D_A = 0$. Such an $[A]$ defines a generator of the chain group of the Floer homology group of M. To define the mod 8 degree $d([A])$, we need the notion of the spectral flow.

The spectral flow for a continuous family of self-adjoint Fredholm operators was studied in [A-P-S 3]. To say that an operator is self-adjoint and Fredholm is to say that its spectrum near 0 is that of a finite-dimensional, self-adjoint matrix. Move on a continuously differentiable path in the space of such operators, and the eigenvalues near 0 move in a continuously differentiable manner. Suppose that the operators at the path's endpoints have empty kernel. Then, the number of eigenvalues which cross zero with positive slope minus the number which cross zero with negative slope is well defined and finite along a suitably generic path. This number is the spectral flow along the path.

When two such generic paths are homotopic (rel. endpoints), their spectral flows agree. Therefore, the spectral flow defines a locally constant function on the space of continuous paths between the two endpoints.

Since the spectral flow is only a locally constant function, there can be non-zero spectral flow around a non-contractible, closed curve in the space of self-adjoint, Fredholm operators. Indeed, as remarked in [A-P-S 3], the spectral flow around closed loops gives an isomorphism between $\mathbf{Z}$ and the fundamental group of the space of self-adjoint Fredholm operators on a real, infinite-dimensional, separable Hilbert space.

Let $\{A_t\}_{0 \le t \le 1}$ be a smooth path of smooth connections on $M \times SU(2)$ such that Ker $D_{A_0} = 0 =$ Ker D_{A_1}. Then $\{D_{A_t}\}_{0 \le t \le 1}$ is a smooth family of self-adjoint Fredholm operators and the spectral flow is defined. The spectral flow defines a locally constant function on the space of continuous paths in $\mathscr{B}^*$ connecting $[A_0]$ and $[A_1]$. This function depends on the homotopy class of the path between $[A_0]$ and $[A_1]$, but its mod 8 reduction does not ([F]).

Proposition 2.1 ([F], [T]). *Let* $[A_0]$ *and* $[A_1]$ *be the* $\mathscr{G}$*-orbits of two irreducible flat connections such that* Ker $D_{A_0} = 0 =$ Ker D_{A_1}. *Then the spectral flow* mod 8 *of a path between* $[A_0]$ *and* $[A_1]$ *depends only on the differentiable structure on M. In particular, it is independent of the choice of Riemannian metric on M.*

The mod 8 spectral flow in Proposition 2.1 gives the difference $d([A_1]) - d([A_0])$. The definition of $d([A])$ itself for an irreducible flat connection with Ker $D_{A_0} = 0$ is given as follows.

Let θ be the trivial connection on $M \times \mathrm{SU}(2)$. Then Ker D_θ is 3-dimensional. We can perturb θ to an irreducible smooth connection θ' with Ker $D_{\theta'} = 0$ (arbitrarily close to θ) so that Ker D_θ splits into the direct sum of three 1-dimensional eigenspaces belonging to the small eigenvalue (small means much smaller than any other eigenvalues of $D_{\theta'}$). Let $p(\theta')$ be the number of the positive eigenvalues among these three. Let $\{A_t\}_{0 \le t \le 1}$ be a smooth path of smooth connections on $M \times \mathrm{SU}(2)$ connecting $A_0 = \theta'$ and $A_1 = A$. The integer mod 8

$$p(\theta') + \text{the spectral flow of } \{D_{A_t}\}_{0 \le t \le 1} \tag{2.2}$$

does not depend on the choice of perturbation θ', and it is $d([A])$.

3. Computations on the cylinder

In this section, we make some explicit calculations which will be basic to the rest of the paper.

Let Σ be a connected oriented closed surface of genus g (≥ 2). Fix a Riemann metric on Σ. Let $\Sigma \times \mathbf{R}^+$ be the product Riemannian manifold, where $\mathbf{R}^+$ denotes the half line $\{s \ge 0\}$ with the standard metric. Let B be an irreducible flat connection on $\Sigma \times \mathrm{SU}(2)$. Let $A = B \times 1$ be the connection on $(\Sigma \times \mathbf{R}^+) \times \mathrm{SU}(2)$ which is the product of B with the trivial connection 1 on $\mathbf{R}^+ \times \mathrm{SU}(2)$. Let $\Omega_0^j(\Sigma \times \mathbf{R}^+) \otimes \mathrm{su}(2)$ denote the space of su(2)-valued smooth j-forms on $\Sigma \times \mathbf{R}^+$ with compact support. For $a \in \Omega_0^j(\Sigma \times \mathbf{R}^+) \otimes \mathrm{su}(2)$ and $0 \le s < \infty$, we write a as $a(s) = p(s) + q(s)\,ds$, where $p(s) \in \Omega^j(\Sigma) \otimes \mathrm{su}(2)$ and $q(s) \in \Omega^{j-1}(\Sigma) \otimes \mathrm{su}(2)$.

Let D_A be the differential operator on $(\Omega_0^1 \oplus \Omega_0^0)\ (\Sigma \times \mathbf{R}^+) \otimes \mathrm{su}(2)$ defined by the equation given in Definition 2.1, where $A = B \times 1$ as above. For $(a, b) \in (\Omega_0^1 \oplus \Omega_0^0)(\Sigma \times \mathbf{R}^+) \otimes \mathrm{su}(2)$ and $0 < s < \infty$, we write

$$(a(s), b(s)) = (p(s), q(s), b(s)) \tag{3.1}$$

where $a(s) = p(s) + q(s)\,ds$ is as above. Thus we regard $(a(s), b(s))$ as an element of $(\Omega^1 \oplus \Omega^0 \oplus \Omega^0)(\Sigma) \otimes \mathrm{su}(2)$. Then D_A can be written in the matrix form

$$D_A \begin{pmatrix} p(s) \\ q(s) \\ b(s) \end{pmatrix} = \sigma\left(\frac{\partial}{\partial s} - D_B\right)\begin{pmatrix} p(s) \\ q(s) \\ b(s) \end{pmatrix} \tag{3.2}$$

where

$$(3.3)\qquad \sigma = \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad D_B = \begin{pmatrix} 0 & d_B & *d_B \\ d_B^* & 0 & 0 \\ -*d_B & 0 & 0 \end{pmatrix}.$$

Here $*$ denotes the Hodge star operator on $\Omega^j(\Sigma) \otimes \mathrm{su}(2)$ ($j = 0, 1$) and d_B denotes the covariant derivative.

The relations

$$(3.4)\qquad \sigma D_B + D_B \sigma = 0 \quad \text{and} \quad \sigma^2 = -1$$

hold.

D_B can be extended to a self-adjoint Fredholm operator from the L_1^2-completion of $(\Omega^1 \oplus \Omega^0 \oplus \Omega^0)(\Sigma) \otimes \mathrm{su}(2)$ to the L^2-completion of $(\Omega^1 \oplus \Omega^0 \oplus \Omega^0)(\Sigma) \otimes \mathrm{su}(2)$. The eigenvalues $\{\mu\}$ of D_B form a discrete subset of $\mathbf{R}$ and each eigenvalue has a finite multiplicity. The set of all the normalized eigenforms $\{\psi_\mu\}$ of D_B forms a complete orthonormal basis of the L^2-completion of $(\Omega^1 \oplus \Omega^0 \oplus \Omega^0)(\Sigma) \otimes \mathrm{su}(2)$. By the above relation, $\sigma(\psi_\mu)$ is a $(-\mu)$-eigenform of D_B. Thus the spectrum of D_B is symmetric about 0.

Let P_+ (resp. P_-) be the subspace of the L^2-completion of $(\Omega^1 \oplus \Omega^0 \oplus \Omega^0)(\Sigma) \otimes \mathrm{su}(2)$ spanned by $\{\psi_\mu\}_{\mu > 0}$ (resp. $\{\psi_\mu\}_{\mu < 0}$).

Let $\mathscr{H}_B$ denote the space of the su(2)-valued 1-forms on Σ harmonic with respect to B,

$$\mathscr{H}_B = \left\{\omega \in \Omega^1(\Sigma) \otimes \mathrm{su}(2) | d_B\omega = d_B^*\omega = 0\right\}.$$

$\mathscr{H}_B$ is a $(6g-6)$-dimensional real vector space isomorphic to the de Rham cohomology group $H_B^1(\Sigma, \mathrm{su}(2))$ with su(2)-valued local coefficient system defined by the holonomy representation of B. Since B is irreducible, d_B has no non-trivial kernel in $\Omega^0(\Sigma) \otimes \mathrm{su}(2)$. Hence each 0-eigenform ψ_0 of D_B has the form $\psi_0 = (\omega, 0, 0) \in (\Omega^1 \oplus \Omega^0 \oplus \Omega^0)(\Sigma) \otimes \mathrm{su}(2)$ for some $\omega \in \mathscr{H}_B$. Thus $\operatorname{Ker} D_B$ is identified with $\mathscr{H}_B$. Note that $\sigma = *$ on $\mathscr{H}_B$.

$\mathscr{H}_B$ has a non-degenerate symplectic structure defined by

$$(3.5)\qquad \langle \omega_1, \omega_2 \rangle = -\int_\Sigma \mathrm{tr}(\omega_1 \wedge \omega_2)$$

for $\omega_1, \omega_2 \in \mathscr{H}_B$. A Lagrangian L of $\mathscr{H}_B$ is a $(3g-3)$-dimensional subspace of $\mathscr{H}_B$ such that $\langle \omega_1, \omega_2 \rangle = 0$ for any $\omega_1, \omega_2 \in L$.

For a subspace W of $(\Omega^1 \oplus \Omega^0 \oplus \Omega^0)(\Sigma) \otimes \mathrm{su}(2)$, we set

$$\Omega_0(\Sigma \times \mathbf{R}^+, W) = \left\{\psi \in \left(\Omega_0^1 \oplus \Omega_0^0\right)(\Sigma \times \mathbf{R}^+) \otimes \mathrm{su}(2) | \psi(0) \in W\right\}.$$

Definition 3.1. Let B be an irreducible flat connection on $\Sigma \times SU(2)$ and $A = B \times 1$ be the connection on $(\Sigma \times \mathbf{R}^+) \times SU(2)$ as above. We define the following operators.

(1) $\mathscr{E}_A$ is the operator from the L_1^2-completion of $\Omega_0(\Sigma \times \mathbf{R}^+, P_+ + \mathscr{H}_B)$ to the L^2-completion of $(\Omega_0^1 \oplus \Omega_0^0)(\Sigma \times \mathbf{R}^+) \otimes su(2)$ defined to be the closure of D_A, and

(2) $\mathscr{E}_A^*$ is the operator from the L_1^2-completion of $\Omega_0(\Sigma \times \mathbf{R}^+, P_+)$ to the L^2-completion of $(\Omega_0^1 \oplus \Omega_0^0)(\Sigma \times \mathbf{R}^+) \otimes su(2)$ defined to be the closure of D_A

PROPOSITION 3.1. *$\mathscr{E}_A$ and $\mathscr{E}_A^*$ are the adjoints of each other.*

Proof. The adjointness relation between the boundary conditions follows from the next equation derived from the Stokes theorem, for $\psi = (a, b) = (p(s) + q(s)\,ds, b(s))$ and $\psi' = (a', b') = (p'(s) + q'(s)\,ds, b'(s)) \in (\Omega_0^1 \oplus \Omega_0^0)(\Sigma \times \mathbf{R}^+) \otimes su(2)$,

$$
\begin{aligned}
(3.1.1)\quad & (D_A\psi, \psi') - (\psi, D_A\psi') \\
&= -\int_{\Sigma\times\mathbf{R}^+} \mathrm{tr}((*\,d_A + d_A b) \wedge *\,a') - \int_{\Sigma\times\mathbf{R}^+} \mathrm{tr}(d_A^* a \wedge *\,b') \\
&\quad + \int_{\Sigma\times\mathbf{R}^+} \mathrm{tr}((a \wedge *(*\,d_A a' + d_A b'))) + \int_{\Sigma\times\mathbf{R}^+} \mathrm{tr}(b \wedge *\,d_A^* a') \\
&= \int_{\Sigma\times\{0\}} \mathrm{tr}(p(0) \wedge p'(0) + b(0) \wedge *\,q'(0) - q(0) \wedge *\,b'(0)) \\
&= (\psi(0), \sigma(\psi'(0))).
\end{aligned}
$$

Note that σ interchanges P_+ and P_-. The rest of the proof goes in the same way as the proof of Proposition (2.5) and (2.1) in [A-P-S 1]. The bounded inverse of $\mathscr{E}_A$

$$Q\colon (\Omega_0^1 \oplus \Omega_0^0)(\Sigma \times \mathbf{R}^+) \otimes su(2) \to \Omega_0(\Sigma \times \mathbf{R}^+, P_+ + \mathscr{H}_B)$$

is constructed as follows. Let $\{\psi_\mu\}$ be an L^2-orthonormal basis of $(\Omega^1 \oplus \Omega^0 \oplus \Omega^0)(\Sigma) \otimes su(2)$ consisting of eigenforms of D_B. To solve $D_A\psi = \varphi$, we expand ψ and φ in terms of $\{\psi_\mu\}$ and $\{\sigma(\psi_\mu)\}$ respectively;

$$\psi(s) = \Sigma c_\mu(s)\psi_\mu, \qquad \varphi(s) = \Sigma d_\mu(s)\sigma(\psi_\mu).$$

We take the explicit solutions

$$
\begin{aligned}
c_\mu(s) &= -\int_s^\infty e^{\mu(s-t)}\, d_\mu(t)\, dt \quad \text{for } \mu \geq 0 \\
&= \int_0^s e^{\mu(s-t)}\, d_\mu(t)\, dt \qquad \text{for } \mu < 0
\end{aligned}
$$

to define Q_μ. Formally $Q = \Sigma Q_\mu$. The proof of the convergence of Q is the same as the proof of Proposition (2.5) of [A-P-S 1] and we refer to it. Similarly we get a bounded inverse R for $\mathscr{E}_A^*$. If we decompose the L^2-completion of $(\Omega_0^1 \oplus \Omega_0^0)(\Sigma \times \mathbf{R}^+) \otimes \mathrm{su}(2)$ into two parts, $\Omega' \oplus \Omega''$, Ω'', involving the zero eigenvalue of D_B and Ω' all the non-zero eigenvalues, then $\mathscr{E}_A$ and $\mathscr{E}_A^*$ decompose accordingly. On Ω'', $D_A^* = (-\sigma)(-\partial/\partial s) = \sigma\partial/\partial s = D_A$ and the adjointness is clear. On Ω', the fundamental solutions Q and R give bounded inverses Q' and R' for $\mathscr{E}_A$ and $\mathscr{E}_A^*$ respectively. Then $R' = (Q')^*$ follows by continuity from the fact that $(\mathscr{E}_A\psi, \psi') = (\psi, \mathscr{E}_A\psi')$ for $\psi, \psi' \in \Omega_0(\Sigma \times \mathbf{R}^+, P_+ + \mathscr{H}_B)$ by Equation (3.1.1). Since the adjoints commute with inverses, the proposition is established. q.e.d.

4. Indices of Fredholm operators with global boundary conditions

Let X be a compact oriented 3-dimensional Riemannian manifold with boundary $\partial X = \Sigma$ a connected closed surface of genus g (≥ 2). Σ inherits its metric from that of X. We assume that, near Σ, X is isometric to the product $\Sigma \times [0,1]$; here $\Sigma \times \{0\} = \partial X$. We orient Σ as $-\partial X$.

Let B be an irreducible flat connection on $\Sigma \times \mathrm{SU}(2)$. Let A be a smooth connection on $X \times \mathrm{SU}(2)$ which restricts to the product $B \times 1$ on $\Sigma \times [0,1] \times \mathrm{SU}(2)$; here 1 denotes the trivial connection on $[0,1] \times \mathrm{SU}(2)$. Let D_A be the differential operator on $(\Omega^1 \oplus \Omega^0)(X) \otimes \mathrm{su}(2)$ defined by the equation in Definition 2.1. D_A has the form $\sigma(\partial/\partial s - D_B)$ on $\Sigma \times [0,1]$; here σ and D_B are as in Section 3.

For a subspace W of $(\Omega^1 \oplus \Omega^0 \oplus \Omega^0)(\Sigma) \otimes \mathrm{su}(2)$, we set

$$\Omega(X, W) = \{\psi \in (\Omega^1 \oplus \Omega^0)(X) \otimes \mathrm{su}(2) | \psi | \partial X \in W\}.$$

Definition 4.1. Let P_+, P_- and $\mathscr{H}_B$ be as in Section 3. We define the following operators:

(1) $\mathscr{E}_A$ is the operator from the L_1^2-completion of $\Omega(X, P_+ + \mathscr{H}_B)$ to the L^2-completion of $(\Omega^1 \oplus \Omega^0)(X) \otimes \mathrm{su}(2)$ defined to be the closure of D_A, and

(2) $\mathscr{E}_A^*$ is the operator from the L_1^2-completion of $\Omega(X, P_+)$ to the L^2-completion of $(\Omega^1 \oplus \Omega^0)(X) \otimes \mathrm{su}(2)$ defined to be the closure of D_A.

PROPOSITION 4.1. *$\mathscr{E}_A$ and $\mathscr{E}_A^*$ are Fredholm operators and are the adjoints of each other.*

Proof. $\mathscr{E}_A$ is a first order elliptic differential operator. On $\Sigma \times [0,1]$, it has the form $\sigma(\partial/\partial s - D_B)$. We construct a parametrix R by patching together the fundamental solution $Q_1 = Q$ constructed in the proof of Proposition 3.1 with an interior parametrix Q_2. More precisely, for $-\infty < u < v < \infty$, let $\beta(u, v)$

denote an increasing C^∞ function of the real variable s, such that

$$\beta = 0 \quad \text{for } s \le u, \qquad \text{and} \qquad \beta = 1 \quad \text{for } s \ge v$$

and define four C^∞ functions r_1, r_2, p_1, p_2 by

$$r_1 = \beta(1/4, 1/2), \qquad p_2 = \beta(1/2, 3/4),$$

$$r_2 = 1 - \beta(3/4, 1) \qquad p_1 = 1 - p_2.$$

Note that $r_j = 1$ on the support of p_j $(j = 1, 2)$. We regard these functions of s as functions on the cylinder $\Sigma \times [0, 1]$ and extend them to X in the obvious way: r_1, p_1 being extended by 0 and r_2, p_2 being extended by 1. Considering r_j, p_j $(j = 1, 2)$ as multiplication operators we put

$$R = p_1 Q_1 r_1 + p_2 Q_2 r_2.$$

R is a linear operator

$$(\Omega^1 \oplus \Omega^0)(X) \otimes \mathrm{su}(2) \to \Omega(X, P_+ + \mathscr{H}_B).$$

R is a right parametrix; that is, $\mathscr{E}_A R - 1$ has a C^∞ kernel. Switching the roles of r_j, p_j gives a left parametrix; hence R is, in fact, a two sided parametrix. Proposition 3.1 shows that R is continuous from the L^2-completion of $(\Omega^1 \oplus \Omega^0)(X) \otimes \mathrm{su}(2)$ to the L^2_1-completion of $\Omega(X, P_+ + \mathscr{H}_B)$. It now follows that $\mathscr{E}_A$ is a Fredholm operator. Essentially the same argument works for $\mathscr{E}_A^*$. q.e.d.

An element of the L^2-completion of $(\Omega^1 \oplus \Omega^0)(X) \otimes \mathrm{su}(2)$, orthogonal to the image of $\mathscr{E}_A$, is necessarily C^∞ (being in $\mathrm{Ker}\, \mathscr{E}_A^*$). Thus $\mathscr{E}_A$ has a well-defined index, computed either in C^∞ or in L^2, and

$$\mathrm{index}\, \mathscr{E}_A = \dim \mathrm{Ker}\, \mathscr{E}_A - \dim \mathrm{Ker}\, \mathscr{E}_A^*.$$

The calculation of index $\mathscr{E}_A$ will be given in Proposition 4.2. For it, we prepare two lemmas. For $0 \le r < \infty$,

$$X(r) = \Sigma \times [-r, 0] \cup X$$

where $\Sigma \times \{0\}$ is identified with ∂X. The product metric on $\Sigma \times [-r, 0]$ should be understood. We extend the connection A on $X \times \mathrm{SU}(2)$ to the connection (still denoted by A) on $X(r) \times \mathrm{SU}(2)$ by setting $A = B \times 1$ on $\Sigma \times [-r, 0] \times \mathrm{SU}(2)$.

LEMMA 4.1. *There is a positive constant $c > 0$ not depending on r $(2 \le r < \infty)$ such that, for any smooth 0-form b on $X(r)$ with $b | \partial X(r) = 0$,*

$$\| \hat{b} \|_{L^2} < c \| d_A^* d_A b \|_{L^2}$$

where $\hat{b} = b | X(2)$ denotes the restriction of b on $X(2) \subset X(r)$.

Proof. On the contrary, assume that there is an infinite sequence of positive numbers $2 < r_1 < r_2 < \cdots$, $r_j \to \infty$, and a sequence of 0-forms on $X(r_j)$, $\{b_j\}_{j=1,2,\dots}$, such that

$$b_j|\partial X(r_j) = 0, \qquad \|d_A^* d_A b_j\|_{L^2} = 1 \quad (j = 1, 2, \dots)$$

and

$$\|\hat{b_j}\|_{L^2} \to \infty \quad \text{as} \quad j \to \infty$$

where $\hat{b_j} = b_j|X(2)$. We set $\bar{b}_j = b_j/\|\hat{b_j}\|_{L^2}$ and $\hat{\bar{b}_j} = \bar{b}_j|X(2)$. Then, by the assumption, $\|d_A^*\bar{b}_j\|_{L^2} \to 0$ as $j \to \infty$. Since $\bar{b}_j|\partial X(r) = 0$, by Stokes's theorem, $\|d_A\bar{b}_j\|_{L^2} \to 0$ as $j \to \infty$. Hence $\|d_A\hat{\bar{b}_j}\|_{L^2} \to 0$ as $j \to \infty$. Thus $\{\hat{\bar{b}_j}\}_{j=1,2,\dots}$ is an L_1^2-bounded sequence. By Rellich's lemma, taking a subsequence if necessary, we may assume that $\{\hat{\bar{b}_j}\}_{j=1,2,\dots}$ has an L^2-strong limit $\hat{\bar{b}_\infty}$. Since $\|\hat{\bar{b}_j}\|_{L^2} = 1$ for $j = 1, 2, \dots$, $\|\hat{\bar{b}_\infty}\|_{L^2} = 1$ and $\hat{\bar{b}_\infty} \neq 0$. Also since $\|d_A\hat{\bar{b}_j}\|_{L^2} \to 0$ as $j \to \infty$, $\{\hat{\bar{b}_j}\}_{j=1,2,\dots}$ is an L_1^2-convergent sequence. Hence $\hat{\bar{b}_\infty}$ is an L_1^2-form and $d_A\hat{\bar{b}_\infty} = 0$. This contradicts the irreducibility of A. q.e.d.

For a calculation of index $\mathscr{E}_A$, we first consider the special case of flat connection. Assume that A is a flat connection on $X \times \mathrm{SU}(2)$ which restricts to $B \times 1$ on $\Sigma \times [0,1] \times \mathrm{SU}(2)$ for an irreducible flat connection B on $\Sigma \times \mathrm{SU}(2)$. Since A is flat, $d_A d_A = 0$, and the complexes $\{\Omega^*(X) \otimes \mathrm{su}(2), d_A\}$ and $\{\Omega^*(X, \Sigma) \otimes \mathrm{su}(2), d_A\}$ become chain complexes, where $\Omega^*(X, \Sigma) \otimes \mathrm{su}(2)$ denotes the space of su(2)-valued smooth forms on X which vanish on $\Sigma = \partial X$. Let $H_A^*(X, \mathrm{su}(2))$ and $H_A^*(X, \Sigma, \mathrm{su}(2))$ be the de Rham cohomology groups of these chain complexes.

LEMMA 4.2. *Let A be a flat connection on $X \times \mathrm{SU}(2)$ which restricts to $B \times 1$ on $\Sigma \times [0,1] \times \mathrm{SU}(2)$ for an irreducible flat connection B on $\Sigma \times \mathrm{SU}(2)$. Then there are natural isomorphisms*

$$\operatorname{Ker} \mathscr{E}_A = H_A^1(X, \mathrm{su}(2)),$$
$$\operatorname{Ker} \mathscr{E}_A^* = H_A^1(X, \Sigma, \mathrm{su}(2)),$$

and

$$\operatorname{Index} \mathscr{E}_A = 3g - 3.$$

Proof. For $2 \le r < \infty$, let $X(r) = \Sigma \times [-r, 0] \cup X$ be as before. Let $x \in H_A^1(X, \mathrm{su}(2))$. Let $a_1 \in \Omega^1(X(1)) \otimes \mathrm{su}(2)$ be a smooth 1-form representing x with $d_A a_1 = 0$. Let $\omega \in \mathscr{H}_B$ be the harmonic 1-form representing i^*x (i: $\Sigma \to X$ the inclusion). For $r \ge 2$, we regard ω as a 1-form on $\Sigma \times [-r, 0]$ constant along $[-r, 0]$. On $\Sigma \times [-1, 0]$, $a_1 = \omega + d_A b_1$ for a 0-form b_1.

For $r \geq 2$, let β_r be a C^∞ increasing function on $[-r, 0]$ such that $\beta_r = 0$ on $[-r, 1/4 - r]$ and $\beta_r = 1$ on $[-r + 1, 0]$. We regard these functions as functions on the cylinder $\Sigma \times [-r, 0]$ and extend them to $X(r)$ by 1.

For $r \geq 2$, we define $a_r \in \Omega^1(X(r)) \otimes \mathrm{su}(2)$ by

$$\begin{aligned} a_r &= a_1 && \text{on } X, \\ &= \omega + d_A(\beta_1 b_1) && \text{on } \Sigma \times [-1, 0], \\ &= \omega && \text{on } \Sigma \times [-r, -1]. \end{aligned}$$

Then $d_A a_r = 0$, $\mathrm{supp}(d_A^* a_r) \subset X(1)$ and $d_A^* a_r = d_A^* a_{r'}$, for $2 \leq r < r'$, on $X(1)$. Since A is smooth and irreducible, there is a unique $b_r \in \Omega^0(X(r)) \otimes \mathrm{su}(2)$ such that

$$d_A^* d_A b_r = d_A^* a_r, \qquad b_r | \partial X(r) = 0$$

(see [T]). By Lemma 4.1, $\|b_r | X(2)\|_{L^2} < c_1$ for a constant $c_1 > 0$ independent of r. On $\Sigma \times [-r, -1]$, since $d_A^* d_A b_r = 0$ and $b_r | \partial X(r) = 0$, b_r can be written as

$$b_r = \Sigma_\lambda b_r^\lambda \left(e^{(s+r)\sqrt{\lambda}} - e^{-(s+r)\sqrt{\lambda}}\right)\phi_\lambda, \qquad -r \leq s \leq -1,$$

for some constants $\{b_r^\lambda\}$, where $\{\phi_\lambda\}$ is an orthonormal basis of the L^2-completion of $\Omega^0(\Sigma) \otimes \mathrm{su}(2)$ consisting of the eigenforms of $d_B^* d_B$. Since $\|b_r | X(2)\|_{L^2} < c_1$ (Lemma 4.1), from the above explicit form of b_r on the cylinder, it follows that $\|b_r\|_{L^2} < c_2$ for a constant $c_2 > 0$ not depending on r. Since $b_r | \partial X(r) = 0$, by Stokes's theorem,

$$\begin{aligned} (d_A b_r, d_A b_r) &= (d_A^* d_A b_r, b_r) = (d_A^* a_r, b_r) \\ &\leq \|d_A^* a_r\|_{L^2} \|b_r\|_{L^2} < c_3 \end{aligned}$$

for a constant $c_3 > 0$ not depending on r. It follows that $\{b_r | X(2)\}_r$ is an L_1^2-bounded sequence. Hence by Rellich's lemma, taking a subsequence if necessary, we obtain an L^2-strong limit $\hat{b}_\infty$, an L^2 0-form on $X(2)$. Since $d_A^* d_A b_r = d_A^* a_r$ and $d_A^* a_r$ is independent of $r \geq 2$, $\hat{b}_\infty$ satisfies the weak equation

$$d_A^* d_A \hat{b}_\infty = d_A^* a_r.$$

By elliptic regularity, $\hat{b}_\infty$ is a smooth 0-form on $X(2)$ and it is an actual solution of the above equation. On $\Sigma \times [-2, -1] \subset X(2)$, $\hat{b}_\infty$ can be written in the form of a linear combination of $\{e^{s\sqrt{\lambda}}\phi_\lambda, e^{-s\sqrt{\lambda}}\phi_\lambda\}_\lambda$ for $-2 \leq s \leq -1$. Since $\hat{b}_\infty$ is the L^2-strong limit of $\{b_r | X(2)\}$ and each b_r has the above explicit form on the cylinder, it follows that $\hat{b}_\infty | \Sigma \times [-2, -1]$ can be written as a linear combination of $\{e^{s\sqrt{\lambda}}\phi_\lambda\}_\lambda$. Hence $\hat{b}_\infty$ can be extended to a 0-form b_∞ on $X(\infty)$ which is smooth, exponentially decays as $s \to -\infty$ and satisfies

$$d_A^* d_A b_\infty = d_A^* a_r.$$

Define $a_\infty \in \Omega^0(X(\infty)) \otimes \mathrm{su}(2)$ by setting $a_\infty = a_r - d_A b_\infty$ on $X(r)$ for $r > 0$. Then $d_A a_\infty = d_A^* a_\infty = 0$. We regard a_∞ as an element of $(\Omega^1 \oplus \Omega^0)(X(\infty)) \otimes \mathrm{su}(2)$. Then $D_A a_\infty = 0$. Hence, on $\Sigma \times (-\infty, 0]$, a_∞ can be written as a linear combination of $e^{s\mu}\psi_\mu$, where $\{\psi_\mu\}$ is the orthonormal basis of the L^2-completion of $(\Omega^1 \oplus \Omega^0 \oplus \Omega^0)(\Sigma) \otimes \mathrm{su}(2)$ consisting of the eigenforms of D_B (Section 3). Since a_∞ is a bounded form, $a_\infty = \omega + \sum_{\mu>0} c_\mu e^{s\mu}\psi_\mu$ for constants $\{c_\mu\}$ on $\Sigma \times (-\infty, 0]$. Thus the restriction $a = a_\infty | X$ satisfies $D_A a = 0$ and $a|\partial X \in P_+ + \mathscr{H}_B$. Hence $a \in \operatorname{Ker} \mathscr{E}_A$. Obviously $d_A a = d_A^* a = 0$ and a represents x. If there is another 1-form a' satisfying these conditions, then a' can be extended to a bounded harmonic 1-form a'_∞ on $X(\infty)$, and $a_\infty - a'_\infty = d_A u$ for a harmonic L^2 0-form u on $X(\infty)$. Since there exists no non-trivial such 0-form, $a_\infty = a'_\infty$ and $a = a'$. The correspondence $x \to a$ gives a homomorphism $H_A^1(X, \mathrm{su}(2)) \to \operatorname{Ker} \mathscr{E}_A$.

Conversely let $(a, b) \in (\Omega^1 \oplus \Omega^0)(X) \otimes \mathrm{su}(2)$ be an element of $\operatorname{Ker} \mathscr{E}_A$. Since, on $\Sigma \times [0,1]$, (a, b) can be written as $\omega + \sum_{\mu>0} c_\mu e^{s\mu}\psi_\mu$ for $\omega \in \mathscr{H}_B$ and $\{c_\mu\}$ constants, it can be extended to a bounded element (a_∞, b_∞) of $\operatorname{Ker} D_A$ on $X(\infty)$. From the equation $* d_A a_\infty + d_A b_\infty = 0$, $d_A^* d_A b_\infty = 0$ follows as A is flat. Hence b_∞ is an L^2 harmonic 0-form on $X(\infty)$. There is no non-zero such 0-form on $X(\infty)$, and $b_\infty = 0$. Hence $d_A a_\infty = d_A^* a_\infty = 0$ and $a = a_\infty | X$ is a harmonic 1-form on X. It defines an element of $H_A^1(X, \mathrm{su}(2))$. This gives the map $\operatorname{Ker} \mathscr{E}_A \to H_A^1(X, \mathrm{su}(2))$ which is the inverse of the above map. Therefore this correspondence is an isomorphism.

As for $\operatorname{Ker} \mathscr{E}_A^* = H_A^1(X, \Sigma, \mathrm{su}(2))$, the above arguments involve the following consequence (as in the special case with $\omega = 0$) that $H_A^1(X, \Sigma, \mathrm{su}(2))$ is isomorphic to the space of the harmonic L^2 1-form on $X(\infty)$. The latter is identical to $\operatorname{Ker} \mathscr{E}_A^*$ by the restriction.

Finally, as for the index of $\mathscr{E}_A$, by the Poincaré duality of the de Rham cohomology group, we see that $\dim \operatorname{Ker} \mathscr{E}_A - \dim \operatorname{Ker} \mathscr{E}_A^*$ is equal to $-(3/2)\chi(\Sigma)$, where $\chi(\Sigma)$ denotes the Euler characteristic of Σ. q.e.d.

Proposition 4.2. *Let* A *be a smooth connection on* $X \times \mathrm{SU}(2)$ *which restricts to a product* $B \times 1$ *on* $\Sigma \times [0,1] \times \mathrm{SU}(2)$ *for an irreducible flat connection* B *on* $\Sigma \times \mathrm{SU}(2)$. *Assume that there is a smooth path of smooth connections on* $X \times \mathrm{SU}(2)$, $\{A_t\}_{0 \le t \le 1}$, *such that* $A_1 = A$, A_0 *is an irreducible flat connection and, for* $0 \le t \le 1$, A_t *restricts to a product* $B_t \times 1$ *on* $\Sigma \times [0,1] \times \mathrm{SU}(2)$ *for an irreducible flat connection* B_t *on* $\Sigma \times \mathrm{SU}(2)$. *Then the Fredholm index of* $\mathscr{E}_A$ *is* $3g - 3$.

Proof. For $0 \leq t \leq 1$, we can construct the Fredholm operator $\mathscr{E}_{A_t}$. Now $\{\mathscr{E}_{A_t}\}_{0 \leq t \leq 1}$ is a continuous one-parameter family of Fredholm operators and we can construct a continuous one-parameter family $\{R_t\}_{0 \leq t \leq 1}$ of parametrices as in the proof of Proposition 3.1. It follows that index $\mathscr{E}_A$ = index $\mathscr{E}_{A_0} = 3g - 3$ by the above lemma. q.e.d.

Definition 4.2. We define the map π_A: Ker $\mathscr{E}_A \to \mathscr{H}_B$ by

$$\pi_A(\psi) = \omega$$

for $\psi \in$ Ker $\mathscr{E}_A$, where $\psi | \partial X = \omega + \psi_+$ for $\omega \in \mathscr{H}_B$ and $\psi_+ \in P_+$.

LEMMA 4.3. *Let π_A be as above. Let $L_A = \pi_A(\text{Ker}\, \mathscr{E}_A)$. Then L_A is a Lagrangian of $\mathscr{H}_B$.*

Proof. Let $\psi, \psi' \in$ Ker $\mathscr{E}_A$ and let $\psi | \partial X = \omega + \psi_+$, $\psi' | \partial X = \omega' + \psi'_+$ for $\omega, \omega' \in \mathscr{H}_B$ and $\psi_+, \psi'_+ \in P_+$. By Equation (3.1.1) in the proof of Proposition 3.1,

$$(\mathscr{E}_A \psi, \psi') - (\psi, \mathscr{E}_A \psi') = (\omega + \psi_+, \sigma(\omega' + \psi'_+)).$$

Since $\psi, \psi' \in$ Ker $\mathscr{E}_A$, the left-hand side is zero. The right side is equal to

$$-\int_\Sigma \operatorname{tr}(\omega \wedge \omega'),$$

as $\psi_+ \in P_+$ and $\sigma(\psi') \in P_-$ and they are orthogonal to each other. It follows that the symplectic pairing vanishes identically on L_A. It remains to show that $\dim L_A = 3g - 3$. Assume that $\psi \in$ Ker π_A. Then $\psi | \partial X \in P_+$ and this implies that $\psi \in$ Ker $\mathscr{E}_A^*$. Conversely an element of Ker $\mathscr{E}_A^*$ corresponds to an element of Ker π_A. Thus Ker π_A is identical to Ker $\mathscr{E}_A^*$. Hence

$$\begin{aligned} \dim L_A &= \dim \operatorname{Ker} \mathscr{E}_A - \dim \operatorname{Ker} \pi_A \\ &= \dim \operatorname{Ker} \mathscr{E}_A - \dim \operatorname{Ker} \mathscr{E}_A^* \\ &= \operatorname{index} \mathscr{E}_A = 3g - 3. \end{aligned}$$

q.e.d.

5. Generic smooth path of smooth connections

Let X be a compact oriented 3-dimensional Riemannian manifold with boundary $\partial X = \Sigma$ a connected surface of genus g (≥ 2) as in Section 4.

Let $\{A_t\}_{0 \leq t \leq 1}$ be a smooth path of smooth connections on $X \times \mathrm{SU}(2)$ connecting irreducible flat connections A_0 and A_1 such that, for $0 \leq t \leq 1$, A_t restricts to a product $B_t \times 1$ on $\Sigma \times [0, 1] \times \mathrm{SU}(2)$ for an irreducible flat connection on $\Sigma \times \mathrm{SU}(2)$. By Definition 4.1, we obtain one-parameter families of Fredholm operators, $\{\mathscr{E}_{A_t}\}_{0 \leq t \leq 1}$ and $\{\mathscr{E}_{A_t}^*\}_{0 \leq t \leq 1}$. Each of these depends smoothly on t. By Lemma 4.3, we obtain a one-parameter family of the Lagrangians, $\{L_{A_t}\}_{0\le t\le 1}$, in $\{\mathscr{H}_{B_t}\}_{0\le t\le 1}$. Each $\mathscr{H}_{B_t}$ is mutually isomorphic as a symplectic vector space for $0\le t\le 1$. If we regard $\{\mathscr{H}_{B_t}\}_{0\le t\le 1}$ to be a symplectic vector bundle over $[0,1]$ with fiber $\mathscr{H}_{B_t}$, there is a trivialization

$$\Theta_t\colon \mathscr{H}_{B_t}\to \mathbf{V}$$

where $\mathbf{V}$ is a fixed non-degenerate $(6g-6)$-dimensional real symplectic vector space and Θ_t is an isomorphism of symplectic vector spaces continuously depending on t. Using Θ_t, we obtain a Lagrangian $\Theta_t(L_{A_t})$ in $\mathbf{V}$ which we denote also by L_{A_t}. Let $\mathscr{L}$ be the space of all the Lagrangians in $\mathbf{V}$. $\mathscr{L}$ is endowed with the topology as a subspace of the Grassmann variety of all the $(3g-3)$-dimensional subspaces in $\mathbf{V}$. Thus we get a map

$$[0,1]\ni t\to L_{A_t}\in\mathscr{L}.$$

LEMMA 5.1. *Let $\{A_t\}_{0\le t\le 1}$ be a smooth path of smooth connections on $X\times SU(2)$ such that, for $0\le t\le 1$, A_t restricts to $B_t\times 1$ on $\Sigma\times[0,1]\times SU(2)$ for an irreducible flat connection B_t on $\Sigma\times SU(2)$. Let $\{L_{A_t}\}$ be the corresponding one-parameter family of the Lagrangians. Assume that, for $0\le t\le 1$, $\operatorname{Ker}\mathscr{E}^*_{A_t}=0$. Then $\{L_{A_t}\}_{0\le t\le 1}$ is a continuous path of Lagrangians.*

Proof. Since $\mathscr{E}^*_{A_t}$ is the adjoint operator of $\mathscr{E}_{A_t}$,

$$\operatorname{Ker}\mathscr{E}_{A_t}=\operatorname{Image}\mathscr{E}^{*\,\perp}_{A_t}.$$

Here the right-hand side denotes the L^2-orthogonal complement of the L^2-closure of the image of $\mathscr{E}^*_{A_t}$ in $\mathbf{H}=$ the L^2-completion of $(\Omega^1\oplus\Omega^0)(X)\otimes su(2)$. The assumption $\operatorname{Ker}\mathscr{E}^*_{A_t}=0$ for $0\le t\le 1$ implies that $\{\operatorname{Image}\mathscr{E}^*_{A_t}\}_{0\le t\le 1}$ is a continuous family of the closed subspaces in the Hilbert space $\mathbf{H}$. Hence $\{\operatorname{Ker}\mathscr{E}_{A_t}=\operatorname{Image}\mathscr{E}^{*\,\perp}_{A_t}\}_{0\le t\le 1}$ is a continuous family of $(3g-3)$-dimensional subspaces in $\mathbf{H}$. Since the map $\pi_{A_t}\colon\operatorname{Ker}\mathscr{E}_{A_t}\to\mathscr{H}_{B_t}$ is injective for $0\le t\le 1$, $\{L_{A_t}\}_{0\le t\le 1}$ is continuous. q.e.d.

Definition 5.1. A smooth path of smooth connections, $\{A_t\}_{0\le t\le 1}$, on $X\times SU(2)$ such that, for $0\le t\le 1$, A_t restricts to $B_t\times 1$ on $\Sigma\times[0,1]\times SU(2)$ for an irreducible flat connection B_t on $\Sigma\times SU(2)$ is called generic if $\operatorname{Ker}\mathscr{E}^*_{A_t}=0$ for $0\le t\le 1$.

Thus by Lemma 5.1, for a generic path of smooth connections $\{A_t\}_{0\le t\le 1}$, the corresponding path of Lagrangians, $\{L_{A_t}\}_{0\le t\le 1}$, is continuous.

To get a generic path we adopt the following perturbations of metrics and connections.

As in Section 4, for $0 \leq r < \infty$, let $X(r)$ be the manifold defined by

$$X(r) = \Sigma \times [-r, 0] \cup X$$

where $\Sigma \times \{0\}$ is identified with ∂X. Let A be a smooth connection on $X \times SU(2)$ which restricts to $B \times 1$ on $\Sigma \times [0, 1] \times SU(2)$ for an irreducible flat connection B on $\Sigma \times SU(2)$. Then A can be extended to a connection, $A(r)$, on $X(r) \times SU(2)$ by putting $A = B \times 1$ on $\Sigma \times [-r, 0] \times SU(2)$.

For $\psi = (a, b) \in (\Omega^1 \oplus \Omega^0)(X) \otimes su(2)$, as in Section 3, on $\Sigma \times [-r, 0]$ we write

$$\begin{aligned} \psi(s) &= (a(s), b(s)) \\ &= (p(s), q(s), b(s)) \end{aligned}$$

where $p(s) \in \Omega^1(\Sigma) \otimes su(2)$, $q(s), b(s) \in \Omega^0(\Sigma) \otimes su(2)$ and $a(s) = p(s) + q(s)\, ds$ for $-r \leq s \leq 0$.

As in Section 3, let $\{\mu\}$ be the set of the eigenvalues of D_B and let $\{\psi_\mu\}_\mu$ be the set of the corresponding eigenforms of D_B which forms an L^2-orthonormal basis of the L^2-completion of $(\Omega^1 \oplus \Omega^0 \oplus \Omega^0)(\Sigma) \otimes su(2)$.

Lemma 5.2. *If $\psi \neq 0 \in \operatorname{Ker} \mathscr{E}^*_{A(r)}$, then, for each $-r \leq s \leq 0$, at least one of $q(s)$ and $b(s)$ is non-zero.*

Proof. On $\Sigma \times [-r, 0]$, ψ can be written as

$$\psi(s) = \sum_{\mu > 0} c_\mu e^{s\mu} \psi_\mu$$

for some constant $\{c_\mu\}$. Let $d_B^* d_B$ be the Laplace operator on $\Omega^0(\Sigma) \otimes su(2)$ with respect to the connection B. Since B is an irreducible flat connection, $d_B^* d_B$ has only positive eigenvalues $\{\lambda\}$ which are related to the non-zero eigenvalues $\{\mu\}_{\mu \neq 0}$ of D_B as $\{\lambda\} = \{\mu^2\}_{\mu \neq 0}$. Moreover the eigenforms of D_B corresponding to the positive eigenvalues, $\{\psi_\mu\}_{\mu > 0}$, are given as forms in $(\Omega^1 \oplus \Omega^0 \oplus \Omega^0)(\Sigma) \otimes su(2)$ by

$$\psi_\mu = \left(d_B \phi_\lambda, \sqrt{\lambda}\, \phi_\lambda, 0\right) \qquad \text{or}$$

$$\psi_\mu = \left(* \, d_B \phi_\lambda, 0, \sqrt{\lambda}\, \phi_\lambda\right)$$

where $\mu = \sqrt{\lambda}$ and $\{\phi_\lambda\}_\lambda$ is a set of the eigenforms of $d_B^* d_B$ which forms an L^2-basis of the L^2-completion of $\Omega^0(\Sigma) \otimes su(2)$.

Therefore $q(s)$ and $b(s)$ can be written as

$$q(s) = \sum f_\lambda e^{s\sqrt{\lambda}} \sqrt{\lambda}\, \phi_\lambda,$$

$$b(s) = \sum g_\lambda e^{s\sqrt{\lambda}} \sqrt{\lambda}\, \phi_\lambda$$

where $\{f_\lambda\}$ and $\{g_\lambda\}$ are sets of constants such that $c_\mu = f_\lambda$ or $c_\mu = g_\lambda$ for each $\mu = \sqrt{\lambda}$. Since ψ is a solution of an elliptic differential equation with smooth coefficients, by the unique continuation theorem, $\psi(s) \neq 0$ for $-r \leq s \leq 0$. Hence the lemma follows from the above expressions. q.e.d.

Definition 5.2. For $r \geq 2$, a simple perturbation of $A(r)$ is defined to be a connection on $X \times \mathrm{SU}(2)$ given by $A(r) + \alpha$, where $\alpha \in \Omega^1(X(r)) \otimes \mathrm{su}(2)$ is a smooth 1-form such that (1) the support of α is contained in $\Sigma \times [-r+1, -r+2]$ and (2), for $-r+1 \leq s \leq -r+2$, $\alpha(s) \in \Omega^1(\Sigma) \otimes \mathrm{su}(2)$; that is, it has no ds-component.

In this section, from now on, we use the notation

$$[\beta, \eta] = \beta \wedge \eta + (-1)^{ij+1} \eta \wedge \beta$$

for an su(2)-valued i-form β and an su(2)-valued j-form η.

Suppose that $\mathscr{E}_A^*$ has a non-trivial kernel $\psi = (a, b) \in (\Omega^1 \oplus \Omega^0)(X) \otimes \mathrm{su}(2)$. Then for a simple perturbation of $A(r)$, $A(r) + \alpha$,

$$D_{A(r)+\alpha}(\psi) = (*[\alpha, a] + [\alpha, b], *[\alpha, *a]) \in (\Omega^1 \oplus \Omega^0)(X) \otimes \mathrm{su}(2). \tag{5.1}$$

The support of $D_{A(r)+\alpha}(\psi)$ is contained in $\Sigma \times [-r+1, -r+2]$. Using the expression $\psi(s) = (p(s), q(s), b(s))$, we write (5.1) as

$$D_{A(r)+\alpha}(\psi) = ([q, *\alpha] + [\alpha, b], *[\alpha, p], *[\alpha, *p]) \in (\Omega^1 \oplus \Omega^0 \oplus \Omega^0)(\Sigma) \otimes \mathrm{su}(2) \tag{5.2}$$

where $*$ denotes the Hodge star operator on $\Omega^i(\Sigma) \otimes \mathrm{su}(2)$ ($i = 0, 1, 2$).

Let $(\omega, 0)$ be an element of $(\Omega^1 \oplus \Omega^0)(\Sigma \times [-r, 0]) \otimes \mathrm{su}(2)$ where ω is an element of $\mathscr{H}_B$, considered to be a 1-form on $\Sigma \times [-r, 0]$, constant with respect to s. We denote $(\omega, 0)$ simply by ω. Consider the L^2-inner product on $\Sigma \times [-r, 0]$,

$$(\omega, D_{A(r)+\alpha}(\psi)) = \int_{-r+1}^{-r+2} ds(\alpha(s), [q(s), *\omega] + [b(s), \omega]) \tag{5.3}$$

where (,) in the integral denotes the L^2-inner product on Σ. We consider the 1-form $[q(s), *\omega] + [b(s), \omega]$ which appeared in the above integral.

Lemma 5.3. *Let q, $b \in \Omega^0(\Sigma) \otimes \mathrm{su}(2)$ be such that at least one of q and b is nonzero. Define the linear map*

$$\tau \colon \mathscr{H}_B \to \Omega^1(\Sigma) \otimes \mathrm{su}(2),$$

for $\omega \in \mathscr{H}_B$, *by*

$$\tau(\omega) = [q, *\omega] + [b, \omega].$$

Then $\dim_{\mathbf{R}} \tau(\mathscr{H}_B) \geq 4g - 4$.

Proof. Assume that $\tau(\omega) = 0$. This implies that

$$\omega = \eta \wedge b + *\eta \wedge q$$

for some $\eta \in \Omega^1(\Sigma)$. We give Σ the complex structure compatible with the conformal structure on Σ coming from the metric. As is well-known ([A-B]), the flat connection B on $\Sigma \times \mathrm{SU}(2)$ defines a holomorphic structure on the complexified adjoint bundle $\mathrm{ad}_{\mathbf{C}} = \Sigma \times (\mathrm{su}(2) \otimes \mathbf{C})$. We set

$$\begin{aligned} \varphi &= \omega + \sqrt{-1} * \omega \\ &= (\eta + \sqrt{-1} * \eta) \wedge (b - \sqrt{-1}q). \end{aligned} \tag{5.3.1}$$

Then φ is a cross-section of $T^*\Sigma \otimes \mathrm{ad}_{\mathbf{C}}$, where $T^*\Sigma$ denotes the holomorphic cotangent bundle of Σ. The fact that ω is d_B-harmonic is equivalent to the fact that φ is holomorphic with respect to the above complex structure on $\mathrm{ad}_{\mathbf{C}}$. If $\omega \neq 0$, then the complex linear span of φ determines a holomorphic sub-line bundle $\mathbf{L}$ in $T^*\Sigma \otimes \mathrm{ad}_{\mathbf{C}}$ and $\mathrm{Ker}\, \tau$ is in one-to-one correspondence with the space of the holomorphic cross-section of $\mathbf{L}$ by (5.3.1). We can find a smooth path of flat connections $\{B'_t\}_{0 \leq t \leq 1}$ on $\Sigma \times \mathrm{SU}(2)$ such that (1) $B_0 = B$ and $B_1 =$ the trivial connection and (2) for $0 \leq t < 1$, B_t is irreducible. Hence we can deform ω continuously to ω_0 which is an SU(2)-valued harmonic 1-form with respect to the trivial connection. It follows that $\mathbf{L}$ is topologically equivalent to $T^*\Sigma$. By the Riemann-Roch theorem and the irreducibility of B, the complex dimension of the space of all the holomorphic cross-sections of $\mathbf{L}$ is equal to $g - 1$. Hence in any case, $\dim_{\mathbf{R}} \mathrm{Ker}\, \tau \leq 2g - 2$ and we get the lemma. q.e.d.

COROLLARY OF LEMMA 5.3. *Let* q, b *and* τ *be as in Lemma* 5.3. *Let* L *be a Lagrangian in* $\mathscr{H}_B$. *Then* $\dim_{\mathbf{R}} \tau(L) \geq 2g - 2$.

Proof. Since $\mathrm{Ker}\, \tau$ is a $*$-closed subspace in $\mathscr{H}_B$, for any Lagrangian L in $\mathscr{H}_B$, $\dim_{\mathbf{R}}(L \cap \mathrm{Ker}\, \tau) \leq (1/2)\dim_{\mathbf{R}} \mathrm{Ker}\, \tau$. The corollary follows from Lemma 5.3. q.e.d.

For $\psi \in \mathrm{Ker}\, \mathscr{E}_{A(r)}$, on $\Sigma \times [-r, 0]$, ψ can be written as

$$\psi(s) = \omega + \psi_+(s) \tag{5.4}$$

where $\omega \in L_A$ and $\psi_+(s) = \sum_{\mu > 0} c_\mu e^{s\mu} \psi_\mu$ for some constants $\{c_\mu\}$. In particular, for $\psi \in \mathrm{Ker}\, \mathscr{E}^*_{A(r)}$, $\psi = \psi_+$. From the expression (5.4), the next two lemmas follow:

LEMMA 5.4. *For* $0 \leq r' < r$,

$$\operatorname{Ker} \mathscr{E}_{A(r)} = \operatorname{Ker} \mathscr{E}_{A(r')}, \quad \text{and}$$
$$\operatorname{Ker} \mathscr{E}^*_{A(r)} = \operatorname{Ker} \mathscr{E}^*_{A(r')}.$$

LEMMA 5.5. *There are positive constants* c_4 *and* c_5 *both of which do not depend on* r *such that, for* $r \geq 2$ *and* $\psi \in \operatorname{Ker} \mathscr{E}_{A(r)}$

$$\|\psi_+(s)\|_{L^2} < c_4 e^{c_5 s} \|\psi\|_{L^2};$$

in particular, for $\psi \in \operatorname{Ker} \mathscr{E}^*_{A(r)}$,

$$\|\psi(s)\|_{L^2} < c_4 e^{c_5 s} \|\psi\|_{L^2}$$

for $-r \leq s \leq 0$.

Let

$$\operatorname{Ker} \mathscr{E}_{A(r)} = \tilde{L}_{A(r)} + \operatorname{Ker} \mathscr{E}^*_{A(r)}$$

be the orthogonal decomposition of $\operatorname{Ker} \mathscr{E}_A$ with respect to the L^2-inner product. Then

$$\pi_{A(r)} | \tilde{L}_{A(r)} \colon \tilde{L}_{A(r)} \to L_A$$

is an isomorphism.

LEMMA 5.6. *Suppose* $\operatorname{Ker} \mathscr{E}^*_A \neq 0$ *and let* ψ *be a non-trivial element of* $\operatorname{Ker} \mathscr{E}^*_{A(r)}$. *Let* P *be the* L^2*-orthogonal projection of the* L^2*-completion of* $(\Omega^1 \oplus \Omega^0)(X(r)) \otimes \mathrm{su}(2)$ *onto* $\tilde{L}_A$. *Then there is a linear subspace* $L'_A \subset L_A$ *with* $\dim_{\mathbf{R}} L'_A \geq 2g - 2$, *such that, for* $\omega \in L'_A$ *with* $\|\omega\|_{L^2} = 1$ *and sufficiently large* $r > 0$, *there is a simple perturbation* $A(r) + \alpha$ (α *depends on* ω) *such that*

$$(5.6.1) \qquad \big(\psi_\omega, P\big(D_{A(r)+\alpha}(\psi)\big)\big) \neq 0$$

where $\psi_\omega \in \tilde{L}_{A(r)}$ *with* $\pi_{A(r)}(\psi_\omega) = \omega$ *and* (,) *denotes the* L^2*-inner product. Moreover* α *can be chosen arbitrarily small.*

Proof. By the corollary of Lemma 5.3, we have a subspace $L'_A \subset L_A$ with $\dim_{\mathbf{R}} L'_A \geq 2g - 2$ such that $L'_A \cap \operatorname{Ker} \tau = \{0\}$; here τ is the map defined in Lemma 5.3. Let $\omega \in L'_A$ with $\|\omega\|_{L^2} = 1$. (5.6.1) is equivalent to

$$(5.6.2) \qquad \big(\psi_\omega, D_{A(r)+\alpha}(\psi)\big) \neq 0.$$

On $\Sigma \times [-r, 0]$, ψ_ω can be written as in (5.4),

$$\psi_\omega(s) = \omega + \psi_{\omega^+}(s).$$

Now (5.6.2) is equal to

$$(\omega, D_{A(r)+\alpha}(\psi)) + (\psi_{\omega^+}, D_{A(r)+\alpha}(\psi)). \tag{5.6.3}$$

By Lemma 5.5, we see that there is a positive constant c_6 not depending on r such that

$$(\psi_{\omega^+}, D_{A(r)+\alpha}(\psi)) < c_6\, e^{-c_5 r}(\omega, D_{A(r)+\alpha}(\psi)).$$

Hence for sufficiently large $r > 0$, we can neglect the second term in (5.6.3). By (5.3) and Lemma 5.3, there is an α which satisfies (5.6.2). Moreover for any such α, $\varepsilon\alpha$ also satisfies (5.6.2) for $\varepsilon > 0$. Hence α can be chosen arbitrarily small. q.e.d.

PROPOSITION 5.1. *Let $\{A_t\}_{0\le t\le 1}$ be a smooth path of smooth connections on $X \times \mathrm{SU}(2)$ connecting two irreducible flat connections A_0 and A_1 such that, for $0 \le t \le 1$, A_t restricts to a product $B_t \times 1$ on $\Sigma \times [0,1] \times \mathrm{SU}(2)$ for an irreducible flat connection B_t on $\Sigma \times \mathrm{SU}(2)$. Suppose $\operatorname{Ker} \mathscr{E}^*_{A_0} = 0 = \operatorname{Ker} \mathscr{E}^*_{A_1}$. Then, for sufficiently large $r > 0$, there is a smooth generic path of smooth connections $\{A'_t\}_{0\le t\le 1}$ on $X(r) \times \mathrm{SU}(2)$ such that (1) $A'_0 = A_0(r)$ and $A'_1 = A_1(r)$ and (2) for $0 \le t \le 1$, A'_t restricts to a product $B'_t \times 1$ on $\Sigma \times [-r, -r+1] \times \mathrm{SU}(2)$ for an irreducible flat connection on $\Sigma \times \mathrm{SU}(2)$. Moreover $\{A'_t\}_{0\le t\le 1}$ can be chosen arbitrarily close to $\{A_t(r)\}_{0\le t\le 1}$.*

Proof. Let $\mathbf{F}$ be the Banach manifold of all the real bounded Fredholm operators of index $3 - 3g$ on real separable Hilbert space. For $k \ge 0$, set

$$\mathbf{F}^{(k)} = \{T \in \mathbf{F} \mid \dim \ker T \ge k\}.$$

Then $\mathbf{F}^{(k)}$ is a closed subvariety of $\mathbf{F}$ of finite codimension $k(3g - 3 + k)$. Also, $\mathbf{F}^{(k)}$ has normal bundle in $\mathbf{F}$ on $\mathbf{F}^{(k)} - \mathbf{F}^{(k+1)}$ whose fiber at $T \in \mathbf{F}^{(k)} - \mathbf{F}^{(k+1)}$ is identified with the finite-dimensional linear space

$$\operatorname{Hom}_{\mathbf{R}}(\operatorname{Ker} T, \operatorname{Coker} T)$$

([Ko]). For $\{A_t\}_{0\le t\le 1}$ and $r \ge 0$, regarding $\{\mathscr{E}^*_{A_t(r)}\}_{0\le t\le 1}$ to be a path of bounded Fredholm operators of index $(3 - 3g)$ from the L^2_1-completion of $\Omega(X(r), P_+)$ to the L^2-completion of $(\Omega^1 + \Omega^0)(X(r)) \otimes \mathrm{su}(2)$, we get a smooth path in $\mathbf{F}$, $\{\mathscr{E}^*_{A_t}\}_{0\le t\le 1}$, such that $\mathscr{E}^*_{A_0}, \mathscr{E}^*_{A_1} \in \mathbf{F}^{(0)} - \mathbf{F}^{(1)}$. The space of all the irreducible representations of $\pi_1(\Sigma)$ to $\mathrm{SU}(2)$ is a smooth manifold. First we perturb $\{A_t\}_{0\le t\le 1}$ so that the curve $\{B_t\}_{0\le t\le 1}$ has nonzero tangent vector at each $0 \le t \le 1$ when it is regarded as a smooth curve in the representation space of $\pi_1(\Sigma)$ to $\mathrm{SU}(2)$. This can be achieved by arbitrarily small perturbations. Then the curve $\{\mathscr{E}^*_{A_t}\}_{0\le t\le 1}$ in $\mathbf{F}$ also has a nonzero tangent vector at each $0 \le t \le 1$. Suppose $\mathscr{E}^*_{A_u} \in \mathbf{F}^{(k)} - \mathbf{F}^{(k+1)}$ for some $0 < u < 1$ and $k \ge 1$. By Lemma 5.4,

$\mathscr{E}^*_{A_u(r)} \in \mathbf{F}^{(k)} - \mathbf{F}^{(k+1)}$ for $r \geq 0$. Let $A_u(r) + \alpha$ be a simple perturbation of $A_u(r)$. Let P be the L^2-orthogonal projection of the L^2-completion of $(\Omega^1 \oplus \Omega^0)(X(r)) \otimes \mathrm{su}(2)$ onto Coker $\mathscr{E}^*_{A_u(r)}$. Then the correspondence

$$\mathrm{Ker}\, \mathscr{E}^*_{A_u(r)} \ni \psi \to P\left(D_{A_u(r)+\alpha}\psi\right) \in \mathrm{Coker}\, \mathscr{E}^*_{A_u(r)}$$

determines an element σ_α of $\mathrm{Hom}_{\mathbf{R}}(\mathrm{Ker}\, \mathscr{E}^*_{A_u(r)}, \mathrm{Coker}\, \mathscr{E}^*_{A_u(r)})$ which is the fiber of the normal bundle of $\mathbf{F}^{(k)}$ at $\mathscr{E}^*_{A_u(r)}$. By Lemma 5.6, for sufficiently large $r > 0$, taking various α, we have at least $(2g-2)$-dimensional perturbations $\{\mathscr{E}^*_{A_u(r)+\alpha}\}_\alpha$ of $\mathscr{E}^*_{A_u(r)}$ in the normal direction of $\mathbf{F}^{(k)}$. In particular, we can choose α so that σ_α is linearly independent of the normal vector to $\mathbf{F}^{(k)}$ at $\mathscr{E}^*_{A_u}$ determined by the tangent vector of the curve $\{\mathscr{E}^*_{A_t(r)}\}_{0 \leq t \leq 1}$ at $t = u$. Moreover we can chose an arbitrarily small such α. Since $\mathbf{F}^{(k)} - \mathbf{F}^{(k+1)}$ is open in $\mathbf{F}^{(k)}$, for a sufficiently small such α, $\mathscr{E}^*_{A_u(r)+\alpha} \in \mathbf{F}^{(k')}$ for $0 \leq k' < k$. The proposition follows from the compactness of $[0, 1]$ and the transversality argument. q.e.d.

6. Operators and invariants associated with splittings of manifolds

Let M be an oriented closed Riemannian 3-manifold. Let M_1 and M_2 be connected submanifolds of codimension 0 such that $M = M_1 \cup M_2$ and $M_1 \cap M_2 = \partial M_1 = \partial M_2 = \Sigma$ is a connected orientable surface of genus g (≥ 2). We orient Σ as the boundary of M_1. We assume that, near Σ, M is isometric to the product Riemannian manifold $\Sigma \times [-1, 1]$ with $\Sigma = \Sigma \times \{0\}$, $\Sigma \times [-1, 0] \subset M_1$ and $\Sigma \times [0, 1] \subset M_2$.

Definition 6.1. Let A be a smooth connection on $M \times \mathrm{SU}(2)$ which restricts to a product $B \times 1$ on $\Sigma \times [-1, 1] \times \mathrm{SU}(2)$ for an irreducible flat connection B on $\Sigma \times \mathrm{SU}(2)$. For $i = 1, 2$, we define the operators:

(1) $\mathscr{E}^i_A = \mathscr{E}_A$ in Definition 4.1(1) for $X = -M_1$ $(i = 1)$ and for $X = M_2$ $(i = 2)$, where $-M_1$ is M_1 with the opposite orientation, and

(2) $\mathscr{E}^{i*}_A = \mathscr{E}^*_A$ in Definition 4.1(2) for $X = -M_1$ $(i = 1)$ and for $X = M_2$ $(i = 2)$.

Let π^i: Ker $\mathscr{E}^i_A \to \mathscr{H}_B$ be the map defined in Definition 4.2 for $\mathscr{E}_A = \mathscr{E}^i_A$ $(i = 1, 2)$. By Lemma 4.3, $L^i_A = \pi^i(\mathrm{Ker}\, \mathscr{E}^i_A)$ is a Lagrangian of $\mathscr{H}_B$.

Let $\{A_t\}_{0 \leq t \leq 1}$ be a smooth one-parameter family of smooth connections on $M \times \mathrm{SU}(2)$ such that, for $0 \leq t \leq 1$, A_t restricts to a product $B_t \times 1$ on $\Sigma \times [-1, 1] \times \mathrm{SU}(2)$ for an irreducible flat connection B_t on $\Sigma \times \mathrm{SU}(2)$.

For $0 \leq t \leq 1$ and $i = 1, 2$, the following operators have been defined:

$\left\{D_{A_t}\right\}_{0 \leq t \leq 1}$, defined in Definition 2.1,

$\left\{\mathscr{E}^i_{A_t}\right\}_{0 \leq t \leq 1}$, defined in Definition 5.1(1),

$\left\{\mathscr{E}^{i*}_{A_t}\right\}_{0 \leq t \leq 1}$, defined in Definition 5.1(2).

The first is a one-parameter family of self-adjoint Fredholm operators, and the second and the third are one-parameter families of Fredholm operators and their adjoints. Each of these depends smoothly on t.

Definition 5.1$'$. Let $\{A_t\}_{0 \leq t \leq 1}$ be as above. $\{A_t\}_{0 \leq t \leq 1}$ is called *generic* if, for $i = 1, 2$ and $0 \leq t \leq 1$, Ker $\mathscr{E}^{i*}_{A_t} = 0$.

We assume that $\{A_t\}_{0 \leq t \leq 1}$ is generic.

For $0 \leq t \leq 1$,

$$\mathscr{H}_{B_t} = \left\{\omega \in \Omega^1(\Sigma) \otimes \mathrm{su}(2) | d_{B_t}\omega = d^*_{B_t}\omega = 0\right\}$$

is a $(6g - 6)$-dimensional real vector space with the non-degenerate symplectic structure defined in Section 3. Let π^i_t: Ker $\mathscr{E}^i_{A_t} \to \mathscr{H}_{B_t}$ be the map defined in Definition 4.2 for $0 \leq t \leq 1$ and $i = 1, 2$. $L^i_{A_t} = \pi^i_t(\text{Ker}\, \mathscr{E}^i_{A_t})$ is a Lagrangian of $\mathscr{H}_{B_t}$ by Lemma 4.3.

As in Section 5, there is a trivialization

$$\Theta_t \colon \mathscr{H}_{B_t} \to \mathbf{V}$$

where $\mathbf{V}$ is a fixed non-degenerate $(6g - 6)$-dimensional real symplectic vector space and Θ_t is an isomorphism of symplectic vector spaces continuously depending on t $(0 \leq t \leq 1)$. We choose and fix such a trivialization. Using Θ_t we obtain a Lagrangian $\Theta_t(L^i_{A_t})$ of $\mathbf{V}$, where $L^i_{A_t} = \pi^i_t(\text{Ker}\, \mathscr{E}^i_{A_t})$ as above $(i = 1, 2)$. For simplicity, from now on, we denote $\Theta_t(L^i_{A_t})$ also by $L^i_{A_t}$.

By the assumption Ker $\mathscr{E}^{i*}_{A_t} = 0$ for $0 \leq t \leq 1$ and Lemma 5.1, $\{L^i_{A_t}\}_{0 \leq t \leq 1}$ is a continuous one-parameter family of Lagrangians in $\mathbf{V}$ $(i = 1, 2)$.

As in Section 5, let $\mathscr{L}$ be the space of all the Lagrangians of $\mathbf{V}$. Let $\mathscr{L}^2 = \mathscr{L} \times \mathscr{L}$ be the space of all the Lagrangian pairs of $\mathbf{V}$. Then $\{(L^1_{A_t}, L^2_{A_t})\}_{0 \leq t \leq 1}$ is a continuous path in $\mathscr{L}^2$.

We study $\mathscr{L}^2$ in the subsequent paragraph. Let $\langle\ ,\ \rangle$ denote the symplectic pairing of $\mathbf{V}$. We choose and fix a symplectic base of $\mathbf{V}$, $\{x_1, \dots, x_{3g-3}, y_1, \dots, y_{3g-3}\}$, where $\langle x_j, y_k\rangle = \delta_{jk}$ and $\langle x_j, x_k\rangle = \langle y_j, y_k\rangle = 0$ for $1 \leq j, k \leq 3g - 3$. For $v_1, \dots, v_s \in \mathbf{V}$, $\{v_1, \dots, v_s\}$ denotes the subspace of $\mathbf{V}$ spanned by $v_1, \dots, v_s$. For a subspace U of $\mathbf{V}$, $U^\perp$ denotes the symplectic orthogonal complement of U; $U^\perp = \{v \in \mathbf{V} | \langle v, x\rangle = 0, \text{ for all } x \in U\}$. Let $\mathrm{Sp}(3g - 3, \mathbf{R})$ be the group of all the symplectic automorphisms of $\mathbf{V}$. Choosing a symplectic base of $\mathbf{V}$ as above, we identify $\mathrm{Sp}(3g - 3, \mathbf{R})$ with the group consisting of all the $(6g-6)$ by $(6g-6)$ real matrices T such that

$$ {}^tT\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} T = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} $$

where I denotes the $(3g-3)$ by $(3g-3)$ unit matrix. Thus $\mathrm{Sp}(3g-3, \mathbf{R})$ is a noncompact Lie group of dimension $2(3g-3)^2 + (3g-3)$. The maximal compact subgroup of $\mathrm{Sp}(3g-3, \mathbf{R})$ is the unitary group $\mathrm{U}(3g-3)$, and $\mathrm{Sp}(3g-3, \mathbf{R}) = \mathrm{U}(3g-3) \times \mathbf{R}^{(3g-3)^2+(3g-3)}$ as C^∞ manifolds. Hence $\pi_1(\mathrm{Sp}(3g-3, \mathbf{R})) = \pi_1(\mathrm{U}(3g-3)) = \mathbf{Z}$ and it is generated by the homotopy class of the loop $\{l(\theta)\}_{0 \le \theta < 2\pi}$, where $l(\theta)$ denotes the rotation

$$ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} $$

in the (x_1, y_1)-plane and the identity on $\{x_1, y_1\}^\perp$.

Let $L_0 = \{x_1, \dots, x_{3g-3}\}$ be the standard Lagrangian of $\mathbf{V}$. For $L \in \mathscr{L}$, there is a (not unique) $h \in \mathrm{Sp}(3g-3, \mathbf{R})$ such that $L = hL_0$ (the translation image of L_0). Thus $\mathscr{L}$ is a homogeneous space $\mathrm{Sp}(3g-3, \mathbf{R})/G_0$, where G_0 is the subgroup of $\mathrm{Sp}(3g-3, \mathbf{R})$ defined by

$$ G_0 = \left\{ \begin{pmatrix} R & S \\ 0 & {}^tR^{-1} \end{pmatrix} \in \mathrm{Sp}(3g-3, \mathbf{R}) \middle| \begin{array}{l} \det R \neq 0 \\ {}^t(R^{-1}S) = R^{-1}S \end{array} \right\}. $$

G_0 is a real Lie group of dimension $(3(3g-3)^2 + (3g-3))/2$.

Lemma 6.1. *$\mathscr{L}$ is a homogeneous space of dimension $((3g-3)^2 + (3g-3))/2$ and $\pi_1(\mathscr{L}) = \mathbf{Z}$.*

Proof. The first statement follows from the above paragraph. From the homotopy exact sequence of the fibration

$$ G_0 \to \mathrm{Sp}(3g-3, \mathbf{R}) \to \mathscr{L}, $$

we have

$$ \pi_1(G_0) \to \pi_1(\mathrm{Sp}(3g-3, \mathbf{R})) \to \pi_1(\mathscr{L}) \to \pi_1(G_0) \to 1. $$

Since $\pi_1(G_0) = \pi_1(\mathrm{GL}(3g-3, \mathbf{R})) = \mathbf{Z}_2$ and $\pi_0(G_0) = \pi_0(\mathrm{GL}(3g-3, \mathbf{R})) = \mathbf{Z}_2$, we obtain the exact sequence

$$ 1 \to \mathbf{Z} \to \pi_1(\mathscr{L}) \to \mathbf{Z}_2 \to 1. $$

Hence $\pi_1(\mathscr{L}) = \mathbf{Z}$. q.e.d.

By the proof of the above lemma, we see that the homotopy class of the loop $\{l(\theta)L_0\}_{0 \le \theta < \pi}$ is a generator of $\pi_1(\mathscr{L})$.

For $0 \le k \le 3g-3$, we set

$$ \mathscr{L}_k^2 = \{(L^1, L^2) \in \mathscr{L}^2 | \dim(L^1 \cap L^2) = k\}, $$

and we set

$$\mathscr{L}^2_{(k)} = \bigcup_{j=k}^{3g-3} \mathscr{L}^2_j.$$

Then $\mathscr{L}^2 = \mathscr{L}^2_{(0)} \supset \mathscr{L}^2_{(1)} \supset \mathscr{L}^2_{(2)} \supset \cdots \supset \mathscr{L}^2_{(3g-3)}$ is a stratification of $\mathscr{L}^2$.

LEMMA 6.2. *$\mathscr{L}^2$ is a C^∞ manifold of dimension $((3g-3)^2 + (3g-3))$. For $0 \le k \le (3g-3)$, $\mathscr{L}^2_k$ is a connected C^∞ submanifold of $\mathscr{L}^2$ of codimension $(k^2+k)/2$, and $\mathscr{L}^2_{(k)}$ is a real analytic subspace of $\mathscr{L}^2$.*

Proof. The first statement is obvious by Lemma 6.1. Let L_0 be the standard Lagrangian and let $L'_k = \{x_1, \ldots, x_k, y_{k+1}, \ldots, y_{3g-3}\}$. Then $(L_0, L'_k) \in \mathscr{L}^2_k$ for $0 < k < 3g-3$. For $(L, L') \in \mathscr{L}^2_k$, we can choose a symplectic base of $\mathbf{V}$, $\{u_1, \ldots, u_{3g-3}, v_1, \ldots, v_{3g-3}\}$, such that $\langle u_j, u_q\rangle = \langle v_j, v_q\rangle = 0$, $\langle u_j, v_q\rangle = \delta_{jq}$ $(1 \le j, q \le 3g-3)$ and

$$L = \{u_1, \ldots, u_{3g-3}\}, \qquad L' = \{u_1, \ldots, u_k, v_{k+1}, \ldots, v_{3g-3}\}.$$

Hence $(L, L') = (hL_0, hL'_k)$ for some $h \in \mathrm{Sp}(3g-3, \mathbf{R})$. Thus $\mathscr{L}^2_k$ is a homogeneous space $\mathrm{Sp}(3g-3, \mathbf{R})/\tilde{G}_k$, where $\tilde{G}_k$ is the subgroup of $\mathrm{Sp}(3g-3, \mathbf{R})$ fixing the pair (L_0, L'_k). By matrix calculation, we have $\dim \tilde{G}_k = (3g-3)^2 + (k^2+k)/2$. Hence $\mathscr{L}^2_k$ is a C^∞ submanifold of codimension $(k^2+k)/2$. The condition that $\dim(L^1 \cap L^2) \ge k$ is written locally by a system of real algebraic equations and the last statement holds. q.e.d.

Thus $\operatorname{codim} \mathscr{L}^2_1 = 1$ and $\operatorname{codim} \mathscr{L}^2_k \ge 3$ for $k \ge 2$.

LEMMA 6.3. *$\pi_1(\mathscr{L}^2, \mathscr{L}^2_0) = \mathbf{Z}$. A generator γ_0 is given by the homotopy class of the path $\{(L_0, l(\theta)L'_1)\}_{-\varepsilon<\theta<\varepsilon}$ for small $\varepsilon > 0$; here $L_0 = \{x_1, \ldots, x_{3g-3}\}$ and $L'_1 = \{x_1, y_2, \ldots, y_{3g-3}\}$.*

Proof. By Lemma 6.1, $\pi_1(\mathscr{L}^2) = \pi_1(\mathscr{L} \times \mathscr{L}) = \mathbf{Z} \times \mathbf{Z}$ and it is generated by the homotopy classes of the two loops $\{(L_0, l(\theta)L_0)\}_{0 \le \theta < \pi}$ and $\{(l(\theta)L_0, L_0)\}_{0 \le \theta < \pi}$. Let $p_1\colon \mathscr{L}^2_0 \to \mathscr{L}$ be the projection onto the first factor. By Lemma 6.1 and Lemma 6.2, $\mathscr{L} = \mathrm{Sp}(3g-3, \mathbf{R})/G_0$ and $\mathscr{L}^2_0 = \mathrm{Sp}(3g-3, \mathbf{R})/\tilde{G}_0$. Hence p_1 is a fiber bundle with fiber $G_0/\tilde{G}_0$ which is contractible. It follows that $\pi_1(\mathscr{L}^2_0) = \pi_1(\mathscr{L}) = \mathbf{Z}$ which is generated by the homotopy class of the loop $\{(l(\theta)L_0, l(\theta)L'_0)\}_{0 \le \theta < \pi}$, where $L'_0 = \{y_1, \ldots, y_{3g-3}\}$. In $\pi_1(\mathscr{L}^2)$, this homotopy class is equal to the sum of the above two generators of $\pi_1(\mathscr{L}^2)$. By the homotopy exact sequence of the pair $(\mathscr{L}^2, \mathscr{L}^2_0)$, we obtain the result. q.e.d.

Now we have prepared for the definition of the invariant $\gamma(\{A_t\})$.

Definition 6.2. Let $\{p(t)\} = \{(L^1_{A_t}, L^2_{A_t})\}$ be the path of $\mathscr{L}^2$ obtained as above from $L^i_{A_t} = \pi^i_t(\operatorname{Ker} \mathscr{E}_{A_t})$ $(0 \le t \le 1$ and $i = 1, 2)$ for a generic smooth path of smooth connections on $M \times \mathrm{SU}(2)$, $\{A_t\}_{0 \le t \le 1}$. We assume $p(0), p(1) \in \mathscr{L}^2_0$. Let $[p(t)]$ be the homotopy class in $\pi_1(\mathscr{L}^2, \mathscr{L}^2_0)$ of the path $\{p(t)\}$. We define the invariant $\gamma(\{A_t\}) \in \mathbf{Z}$ by

$$[p(t)] = \gamma(\{A_t\})\gamma_0$$

where γ_0 is the generator of $\pi_1(\mathscr{L}^2, \mathscr{L}^2_0)$ given in Lemma 6.3.

7. Proof of Theorem 1.1

Let M, M_1 and M_2 and Σ be as in Theorem 1.1. For $0 \le r < \infty$, we define the elongated manifolds $M_1(r)$ and $M_2(r)$ by

$$M_1(r) = M_1 \cup \Sigma \times [-r, 0] \quad \text{and}$$
$$M_2(r) = \Sigma \times [0, r] \cup M_2$$

where $\Sigma \times \{-r\}$ and $\Sigma \times \{r\}$ are identified with ∂M_1 and ∂M_2 respectively. The product metrics on $\Sigma \times [-r, 0]$ and $\Sigma \times [0, r]$ should be understood. For $0 \le r < \infty$, we define the manifold $M(r)$ by

$$M(r) = M_1(r) \cup M_2(r)$$

where $\partial M_1(r) = \Sigma \times \{0\}$ is identified with $\partial M_2(r) = \Sigma \times \{0\}$. The volume and the diameter of $M_1(r)$, $M_2(r)$ and $M(r)$ tend to infinity as $r \to \infty$, while the Riemannian metrics on M_1, M_2 and Σ all remain constant. For $0 \le r' < r$, we regard $M_i(r')$ as a submanifold of $M(r)$ in a natural way $(i = 1, 2)$.

LEMMA 7.1. *For $i = 1, 2$, and $1 \le r < \infty$, let $\{A_j\}_{j=1,2,\dots}$ be a sequence of smooth connections on $M_i(r) \times \mathrm{SU}(2)$ such that A_j converges to a smooth connection A on $M_i(r) \times \mathrm{SU}(2)$ as $j \to \infty$. Let $\{\psi_j\}_{j=1,2,\dots}$ be a sequence of forms in $(\Omega^1 \oplus \Omega^0)(M_i(r)) \otimes \mathrm{su}(2)$ such that $D_{A_j}\psi_j = \lambda_j \psi_j$ and $\{\|\psi_j\|_{L^2}\}$ is bounded, where $\{\lambda_j\}_{j=1,2,\dots}$ is a sequence of real numbers such that $\lambda_j \to \lambda$ as $j \to \infty$. Let $\overline{\psi}_j = \psi_j | M_i(r-1)$ be the restriction of ψ_j on the submanifold $M_i(r-1) \subset M_i(r)$. Then there is an L^2_1-strongly convergent subsequence $\{\overline{\psi}_{j'}\}_{j'}$ of $\{\overline{\psi}_j\}_j$ whose limit $\overline{\psi}_\infty$ satisfies $D_A \overline{\psi}_\infty = \lambda \overline{\psi}_\infty$.*

Proof. For $j = 1, 2, \dots$, let

$$\nabla_j \colon \Omega^k(M_i(r)) \otimes \mathrm{su}(2) \to T^*M_i(r) \otimes \Omega^k(M_i(r)) \otimes \mathrm{su}(2)$$

be the covariant derivative operator associated to the connection A_j. Since A_j converges to A, we may assume that there is a C^∞ function β on $M_i(r)$ such that $0 \le \beta \le 1$, $0 \le |\nabla_j \beta| \le 1$, $\beta = 1$ on $M_i(r-1) \subset M_i(r)$ and $\beta = 0$ near

$\partial M_i(r)$ $(j = 1, 2, \dots)$. We set $\tilde{\psi}_j = \beta\psi_j$. The leading term of the differential operator $(D_{A_j})^2$ is the Laplace operator on $(\Omega^1 \oplus \Omega^0)(M_i(r)) \otimes \mathrm{su}(2)$. Hence by the Weitzenbock formula and the assumption on ψ_j, it follows that $\{(\nabla_j^* \nabla_j \tilde{\psi}_j, \tilde{\psi}_j)\}_j$ is bounded. Since $(\nabla_j^* \nabla_j \tilde{\psi}_j, \tilde{\psi}_j) = \|\nabla_j \tilde{\psi}_j\|^2_{L^2}$, $\{\tilde{\psi}_j\}_j$ is an L_1^2-bounded sequence. By Rellich's lemma, there is an L^2-strongly convergent subsequence $\{\tilde{\psi}_{j'}\}_{j'}$. By the assumption that $D_{A_j}\psi_j = \lambda_j\psi_j$ and $\lambda_j \to \lambda$, it follows that $\{\tilde{\psi}_{j'}\}_{j'}$ is also an L_1^2-strongly convergent sequence. The limit $\tilde{\psi}_\infty$ of this sequence satisfies $D_A\tilde{\psi}_\infty = \lambda\tilde{\psi}_\infty$ on $M(r-1)$, and we set $\bar{\psi}_\infty = \tilde{\psi}_\infty | M_i(r-1)$. q.e.d.

For a smooth connection A on $M \times \mathrm{SU}(2)$ which restricts to a product $B \times 1$ on $\Sigma \times [-1, 1] \times \mathrm{SU}(2)$ for an irreducible flat connection B on $\Sigma \times \mathrm{SU}(2)$, A can be extended to a smooth connection denoted by $A(r)$ (in each case) on $M(r) \times \mathrm{SU}(2)$ and $M_i(r) \times \mathrm{SU}(2)$ $(i = 1, 2)$ by setting $A(r) = B \times 1$ on the attaching cylinder parts.

For $0 \le r < \infty$, we define the operators $D_{A(r)}$, $\mathscr{E}^i_{A(r)}$ and $\mathscr{E}^{i*}_{A(r)}$ as in Definitions 2.1 and 6.1 (1), (2) respectively.

Let $\{\mu\}$ and $\{\psi_\mu\}$ be the set of eigenvalues and orthonormal eigenforms of D_B respectively.

First we consider the behavior of the kernels of the above operators for $r \gg 0$.

Let Π: $\mathrm{Ker}\, \mathscr{E}^1_A + \mathrm{Ker}\, \mathscr{E}^2_A \to \mathscr{H}_B$ be the map defined by

$$\Pi\big((\psi^1, \psi^2)\big) = \omega^1 - \omega^2,$$

for $\psi^i \in \mathrm{Ker}\, \mathscr{E}^i_A$ $(i = 1, 2)$, where ω^i is the harmonic part of $\psi^i | \partial M_i$.

LEMMA 7.2. *If* $\mathrm{Ker}\, \Pi$ *is trivial, then there is* $r(0) > 0$ *such that, for* $r > r(0)$, $\mathrm{Ker}\, D_{A(r)}$ *is trivial.*

Proof. Assume that $D_{A(r)}$ has a nontrivial kernel ψ. Let $\psi^i = \psi | M_i$ be the restriction of ψ on M_i $(i = 1, 2)$. On $\Sigma \times [-r, r]$, ψ can be written as

$$\psi(s) = \omega + \psi_+(s) + \psi_-(s), \qquad -r \le s \le r, \tag{7.2.1}$$

where $\omega \in \mathscr{H}_B$ (considered as a 1-form on $\Sigma \times [-r, r]$, constant with respect to s), $\psi_+(s) = \sum_{\mu > 0} c_\mu e^{s\mu}\psi_\mu$ and $\psi_-(s) = \sum_{\mu < 0} c_\mu e^{s\mu}\psi_\mu$ for constants $\{c_\mu\}$. Since ψ is a solution of an elliptic differential equation with smooth coefficients, by the unique continuation theorem, ψ has positive mass on $M_1 \cup M_2$. Hence we can normalize ψ as $\|\psi^1\|_{L^2} + \|\psi^2\|_{L^2} = 1$. From the above form of $\psi(s)$, it follows that there is a positive constant c_7 not depending on r such that

$$\|\psi_-(-r)\|_{L^2} < c_7 \|\psi^1\|_{L^2} \quad \text{and}$$
$$\|\psi_+(r)\|_{L^2} < c_7 \|\psi^2\|_{L^2}.$$

It follows that there are positive constants c_8 and c_9 both not depending on r such that

$$\|\psi_-(r)\|_{L^2} < c_8 e^{-c_9 r} \quad \text{and} \tag{7.2.2}$$
$$\|\psi_+(-r)\|_{L^2} < c_8 e^{-c_9 r}.$$

Assume that there is a sequence $r_1 < r_2 < \cdots$, $r_j \to \infty$ such that $\operatorname{Ker} D_{A(r_j)} \neq 0$. Let $\{\psi_j\}$ be a sequence such that $\psi_j \in \operatorname{Ker} D_{A(r_j)}$ and $\|\psi_j^1\|_{L^2} + \|\psi_j^2\|_{L^2} = 1$. By Lemma 7.1, taking a subsequence if necessary, we may assume that this sequence strongly converges to $(\psi_\infty^1, \psi_\infty^2)$ in L_1^2. Since $\|\psi_\infty^1\|_{L^2} + \|\psi_\infty^2\|_{L^2} = 1$, $(\psi_\infty^1, \psi_\infty^2) \neq (0,0)$. By the inequality (7.2.2), $\psi_\infty^1|\partial M_1 \in P_- + \mathscr{H}_B$ and $\psi_\infty^2|\partial M_2 \in P_+ + \mathscr{H}_B$. Hence $\psi_\infty^i \in \operatorname{domain} \mathscr{E}_A^i$ and $\psi_\infty^i \in \operatorname{Ker} \mathscr{E}_A^i$ $(i = 1,2)$. Since the harmonic part of $\psi_j^1|\partial M_1$ in the above decomposition is equal to that of $\psi_j^2|\partial M_2$ $(j = 1,2,\dots)$, $\Pi((\psi_\infty^1, \psi_\infty^2)) = 0$ and $\operatorname{Ker}\Pi \neq 0$. This contradicts the assumption of the lemma. q.e.d.

The next lemma is a restatement of Lemma 5.4.

LEMMA 7.3. *For $0 \leq r' < r < \infty$ and $i = 1,2$,*

$$\operatorname{Ker} \mathscr{E}_{A(r)}^i = \operatorname{Ker} \mathscr{E}_{A(r')}^i \quad \textit{and}$$
$$\operatorname{Ker} \mathscr{E}_{A(r)}^{i*} = \operatorname{Ker} \mathscr{E}_{A(r')}^{i*}.$$

Next we consider the behavior of the small eigenvalues and the corresponding eigenforms of $\mathscr{E}_{A(r)}^i$ for $r \gg 0$.

For $\psi \in (\Omega^1 \oplus \Omega^0)(M_i(r)) \otimes \mathrm{su}(2)$, on $\Sigma \times [-r,0]$ for $i = 1$ and on $\Sigma \times [0,r]$ for $i = 2$, ψ can be written as

$$\psi(s) = \psi_0(s) + \psi_+(s) + \psi_-(s)$$

where $\psi_0(s) \in \mathscr{H}_B$, $\psi_+(s) \in P_+$ and $\psi_-(s) \in P_-$.

LEMMA 7.4. *Let $\delta > 0$ be such that $\delta < \min|\mu|$, where μ runs through all the nonzero eigenvalues of D_B. Then there are positive constants $c_{10}(\delta)$ and $c_{11}(\delta)$ both not depending on r such that, for any λ-eigenform ψ of $\mathscr{E}_{A(r)}^i$ with $|\lambda| < \delta$ $(i = 1,2)$,*

$$\|\psi_+(0) + \psi_-(0)\|_{L^2} < c_{10}(\delta) e^{-c_{11}(\delta) r} \|\psi\|_{L^2}.$$

Proof. On $\Sigma \times \mathbf{R}$, a λ-eigenform of D_A $(|\lambda| < \min\{|\mu| \mid \mu \neq 0\})$ is a linear combination of

$$\varphi_\mu^+(s) = \left(\left(\mu + \sqrt{\mu^2 - \lambda^2}\right)\psi_\mu - \lambda\sigma(\psi_\mu)\right) e^{\sqrt{\mu^2 - \lambda^2}\, s},$$

and

$$\varphi_\mu^-(s) = \Big((-\lambda)\psi_\mu + \big(\mu + \sqrt{\mu^2-\lambda^2}\big)\sigma(\psi_\mu)\Big)e^{-\sqrt{\mu^2-\lambda^2}s} \quad \text{for } \mu \neq 0,$$

and

$$\varphi_0(s) = (\cos\lambda s)\omega - (\sin\lambda s) * \omega \qquad \text{for } \mu = 0, \omega \in \mathscr{H}_B.$$

We consider the case that $i = 2$ and ψ is a λ-eigenform of $\mathscr{E}^2_{A(r)}$. The other case can be treated in essentially the same way. We set

$$\varphi_\mu = \Big(\big(\mu + \sqrt{\mu^2-\lambda^2}\big)\varphi_\mu^+ + \lambda\varphi_\mu^-\Big)/\|\ \|_{L^2}.$$

Then the boundary condition of ψ implies that, on $\Sigma \times [0,r]$, ψ can be written as

$$\psi(s) = \sum_{\mu\neq 0} c_\mu\varphi_\mu(s), \qquad 0 \leq s \leq r$$

for some constants $\{c_\mu\}$. There is a constant c_{12} not depending on r such that

$$\|\psi(r)\|_{L^2} < c_{12}\|\psi\|_{L^2}.$$

By the above forms of $\varphi_\mu^+(s)$ and $\varphi_\mu^-(s)$, we see that there are positive constants $c_{10}(\delta)$ and $c_{11}(\delta)$ both not depending on r such that

$$\Big(\sum_{\mu\neq 0} c_\mu^2\Big)e^{c_{11}(\delta)r} < c_{10}(\delta)\|\psi\|_{L^2}.$$

Since $\|\psi_+(0) + \psi_-(0)\|_{L^2} = \sqrt{\sum_{\mu\neq 0} c_\mu^2}$, the lemma follows. q.e.d.

For $r > 0$, we define a self-adjoint Fredholm operator $\mathscr{K}_{A(r)}$ which is related to $\gamma(\{A_t\})$ in Theorem 1.1 as follows:

Let Γ be the space of smooth sections of the trivial $\mathscr{H}_B$-bundle $(\Sigma \times [-r,r]) \times \mathscr{H}_B$. Let $L_A^i = \text{image}(\pi^i\colon \operatorname{Ker}\mathscr{E}_A^i \to \mathscr{H}_B)$ for $i = 1, 2$, where π^i is the map defined in Definition 4.2.

Set

$$\Gamma\big(L_A^1, L_A^2\big) = \big\{\psi \in \Gamma \,|\, \psi(-r) \in L_A^1 \text{ and } \psi(r) \in L_A^2\big\}.$$

Definition 7.1. Let $\mathscr{K}_{A(r)}$ be the operator from the L_1^2-completion of $\Gamma(L_A^1, L_A^2)$ to the L^2-completion of Γ defined to be the closure of the operator sending $\psi \in \Gamma(L_A^1, L_A^2)$ to $*(\partial\psi/\partial s)$, where s is the coordinate of $[-r,r]$.

Note that $\mathscr{K}_{A(r)}$ is the restriction of $D_{A(r)}$ to the part of $(\Omega^1 \oplus \Omega^0)(\Sigma \times [-r,r]) \otimes \mathrm{su}(2)$ involving the harmonic part of $\Omega^1(\Sigma) \otimes \mathrm{su}(2)$.

LEMMA 7.5. *$\mathscr{K}_{A(r)}$ is a self-adjoint Fredholm operator. For $0 < r < r' < \infty$, the eigenvalues $\{\lambda\}$ and $\{\lambda'\}$ of $\mathscr{K}_{A(r)}$ and $\mathscr{K}_{A(r')}$, respectively, are related as $\lambda' = (r/r')\lambda$.*

Proof. The self-adjointness of the boundary condition for $\mathscr{K}_{A(r)}$ follows from (3.1.1) and the fact that L_A^1 and L_A^2 are Lagrangians. That $\mathscr{K}_{A(r)}$ is Fredholm follows from essentially the same argument as in the proof of Proposition 4.1(1). The second statement follows from the simple variable change. q.e.d.

The next lemma relates the small eigenvalues and the corresponding eigenforms of $\mathscr{K}_{A(1)}$ to those of $D_{A(r)}$ for $r \gg 0$. In the lemma and its proof, we use the map

$$h_r\colon \Sigma \times [-1,1] \to \Sigma \times [-r, r]$$

defined by $h_r(x, s) = (x, rs)$ for $x \in \Sigma$ and $s \in [-1, 1]$ and its induced isomorphism

$$h_r^*\colon \Omega^k(\Sigma \times [-r, r]) \otimes \mathrm{su}(2) \to \Omega^k(\Sigma \times [-1,1]) \otimes \mathrm{su}(2) \qquad (k = 0, 1).$$

LEMMA 7.6. *Let λ be an eigenvalue of $\mathscr{K}_{A(1)}$ of multiplicity n such that $|\lambda| < \pi$. Then for sufficiently large $r > 0$, there are precisely n linearly independent eigenforms $\{\psi_{\xi_\kappa(r)}\}_{\kappa=1,\dots,n}$ of $D_{A(r)}$ belonging to the eigenvalues $\{\xi_\kappa(r)\}_{\kappa=1,\dots,n}$ such that, for $\kappa = 1, \dots, n$,*

(i) $|\lambda - r\xi_\kappa(r)|$ is $O(1/r)$, and

(ii) $\{h_r^*(\omega_\kappa(r))\}_r$ converges to a λ-eigenform of $\mathscr{K}_{A(1)}$ as $r \to \infty$, where $\omega_\kappa(r)$ denotes the harmonic part of the restriction $\psi_{\xi_\kappa(r)}|\Sigma \times [-r, r]$.

Proof. Let ϕ be a λ-eigenform of $\mathscr{K}_{A(1)}$. Then ϕ is a form on $\Sigma \times [-1, 1]$:

$$\phi = (\cos \lambda s)\omega - (\sin \lambda s) * \omega, \qquad -1 \le s \le 1$$

for $\omega \in \mathscr{H}_B$, and

$$\omega^1 = (\cos \lambda)\omega + (\sin \lambda) * \omega \in L^1,$$

$$\omega^2 = (\cos \lambda)\omega - (\sin \lambda) * \omega \in L^2.$$

Put $\phi_r = (h_r^*)^{-1}\phi$. Let $\phi^i \in \operatorname{Ker} \mathscr{E}_A^i$ be such that $\pi^i(\phi^i) = \omega^i$ $(i = 1, 2)$. Then ϕ^i is the restriction of $\tilde{\phi}^i \in \operatorname{Ker} \mathscr{E}_{A(r)}^i$ (Lemma 7.3). On $\Sigma \times [-r, 0]$ (resp. $\Sigma \times [0, r]$), $\tilde{\phi}^1$ (resp. $\tilde{\phi}^2$) can be written as

$$\tilde{\phi}^1(s) = \omega^1 + \tilde{\phi}_-^1(s),$$

$$\tilde{\phi}^2(s) = \omega^2 + \tilde{\phi}_+^2(s)$$

where $\tilde{\phi}^1_-(s)$ (resp. $\tilde{\phi}^2_+(s)$) is a linear combination of $e^{s\mu}\psi_\mu$ for $\mu < 0$ (resp $\mu > 0$). If we normalize as $\|\phi^i\|_{L^2} = 1$ $(i = 1, 2)$, then $\|\tilde{\phi}^1_-(0)\|_{L^2}$ and $\|\tilde{\phi}^2_+(0)\|_{L^2}$ are both $O(e^{-c_{11}(\delta)r})$ by Lemma 7.4. For $r \geq 1$ and $i = 1, 2$, let β^i_r be a C^∞ function defined on $M_i(r)$ such that $0 \leq \beta^i_r \leq 1$, $0 \leq |\nabla\beta^i_r| \leq 1$, $\beta^i_r = 1$ on $M_i(r-1)$ and $\beta^i_r = 0$ near $\partial M_i(r)$. We define a form ψ'_r on $M(r)$ by

$$\begin{aligned} \psi'_r &= \tilde{\phi}^i && \text{on } M_i \qquad (i = 1, 2) \\ &= \phi_r + \beta^1_r\tilde{\phi}^1_- + \beta^2_r\tilde{\phi}^2_+ && \text{on } \Sigma \times [-r, r]. \end{aligned}$$

Here $\beta^1_r\tilde{\phi}^1_-$ and $\beta^2_r\tilde{\phi}^2_+$ are considered to be forms on $\Sigma \times [-r, r]$ by extending by 0 on $\Sigma \times [0, r]$ and on $\Sigma \times [-r, 0]$, respectively. We set

$$\psi_r = \psi'_r / \|\psi'_r\|_{L^2}.$$

Then ψ is an element of the L^2_1-completion of $(\Omega^1 \oplus \Omega^0)(M(r)) \otimes \mathrm{su}(2)$. For $r \gg 0$, the greater part of the mass of ψ'_r lies on ϕ_r and we see that $\|D_{A(r)}\psi_r - (\lambda/r)\psi_r\|_{L^2}$ is $O(\lambda/r^2)$.

Let $\{\lambda(r)\}$ and $\{\psi_{\lambda(r)}\}$ be the set of eigenvalues and the set of orthonormal eigenforms of $D_{A(r)}$, respectively. Then

$$\psi_r = \sum c_{\lambda(r)}\psi_{\lambda(r)}$$

for constants $\{c_{\lambda(r)}\}$, and

$$\left\| D_{A(r)}\psi_r - (\lambda/r)\psi_r \right\|_{L^2} \geq \min|\lambda(r) - \lambda/r|.$$

Hence there must be an eigenvalue $\xi(r)$ of $D_{A(r)}$ such that $|\xi(r) - \lambda/r|$ is $O(1/r^2)$. For a sequence $0 \ll r_1 < r_2 < \cdots, r_j \to \infty$, we consider a sequence of eigenforms of $D_{A(r_j)}$, $\{\psi_{\xi(r_j)}\}_j$, belonging to a sequence of such eigenvalues $\{\xi(r_j)\}_j$. Since $\lambda/r_j \to 0$ as $j \to \infty$, when we normalize as $\|\psi_{\xi(r_j)}|M_1 \cup M_2\|_{L^2} = 1$, by Lemma 7.1, taking a subsequence if necessary, we may assume that $\{\psi_{\xi(r_j)}|M_i\}_j$ converges strongly in L^2_1 to a form ψ^i_∞ in $\mathrm{Ker}\,\mathscr{E}^i_A$ $(i = 1, 2)$. Hence, taking a subsequence if necessary, we may assume that $\{h^*_r(\omega(r_j))\}_j$ converges to a λ-eigenform of $\mathscr{K}_{A(1)}$, where $\omega(r_j)$ denotes the harmonic part of $\psi_{\xi(r_j)}|\Sigma \times [-r, r]$. By the method of construction, the harmonic part of $h^*_r(\psi_r|\Sigma \times [-r, r])$ is the nontrivial λ-eigenform ϕ of $\mathscr{K}_{A(1)}$. Therefore there must be a sequence of eigenforms $\{\psi_{\xi(r_j)}\}_j$ satisfying the above conditions such that $\{h^*_r(\omega(r_j))\}_j$ converges to a nontrivial λ-eigenform of $\mathscr{K}_{A(1)}$. Since we can construct such a sequence of eigenforms of $D_{A(r)}$ for arbitrary λ-eigenform ϕ of $\mathscr{K}_{A(1)}$, by dimension counting, the number of linearly independent such eigenforms is n for $r \gg 0$. q.e.d.

Now let A_0 and A_1 be smooth irreducible flat connections on $M \times SU(2)$ with $\operatorname{Ker} D_{A_0} = 0 = \operatorname{Ker} D_{A_1}$ such that A_0 and A_1 restrict to irreducible flat connections B_0 and B_1 on $\Sigma \times SU(2)$, respectively. Since the space of all the irreducible representations of $\pi_1(\Sigma)$ to SU(2) is arcwise connected, there is a smooth path of smooth irreducible flat connections on $\Sigma \times SU(2)$, $\{B_t\}_{0 \le t \le 1}$, connecting B_0 and B_1. Also by Proposition 5.1, we may assume that there are a Riemannian metric on M and a smooth path of smooth connections, $\{A_t\}_{0 \le t \le 1}$, on $M \times SU(2)$ such that (1) M is isometric to a product $\Sigma \times [-1, 1]$ near Σ, (2) A_t restricts to the product $B_t \times 1$ on $\Sigma \times [-1, 1] \times SU(2)$ for $0 \le t \le 1$ and (3) $\operatorname{Ker} \mathscr{E}^{i*}_{A_t} = 0$ for $0 \le t \le 1$ and $i = 1, 2$.

Since A_j $(j = 0, 1)$ is irreducible flat, $\operatorname{Ker} D_{A_j}$ is isomorphic to $H^1_{A_j}(M, \mathrm{su}(2))$ and it is independent of the Riemannian metric on M. In particular, $\operatorname{Ker} D_{A_j(r)} = 0$ for $r \ge 0$ $(j = 0, 1)$. The sequence of the path of operators $\{D_{A_t(r')}\}_{0 \le t \le 1}$ for $0 \le r' < r$ gives a homotopy between $\{D_{A_t}\}$ and $\{D_{A_t(r)}\}$. It follows that $SF(M, \{A_t\})$ is equal to the spectral flow of $\{D_{A_t(r)}\}$ for $r \ge 0$.

For $0 \le t \le 1$, let $\{\mu_t\}$ and $\{\psi_{\mu_t}\}$ be the set of eigenvalues and eigenforms of D_{B_t} respectively. Since [0, 1] is compact and $\{\mu_t\}$ depends continuously on t, there is a $\delta > 0$ such that $\delta < \min\{|\mu_t|\}$ for $0 \le t \le 1$, where μ_t runs through all the nonzero eigenvalues of D_{B_t}. We choose and fix such a δ. Then the constants $c_{10}(\delta), c_{11}(\delta)$ in Lemma 7.4 can be chosen independent of t.

Since $\operatorname{Ker} \mathscr{E}^{i*}_{A_t} = 0$ for $0 \le t \le 1$, by Lemma 5.1, $\{L^i_{A_t}\}_{0 \le t \le 1}$ is a continuous path of Lagrangians $(i = 1, 2)$. Hence $\{\mathscr{K}_{A_t(r)}\}_{0 \le t \le 1}$ is a continuous one-parameter family of self-adjoint Fredholm operators for $r > 0$. Since $\operatorname{Ker} \mathscr{K}_{A_t(r)}$ coincides with the space of constant sections in $\Gamma(L^1_{A_t}, L^2_{A_t})$, it is isomorphic to $L^1_{A_t} \cap L^2_{A_t}$. The nondegeneracy of the irreducible flat connections $A_0(r)$ and $A_1(r)$ implies that $\operatorname{Ker} \mathscr{K}_{A_0(r)} = 0 = \operatorname{Ker} \mathscr{K}_{A_1(r)}$. Hence the spectral flow of $\{\mathscr{K}_{A_t(r)}\}_{0 \le t \le 1}$ is well-defined.

LEMMA 7.7. *For $r > 0$, the spectral flow of $\{\mathscr{K}_{A_t(r)}\}_{0 \le t \le 1}$ is equal to $\gamma(\{A_t\})$.*

Proof. A λ-eigenform ψ_t of $\mathscr{K}_{A_t(r)}$ has the form

$$\psi_t = (\cos \lambda s)\omega_t - (\sin \lambda s) * \omega_t$$

for $\omega_t \in \mathscr{H}_B$ and $-r \le s \le r$; here

$$\omega^1_t = (\cos \lambda r)\omega_t + (\sin \lambda r) * \omega_t \in L^1_t$$

and

$$\omega^2_t = (\cos \lambda r)\omega_t - (\sin \lambda r) * \omega_t \in L^2_t.$$

The 0-eigenvalue of $\mathscr{K}_{A_t(r)}$ appears exactly at those t such that $L^1_{A_t} \cap L^2_{A_t} \ne 0$ and $\dim L^1_{A_t} \cap L^2_{A_t}$ is its multiplicity. To see the sign changes of λ near the

0-eigenvalue, let $\{\psi_t\}$ be a sequence of eigenforms of $\mathscr{K}_{A_t(r)}$ such that $\psi_t \to \psi_{t_0}$, a 0-eigenform, as $t \to t_0$ and $\mathscr{K}_{A_t(r)}\psi_t = \lambda_t \psi_t$ where $\lambda_t \to 0$ as $t \to t_0$. Consider the symplectic pairing

$$\begin{aligned}\langle \omega_t^1, \omega_t^2 \rangle &= -\int_\Sigma \operatorname{tr}(\omega_t^1 \wedge \omega_t^2)\\ &= (\sin 2\lambda_t r)\left(\int_\Sigma \omega_t \wedge * \omega_t\right).\end{aligned}$$

For λ_t near 0, the sign of λ_t is equal to that of $\langle \omega_t^1, \omega_t^2 \rangle$. Hence the sign change of λ_t at $t = t_0$ (if it occurs) is equal to that of $\langle \omega_t^1, \omega_t^2 \rangle$. In the definition of $\gamma(\{A_t\})$ (Definition 6.2), the generator γ_0 of $\pi_1(\mathscr{L}^2, \mathscr{L}_0^2)$ is given by the homotopy class of the path $\{(L_0, l(\theta)L_1')\}_{-\varepsilon < \theta < \varepsilon}$ in the notation of Section 6. The sign change of θ coincides with that of the symplectic pairing $\langle \omega_\theta, \omega_\theta' \rangle$ where ω_θ and ω_θ' $(-\varepsilon < \theta < \varepsilon)$ are sequences of elements of L_0 and $l(\theta)L_1'$, respectively, such that $\omega_\theta, \omega_\theta' \to \omega_0 \in L_0 \cap L_1'$ as $\theta \to 0$. It follows that the spectral flow of $\mathscr{K}_{A_t(r)}$ is equal to $\gamma(\{A_t\})$. q.e.d.

The following lemma completes the proof of Theorem 1.1.

LEMMA 7.8. *For sufficiently large $r \gg 0$, the spectral flow of $\{D_{A_t(r)}\}_{0 \le t \le 1}$ is equal to $\gamma(\{A_t\})$.*

Proof. For $0 \le t \le 1$, let Π_t: $\operatorname{Ker} \mathscr{E}_{A_t}^1 + \operatorname{Ker} \mathscr{E}_{A_t}^2 \to \mathscr{H}_B$ be the map defined just before Lemma 7.2. By Lemma 7.2, if $r \to \infty$, the set $\{t \in [0,1] | \operatorname{Ker} D_{A_t(r)} \neq 0\}$ concentrates to the set $T_0 = \{t \in [0,1] | \operatorname{Ker} \Pi_t \neq 0\}$. Since the sign change of the eigenvalues of $D_{A_t(r)}$ occurs near 0-eigenvalue, the points at which the sign of the eigenvalues change concentrate to the points of T_0. $\operatorname{Ker} \Pi_t$ is the space spanned by those $(\psi^1, \psi^2) \in \operatorname{Ker} \mathscr{E}_{A_t}^1 + \operatorname{Ker} \mathscr{E}_{A_t}^2$ with $\pi^1(\psi^1) = \pi^2(\psi^2) \neq 0$. The latter space is nontrivial precisely at those t with $L_{A_t}^1 \cap L_{A_t}^2 \neq 0$. At these points, by Lemma 7.6, the sign changes of the eigenvalues of $D_{A_t(r)}$ are equal to the sign changes of the eigenvalues of $\mathscr{K}_{A_t(1)}$ for sufficiently large $r > 0$. Hence the spectral flow of $\{D_{A_t(r)}\}_{0 \le t \le 1}$ is equal to the spectral flow of $\{\mathscr{K}_{A_t(1)}\}_{0 \le t \le 1}$ for sufficiently large $r > 0$. The lemma follows from Lemma 7.7. q.e.d.

8. Dehn surgery along the figure eight knot

In this section we prove Theorem 1.2. Let S^1 and D^2 be the unit circle and the unit disk in the complex plane respectively oriented by the complex structure. Let $S^1 \times S^1$ be the 2-torus. Let m and ℓ be the homotopy classes in

$\pi_1(S^1 \times S^1)$ represented by $S^1 \times 1$ and $1 \times S^1$ respectively. Set

$$R(S^1 \times S^1) = \mathrm{Hom}(\pi_1(S^1 \times S^1), \mathrm{SU}(2))/\mathrm{ad}\,\mathrm{SU}(2).$$

For $\rho \in R(S^1 \times S^1)$, we can write, up to conjugation,

$$\rho(m) = \begin{pmatrix} e^{iu} & 0 \\ 0 & e^{-iu} \end{pmatrix} \qquad \rho(\ell) = \begin{pmatrix} e^{iv} & 0 \\ 0 & e^{-iv} \end{pmatrix} \tag{8.1}$$

for $-\pi \le u, v < \pi$. The correspondence $\rho \to (e^{iu}, e^{iv})$ induces a bijection between $R(S^1 \times S^1)$ and $S^1 \times S^1/\sim$, where $(z, w) \sim (\bar{z}, \bar{w})$ (- denotes the complex conjugate) for $z, w \in S^1$. Thus $R(S^1 \times S^1)$ is the 2-sphere with distinguished 4-points $Q = \{(\pm 1, \pm 1)\}$ each of which corresponds to a representation of $\pi_1(S^1 \times S^1)$ into the center of SU(2).

Let $P_0 = R(S^1 \times S^1) - Q$. P_0 is a four-punctured sphere. Let $q\colon \tilde{P}_0 \to P_0$ be the universal abelian covering of P_0. $\tilde{P}_0$ can be identified with $\mathbf{R}^2 - \tilde{Q}$, where $\tilde{Q}$ denotes the set of those points whose coordinates are integer multiples of π. The variables (u, v) in (8.1) can be considered as coordinates of $\tilde{P}_0$. Let Λ be the discrete subgroup of the euclidean motions of $\tilde{P}_0$ generated by three motions, $(u, v) \to (u + 2\pi, v)$, $(u, v) \to (u, v + 2\pi)$ and $(u, v) \to (-u, -v)$. Then $\tilde{P}_0/\Lambda = P_0$.

Let K be the figure eight knot in S^3. Let $U(K)$ be a closed tubular neighborhood of K. Let $N = S^3 - \mathring{U}(K)$. Let $(\bar{m}, \bar{\ell})$ be the meridian-longitude pair on ∂N. Thus $\bar{m}$ is a simple closed curve on ∂N, null-homotopic in $U(K)$, and $\bar{\ell}$ is one null-homologous in N. We denote their homotopy classes in $\pi_1(\partial N)$ by the same letters.

For an integer k, let $f_k\colon S^1 \times S^1 \to \partial N$ be a diffeomorphism such that $f_{k*}(m) = \bar{m} + k\bar{\ell}$ and $f_{k*}(\ell) = \bar{\ell}$. Let $N_k = N \cup_f (D^2 \times S^1)$ be the manifold obtained by attaching $D^2 \times S^1$ to N along their boundaries by f_k: $S^1 \times S^1 = \partial(D^2 \times S^1) \to \partial N$. Then N_k is a **Z**-homology 3-sphere. We give N_k an orientation compatible with that of N.

Set

$$R(N) = \mathrm{Hom}(\pi_1(N), \mathrm{SU}(2))^*/\mathrm{ad}\,\mathrm{SU}(2),$$

$$R(N_k) = \mathrm{Hom}(\pi_1(N_k), \mathrm{SU}(2))^*/\mathrm{ad}\,\mathrm{SU}(2),$$

where * denotes the irreducible representations.

$\pi_1(N)$ has the presentation

$$\pi_1(N) = \{s, t \mid s t^{-1} s^{-1} t s = t s t^{-1} s^{-1} t\}$$

and $\bar{m} = s$, $\bar{\ell} = t s^{-1} t^{-1} s^2 t^{-1} s^{-1} t$. Using this presentation, Burde [B] has shown that the SO(3)-representation space of $\pi_1(N)$

$$\bar{R}(N) = \mathrm{Hom}(\pi_1(N), \mathrm{SO}(3))^*/\mathrm{ad}\,\mathrm{SO}(3)$$

is a real algebraic curve $\bar{C}$ in the plane. The algebraic equation of $\bar{C}$ is given as follows [B]: For a representation $\bar{\rho}$: $\pi_1(N) \to \mathrm{SO}(3)$, let α be the rotation angle of $\bar{\rho}(\mathscr{s})$ (= that of $\bar{\rho}(\mathscr{t})$) and let β be the (unoriented) angle between the axis of $\bar{\rho}(\mathscr{s})$ and $\bar{\rho}(\mathscr{t})$. Put $\tau = \cos\beta$ and $y = (\cot(\alpha/2))^2$. When these parameters are used, the algebraic equation of $\bar{C}$ is given by

$$(8.2) \qquad \bar{C}\colon y^2 + (2\tau + 4)y + 4\tau^2 + 2\tau - 1 = 0.$$

$R(N)$ is a circle containing two binary dihedral representations. Let D = {binary dihedral representations} denote these two points in $R(N)$. Then $R(N) - D$ consists of two disjoint arcs C and C'. For each $\rho' \in C'$, there is a $\rho \in C$ such that $\rho'(\mathscr{s}) = (-1)\rho(\mathscr{s})$ and $\rho'(\mathscr{t}) = (-1)\rho(\mathscr{t})$, where -1 is the generator of the center of SU(2).

The map f_K: $S^1 \times S^1 \to \partial N$ induces the map f_k^*: $R(N) \to R(S^1 \times S^1)$ sending $\rho \in R(N)$ to the restriction $\rho | f_{k*}\pi_1(S^1 \times S^1)$. From the analysis in [B], it can be seen that $f_k^*(R(N)) \subset P_0 = R(S^1 \times S^1) - Q$. Thus $f_k^*(C \cup C')$ consists of two curves in P_0, C_k and C_k'. Let $\tilde{C}_k$ and $\tilde{C}_k'$ be their inverse images in $\tilde{P}_0$. For an SU(2)-representation ρ of $\pi_1(N)$, up to conjugation, we can write

$$\rho(\mathscr{m}) = \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} \quad \rho(\ell) = \begin{pmatrix} e^{i\zeta} & 0 \\ 0 & e^{-i\zeta} \end{pmatrix}$$

for $-\pi \le \alpha,\ \zeta < \pi$. Then the curves $\tilde{C}_k$ and $\tilde{C}_k'$ are the graphs given by $(u, v) = (\alpha/2 + k\zeta, \zeta)$.

For k an integer < 0, a fundamental segment E_k of $\tilde{C}_k$ with respect to the action of Λ is as in Figure 8.1 (for $k = -1$, see [B]), where e_0 corresponds to a

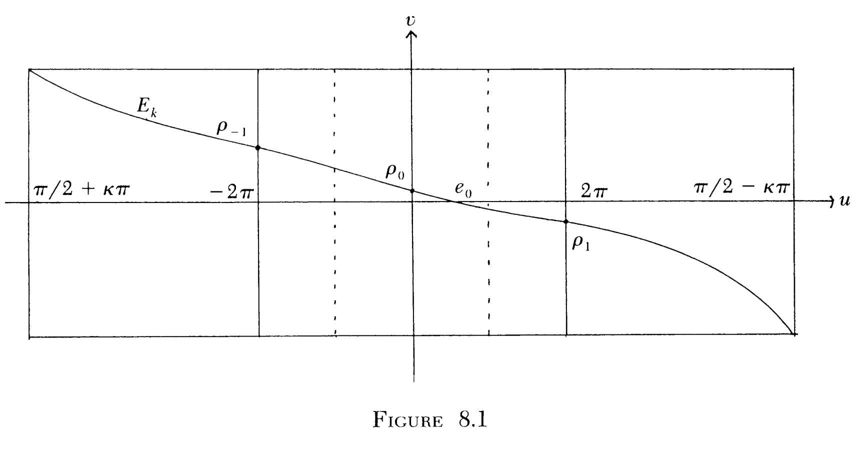

FIGURE 8.1

binary dihedral representation. A Λ-fundamental segment E'_k of $\tilde{C}'_k$ is obtained by shifting E_k to the right by the amount π. The graph in Figure 8.1 is not rigorous, but the important features of E_k and E'_k are that they have no self-intersection points and they intersect transversely with these lines

$$\mathbf{L} = \{u = 2m\pi | m \text{ integers}\}.$$

$R(N_k)$ consists of finite points and it is in one-to-one correspondence with the set of the intersection points $(E_k \cup E'_k) \cap \mathbf{L}$. By Figure 8.1, we see that there are precisely k such points on each of E_k and E'_k. Their u-coordinates are:

$$(8.3) \qquad \begin{aligned} &\{2\pi j | j \in \mathbf{Z}, 1/4 + k/2 < j < 1/4 - k/2\}, \\ &\{2\pi j | j \in \mathbf{Z}, 3/4 + k/2 < j < 3/4 - k/2\}. \end{aligned}$$

For each j in (8.3), let ρ_j and ρ'_j be the points of $R(N_k)$ corresponding to the above intersection points on $E_k \cap \mathbf{L}$ and $E'_k \cap \mathbf{L}$ respectively.

For j in (8.3), let K_j and K'_j be the flat connections on $N_k \times \mathrm{SU}(2)$ associated to ρ_j and ρ'_j respectively.

The de Rham cohomology group $H^1_{K_j}(N, \mathrm{su}(2))$ is the Zariski tangent space of $R(N)$ at ρ_j. $H^1_{K_j}(S^1 \times S^1, \mathrm{su}(2))$ is identified with the tangent space of P_0 at ρ_j. The image $f^*_k(H^1_{K_j}(N, \mathrm{su}(2))$ by the map f_k is identified with the tangent space of the curve E_k at ρ_j. Let ι: $S^1 \times S^1 \to D^2 \times S^1$ be the inclusion map. Then $\iota^*(H^1_{K_j}(D^2 \times S^1, \mathrm{su}(2))$ is identified with the tangent space of the vertical lines $\mathbf{L}$ at ρ_j. Since the E_k cut $\mathbf{L}$ transversely, by the Mayer-Vietoris exact sequence of de Rham cohomology groups, $H^1_{K_j}(N_k, \mathrm{su}(2)) = 0$. Since $\operatorname{Ker} D_{K_j}$ is identified with this de Rham cohomology group, $\operatorname{Ker} D_{K_j} = 0$. The same arguments hold for ρ_j, E'_k and K'_j.

Thus the set of the gauge equivalence classes of the flat connections on $N_k \times \mathrm{SU}(2)$, $\{[K_j], [K'_j]\}$, is the set of the non-degenerate critical points of the Chern-Simons functional and they form a basis of the chain complex of $I_*(N_k)$.

(I) *Computation of* $d([K_j]) - d([K_j])$ *and* $d([K'_{j+1}]) - d([K'_j])$. First we compute the difference $d([K_{j+1}]) - d([K_j])$ for j, $j+1$ in (8.3).

Let $\{A_t\}_{0 \le t \le 1}$ be a smooth path of smooth connections on $N_k \times \mathrm{SU}(2)$ such that (1) $A_0 = K_j$, $A_1 = K_{j+1}$ and (2) for $0 \le t \le 1$, $A_t | N \times \mathrm{SU}(2)$ is the irreducible flat connection corresponding to the representation ρ^t, where $\{\rho^t\}_{0 \le t \le 1}$ is the segment of E_k connecting $\rho^0 = \rho_j$ and $\rho^1 = \rho_{j+1}$, and (3) near Σ, A_t restricts to $B_t \times 1$; here B_t is the flat connection on $\Sigma \times \mathrm{SU}(2)$ associated to the restriction $\rho^t | \pi_1(\Sigma)$.

To apply Theorem 1.1, we need a technical modification. Since $\pi_1(N)$ is generated by the homotopy classes represented by the meridian loop in ∂N and

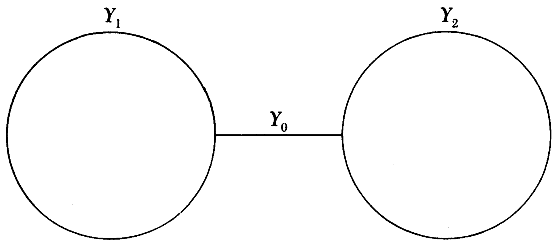

FIGURE 8.2

its conjugate, we can find a simple closed curve ν in N such that $\nu \cap \partial N =$ {base point} and $\pi_1(\nu \cup \partial N)$ carries the generator of $\pi_1(N)$. Let $Y = Y_1 \cup Y_0 \cup Y_2$ be the 1-complex pictured in Figure 8.2.

There is an embedding $\Psi\colon Y \to N_k$ such that $\Psi(Y_1)$ is isotopic to the core curve of the attaching solid torus $D^2 \times S^1$ in N_k, and $\Psi(Y_2)$ is isotopic to ν. Let $N_\Psi(Y)$ be a regular neighborhood of $\Psi(Y)$ in N_k. We set $M_1 = N_\Psi(Y)$, $M_2 = \overline{N_k - N_\Psi(Y)}$ and $\Sigma = M_1 \cap M_2 = \partial N_\Psi(Y)$. Obviously we may assume $M_2 \subset \operatorname{Int} N$. By the above choice of ν, $\pi_1(\Sigma)$ carries the generator of $\pi_1(N)$. It follows that, for each ρ^t $(0 \le t \le 1)$, the restriction $\rho^t | \pi_1(\Sigma)$ is irreducible. We give a Riemannian metric on N_k such that, near Σ, it is isometric to the product Riemannian manifold $\Sigma \times [-1, 1]$ for some metric on Σ.

(I-1) *Claim.* Ker $\mathscr{E}_{A_t}^{2*} = 0$. Since $A_t | M_2 \times \mathrm{SU}(2)$ is an irreducible flat connection for $0 \le t \le 1$, by Lemma 4.2, Ker $\mathscr{E}_{A_t}^{2*}$ is isomorphic to the relative de Rham cohomology group $H^1_{A_t}(M_2, \Sigma, \mathrm{su}(2))$. By the excision property, it is isomorphic to $H^1_{A_t}(N, \overline{N - M_2}, \mathrm{su}(2))$. As $H^0_{A_t}(\overline{N - M_2}, \mathrm{su}(2)) = 0$ by the irreducibility, we have the exact sequence

$$0 \to H^1_{A_t}\big(N, \overline{N - M_2}, \mathrm{su}(2)\big) \to H^1_{A_t}(N, \mathrm{su}(2)) \to H^1_{A_t}\big(\overline{N - M_2}, \mathrm{su}(2)\big).$$

$N - M_2$ is homotopy equivalent to $\nu \cup \partial N$ and the Mayer-Vietoris sequence shows that $H^1_{A_t}(\overline{N - M_2}, \mathrm{su}(2))$ contains $H^1_{A_t}(\partial N, \mathrm{su}(2))$ as a direct summand. Since the image $(H^1_{A_t}(N, \mathrm{su}(2)) \to H^1_{A_t}(\partial N, \mathrm{su}(2)))$ is identified with the tangent space of the curve E_k at ρ^t as noted before, this map is injective. It follows that the last map in the above exact sequence is injective. Hence $H^1_{A_t}(N, \overline{N - M_2}, \mathrm{su}(2)) = 0$ for $0 \le t \le 1$.

By Proposition 5.1, perturbing $\{A_t\}_{0 \le t \le 1}$ and the metric slightly if necessary, we may assume that Ker $\mathscr{E}_{A_t}^{1*} = 0$ for $0 \le t \le 1$. Hence $\gamma(\{A_t\})$ can be defined as in Section 6.

(I-2) *Computation of* $\gamma(\{A_t\})$. We adopt an indirect method to compute $\gamma(\{A_t\})$ by making use of its homotopy invariance and the computation of the

Floer homology group of the Brieskorn homology 3-sphere by Fintushel-Stern [F-S].

Let $H = \Sigma(2,3,19)$ be the Brieskorn homology 3-sphere

$$H = \{z_1^2 + z_2^3 + z_3^{19} = \varepsilon\} \cap S^5 \qquad (\varepsilon > 0 \text{ small})$$

where $(z_1, z_2, z_3) \in \mathbf{C}^3$ and S^5 is the unit sphere in $\mathbf{C}^3$. It is oriented as the boundary of the non-singular algebraic variety $\{z_1^2 + z_2^3 + z_3^{19} = \varepsilon\} \cap D^6$, where D^6 is the unit disk in $\mathbf{C}^3$.

Let $T = \{z_3 = 0\} \cap H$. T is a circle in H. Let $U(T)$ be a closed tubular neighborhood of T in H. Let $V = \overline{H - U(T)}$. Then V is an oriented compact connected 3-manifold with ∂V diffeomorphic to $S^1 \times S^1$. $\pi_1(V)$ has the presentation

$$\pi_1(V) = \{\mathscr{w}, \mathscr{z} | \mathscr{w}^2 = \mathscr{z}^3\}.$$

We set

$$\overline{R}(V) = \operatorname{Hom}(\pi_1(V), \mathrm{SO}(3))^* / \operatorname{ad} \mathrm{SO}(3),$$
$$R(V) = \operatorname{Hom}(\pi_1(V), \mathrm{SU}(2))^* / \operatorname{ad} \mathrm{SU}(2).$$

$R(V)$ is an open arc, and each point ρ of $\overline{R}(V)$ is parametrized by the (unoriented) angle between the axis of the rotations $\rho(\mathscr{w})$ of order 2 and $\rho(\mathscr{z})$ of order 3. $R(V)$ is also an open arc.

Set $m' = \mathscr{w}\mathscr{z}$ and $\ell' = \mathscr{w}^2 = \mathscr{z}^3$ in $\pi_1(V)$. Let $g\colon \partial V \to V$ be the inclusion. Then $g_*\pi_1(\partial V)$ is generated by m' and ℓ'. Note that $U(T)$ is diffeomorphic to $D^2 \times S^1$ and H can be written as $H = V \cup_h (D^2 \times S^1)$, where $D^2 \times S^1$ is attached to V along their boundaries by the diffeomorphism $h\colon S^1 \times S^1 = \partial(D^2 \times S^1) \to \partial V$ such that $h_*(m) = 19m' - 3\ell'$ and $h_*(\ell) = -6m' + \ell'$ (see [E-N]). Let $h^*\colon R(V) \to R(S^1 \times S^1)$ be the map induced by h. Then $h^*(R(V)) \subset P_0 = R(S^1 \times S^1) - Q$. A Λ-fundamental segment of $q^{-1}(h^*(R(V))$ $\subset \tilde{P}_0$ is a straight line J as shown in Figure 8.3.

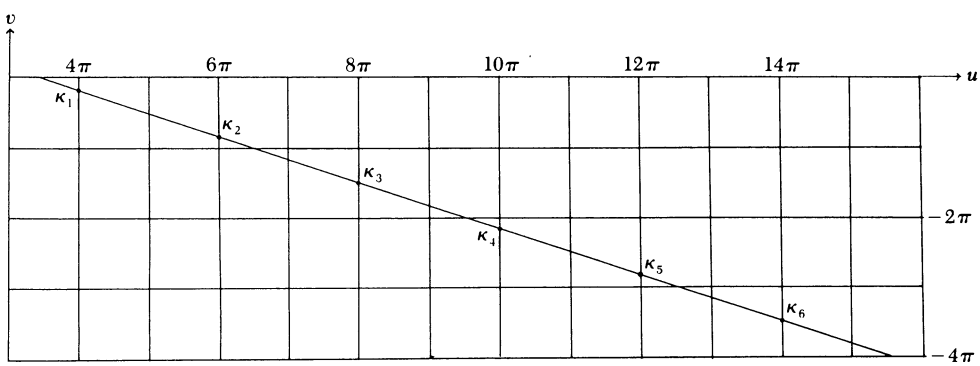

FIGURE 8.3

$R(H)$ consists of 6 points and it is in one-to-one correspondence with the set of the points $J \cap \mathbf{L}$. We label the six points of $R(H)$ as $\kappa_1, \kappa_2, \ldots, \kappa_6$ corresponding to the above six points on J in the order from left to right in Figure 8.3.

Let $S_1, S_2, \ldots, S_6$ be the flat connections on $H \times \mathrm{SU}(2)$ corresponding to the above representations. By the same argument as in the case of N_k, we see that all the gauge equivalence classes $\{[S_j]\}_{1 \leq j \leq 6}$ are non-degenerate critical points of the Chern-Simons functional and they form a basis of the chain complex of $I_*(H)$. The mod 8 degree of $[S_j]$ is computed as follows ([F-S])

$$\begin{array}{ll} \text{degree } 0 & \{[S_3]\} \\ \text{degree } 2 & \{[S_2], [S_4]\} \\ \text{degree } 4 & \{[S_6]\} \\ \text{degree } 6 & \{[S_1], [S_5]\}. \end{array} \tag{8.4}$$

For $1 \leq i \leq 5$, let $\{G_t\}_{0 \leq t \leq 1}$ be a smooth path of smooth connections on $H \times \mathrm{SU}(2)$ such that (1) $G_0 = S_i$, $G_1 = S_{i+1}$ and (2) for each $0 \leq t \leq 1$, $G_t | V \times \mathrm{SU}(2)$ is the flat connection corresponding to κ^t where $\{\kappa^t\}_{0 \leq t \leq 1}$ is the segment in J connecting $\kappa^0 = \kappa_i$ and $\kappa^1 = \kappa_{i+1}$ and (3) near Σ, G_t restricts to $B'_t \times 1$; here B'_t is the flat connection associated with the restriction $\kappa^t | \pi_1(\Sigma)$.

Now we make a technical modification similar to the case of N_k. Let ω be a simple closed curve in V such that $\omega \cap \partial V = \{\text{base point}\}$ and ω represents the homotopy class ϖ. Let Y be the 1-complex in Figure 8.2. Let $\Phi\colon Y \to H$ be an embedding such that $\Phi(Y_1)$ is isotopic to T (= the core curve of $D^2 \times S^1$) and $\Phi(Y_2)$ is isotopic to ω. Let $N_\Phi(Y)$ be a regular neighborhood of $\Phi(Y)$ in H. We set $H_1 = N_\Phi(Y)$ and $H_2 = \overline{H - N_\Phi(Y)}$. We may assume that $H_2 \subset \operatorname{Int} V$. We give a Riemannian metric on H such that, near $\Sigma = H_1 \cap H_2$, it is isometric to the product $\Sigma \times [-1, 1]$ for some metric on Σ. Note that this metric on Σ can be chosen to be the same metric as in the case of N_k and we do so. By essentially the same argument as in Claim (I-1), we can show that $\operatorname{Ker} \mathscr{E}_{G_t}^{2*} = 0$ for $0 \leq t \leq 1$. Also we may assume that $\operatorname{Ker} \mathscr{E}_{G_t}^{1*} = 0$ for $0 \leq t \leq 1$ and that $\gamma(\{G_t\})$ can be defined.

Now we compare $\gamma(\{A_t\})$ with $\gamma(\{G_t\})$. We consider both of M_1 and H_1 to be the same Riemannian manifold with boundary Σ which is homeomorphic to the handlebody of genus 2.

The endpoints of the arcs $\{\rho^t\}_{0 \leq t \leq 1}$ in E_k and $\{\kappa^t\}_{0 \leq t \leq 1}$ in J sit on $\mathbf{L}$. Let $\pi_1^{\mathrm{reg}}(\tilde{P}_0, \mathbf{L})$ be the set of the smooth regular homotopy classes of smoothly immersed curves in $\tilde{P}_0$ which intersect with $\mathbf{L}$ transversely at the endpoints. We require that the homotopy keep the transversality at the endpoints. Then $\{\rho^t\}$

and $\{\kappa^t\}$ determine homotopy classes $[\rho^t]$ and $[\kappa^t]$ in $\pi_1^{\mathrm{reg}}(\tilde{P}_0, \mathbf{L})$, respectively. Obviously the discrete subgroup Λ acts on $\pi_1^{\mathrm{reg}}(\tilde{P}_0, \mathbf{L})$.

Proposition 8.1. *Assume that $[\rho^t]$ and $[\kappa^t]$ are the same homotopy class in $\pi_1^{\mathrm{reg}}(\tilde{P}_0, \mathbf{L})$ up to the Λ-action. Then*

$$\gamma(\{A_t\}) = \gamma(\{G_t\}) \qquad (\mathrm{mod}\ 8).$$

Proof. Let $N(Y)$ be the 3-dimensional thickening of the 1-complex Y in Figure 8.2 with boundary Σ. $N(Y)$ is a handlebody of genus 2. We extend the embeddings $\Psi\colon Y \to N_k$ and $\Phi\colon Y \to H$ to diffeomorphisms $\Psi\colon N(Y) \to N_\Psi(Y)$ and $\Phi\colon N(Y) \to N_\Phi(Y)$, respectively. We assume that $N(Y)$ is endowed with a Riemannian metric which is a product near Σ and that both Ψ and Φ are isometries. We consider the restrictions of the connections $\{A_t\}$ and $\{G_t\}$ on $N_\Psi(Y) \times \mathrm{SU}(2)$ and $N_\Phi(Y) \times \mathrm{SU}(2)$ respectively to be the connections on $N(Y) \times \mathrm{SU}(2)$.

Let $U(Y_1)$ be a closed tubular neighborhood of Y_1 in Int $N(Y)$. We set $N_0(Y) = \overline{N(Y) - U(Y_1)}$. Then $\partial N_0(Y) = \Sigma \cup \partial U(Y_1)$ (disjoint union). We assume that $\Psi(U(Y_1))$ (resp. $\Phi(U(Y_1))$) is the solid torus in N_k (resp. in H) attached to N (resp. V).

Under the assumption of the proposition there is a smooth map $p\colon [0,1] \times [0,1] \to \tilde{P}_0$ such that $p(t,0) = \rho^t$, $p(t,1) = \kappa^t$ for $0 \le t \le 1$ and $\partial p(t,s)/\partial t \ne 0$ for $0 \le t, s \le 1$. $N_0(Y)$ is homotopy equivalent to the one-point union $S^1 \vee (S^1 \times S^1)$. For $0 \le t, s \le 1$, $p(t,s)$ gives a nontrivial flat connection on $(S^1 \times S^1) \times \mathrm{SU}(2)$. From this, it follows that we can choose $\{A_t\}$ and $\{G_t\}$ so that there may be smooth connections $\mathbf{A}$ on $N(Y) \times [0,1] \times [0,1] \times \mathrm{SU}(2)$ and $\mathbf{B}$ on $\Sigma \times [0,1] \times [0,1] \times \mathrm{SU}(2)$ satisfying the following conditions:

(1) $\mathbf{A}|N(Y) \times \{t\} \times \{0\} \times \mathrm{SU}(2) = A_t, \qquad 0 \le t \le 1,$

(2) $\mathbf{A}|N(Y) \times \{t\} \times \{1\} \times \mathrm{SU}(2) = G_t, \qquad 0 \le t \le 1,$

(3) $\mathbf{A}|N(Y) \times \{i\} \times \{s\} \times \mathrm{SU}(2)$ is an irreducible flat connection for $i = 0, 1$ and $0 \le s \le 1$,

(4) $\mathbf{A}|N_0(Y) \times \{t\} \times \{s\} \times \mathrm{SU}(2)$ is an irreducible flat connection and, on $\partial U(Y_1) \times \{t\} \times \{s\} \times \mathrm{SU}(2)$, it is the flat connection associated to $p(t,s)$, for $0 \le t, s \le 1$,

(5) $B_{t,s} = \mathbf{B}|\Sigma \times \{t\} \times \{s\} \times \mathrm{SU}(2)$ is an irreducible flat connection for $0 \le t, s \le 1$, and

(6) $A_{t,s} = \mathbf{A}|N(Y) \times \{t\} \times \{s\} \times \mathrm{SU}(2)$ restricts to the product $B_{t,s} \times 1$ near $\Sigma \times \{t\} \times \{s\}$, for $0 \le t, s \le 1$.

We can construct such an $\mathbf{A}$ by first constructing it on $N_0(Y) \times \mathrm{SU}(2)$ using $p(t,s)$ and its flat extension on $(S^1 \vee (S^1 \times S^1)) \times \mathrm{SU}(2)$ and then extending it smoothly on $U(Y_1) \times \mathrm{SU}(2)$.

We choose and fix a trivialization as in Section 5:

$$\Theta = \Theta_{t,s}\colon \mathscr{H}_{B_{t,s}} = H^1_{B_{t,s}}(\Sigma, \mathrm{su}(2)) \to \mathbf{V}$$

where $\mathbf{V}$ is a 6-dimensional symplectic vector space and Θ is a symplectic isomorphism depending continuously on t, s for $0 \le t, s \le 1$.

Now $\mathbf{A}$ is a homotopy of smooth connections on $N(Y) \times \mathrm{SU}(2)$ connecting $\{A_t\}$ and $\{G_t\}$. For $0 \le t, s \le 1$, the connection $A_{t,s}$ has property (6) and the operators $\mathscr{E}_{A_{t,s}}$ and $\mathscr{E}^*_{A_{t,s}}$ can be defined as in Section 6. By perturbing $\mathbf{A}$ and the metric on $N(Y)$ as in Section 5 if necessary, we may assume that $\operatorname{Ker} \mathscr{E}^*_{A_{t,s}} = 0$ for $0 \le t, s \le 1$. In fact, in this case, using the fact that $N(Y)$ is a handlebody, we can directly construct an $\mathbf{A}$ having such a property (we omit the details). Hence by Lemma 5.1, we get the continuous 2-parameter family of Lagrangians $\{L_{A_{t,s}}\}_{0 \le t,s \le 1}$ of $\mathbf{V}$ as in Section 5.

Let $\{(L^1_{A_t}, L^2_{A_t})\}_{0 \le t \le 1}$ and $\{(L^1_{G_t}, L^2_{G_t})\}_{0 \le t \le 1}$ be the path of Lagrangian pairs associated with the connections $\{A_t\}$ and $\{G_t\}$ on $N_k \times \mathrm{SU}(2)$ and on $H \times \mathrm{SU}(2)$, respectively.

Note that ∂N is a 2-torus, and, for $0 \le t, s \le 1$, $H^1_{A_{t,s}}(\partial N, \mathrm{su}(2))$ is a 2-dimensional real vector space endowed with non-degenerate symplectic structure defined by the cup product and taking the trace of the coefficient. Let $\mathbf{U}$ be a standard non-degenerate 2-dimensional symplectic vector space. We choose and fix a trivialization

$$\Xi = \Xi_{t,s}\colon H^1_{A_{t,s}}(\partial N, \mathrm{su}(2)) \to \mathbf{U}.$$

Then

$$\Theta + \Xi\colon H^1_{A_{t,s}}(\partial N_0(Y), \mathrm{su}(2)) \to \mathbf{V} + \mathbf{U}$$

gives a trivialization of

$$H^1_{A_{t,s}}(\partial N_0(Y), \mathrm{su}(2)) = \mathscr{H}_{B_{t,s}} + H^1_{A_{t,s}}(\partial N, \mathrm{su}(2)).$$

The 2-parameter family $\{L_{A_{t,s}}\}$ gives a homotopy connecting $\{L^1_{A_t}\} = \{L_{A_{t,0}}\}$ and $\{L^1_{G_t}\} = \{L_{A_{t,1}}\}$. For $0 \le t, s \le 1$, the tangent space $X_{t,s}$ of the curve $\{p(t,s)\}_{0 \le t \le 1}$ at $p(t,s)$ is a 1-dimensional subspace of the tangent space of $\tilde{P}_0$ at $p(t,s)$. We can consider $X_{t,s}$ to be a 1-dimensional subspace of $H^1_{A_{t,s}}(\partial U(Y_1), \mathrm{su}(2))$. We denote $\Xi(X_{t,s}) \subset \mathbf{U}$ also by $X_{t,s}$. Then

$$\{L_{A_{t,s}} + X_{t,s}\}_{0 \le t,s \le 1} \tag{8.1.1}$$

is a continuous 2-parameter family of Lagrangians of $\mathbf{V} + \mathbf{U}$.

Since $A_t|N \times \mathrm{SU}(2)$ and $G_t|V \times \mathrm{SU}(2)$ are both irreducible flat connections for $0 \leq t \leq 1$, by the Mayer-Vietoris theorem, we have the exact sequences

$$(8.1.2)\qquad \begin{aligned} &0 \to H^1_{A_t}(N, \mathrm{su}(2)) \\ &\quad \to H^1_{A_t}(N_0(Y), \mathrm{su}(2)) + H^1_{A_t}(M_2, \mathrm{su}(2)) \to \mathscr{H}_{B_t} \to 0, \\ &0 \to H^1_{G_t}(V, \mathrm{su}(2)) \\ &\quad \to H^1_{G_t}(N_0(Y), \mathrm{su}(2)) + H^1_{G_t}(H_2, \mathrm{su}(2)) \to \mathscr{H}_{B'_t} \to 0. \end{aligned}$$

Now $\mathrm{Im}(H^1_{A_t}(N, \mathrm{su}(2)) \to H^1_{A_t}(\partial N, \mathrm{su}(2)))$ and $\mathrm{Im}(H^1_{A_t}(V, \mathrm{su}(2)) \to H^1_{A_t}(\partial V, \mathrm{su}(2)))$ are identified with $X_{t,0}$ and $X_{t,1}$, respectively. Let $*$ be the Hodge star operator on $H^1_{A_{t,s}}(\partial U(Y_1), \mathrm{su}(2))$, where we consider the said space as the space of the harmonic 1-forms on $\partial U(Y_1)$ for $0 \leq t, s \leq 1$. We denote $\Xi(*X_{t,s}) \subset \mathbf{U}$ also by $* X_{t,s}$.

By definition,

$$L^2_{A_t} = \mathrm{Im}\left(H^1_{A_{t,0}}(M_2, \mathrm{su}(2)) \to \mathscr{H}_{B_{t,0}}\right),$$

$$L^2_{G_t} = \mathrm{Im}\left(H^1_{A_{t,1}}(H_2, \mathrm{su}(2)) \to \mathscr{H}_{B_{t,1}}\right).$$

We set, for $0 \leq t, s \leq 1$,

$$Z_{t,s} = (\Theta + \Xi)\left(\mathrm{Im}\left(H^1_{A_{t,s}}(N_0(Y), \mathrm{su}(2))\right) \to H^1_{A_{t,s}}(\partial N_0(Y), \mathrm{su}(2))\right),$$

where the map is the homomorphism induced by the inclusion. Then by the exact sequence (8.1.2) we see that both of

$$(8.1.3)\qquad \left(Z_{t,0}, L^2_{A_t} + * X_{t,0}\right) \quad \text{and} \quad \left(Z_{t,1}, L^2_{G_t} + * X_{t,1}\right)$$

are complementary pairs of Lagrangians of $\mathbf{V} + \mathbf{U}$ for $0 \leq t \leq 1$. Consider the two-parameter family of Lagrangians of $\mathbf{V} + \mathbf{U}$, $\{* Z_{t,s}\}_{0 \leq t, s \leq 1}$. Combining it with (8.1.1), we obtain the two-parameter continuous family of Lagrangian pairs of $\mathbf{V} + \mathbf{U}$,

$$\{q(t,s)\}_{0 \leq t, s \leq 1} = \left\{\left(L_{A_{t,s}} + X_{t,s}, * Z_{t,s}\right)\right\}_{0 \leq t, s \leq 1}.$$

Now we form the space $\mathscr{L}^2$ of all the Lagrangian pairs of $\mathbf{V} + \mathbf{U}$ and the subspaces $\mathscr{L}^2_k$ for $k = 0, 1, \ldots, 4$, as in Section 6. By property (3) of $\mathbf{A}$, we see that $q(0, s)$, $q(1, s) \in \mathscr{L}^2_0$ for $0 \leq s \leq 1$. Hence the paths $\{q(t, 0)\}_{0 \leq t \leq 1}$ and

$\{q(t,1)\}_{0\le t\le 1}$ are mutually homotopic relative to $\mathscr{L}_0^2$ and they determine the same homotopy class in $\pi_1(\mathscr{L}^2, \mathscr{L}_0^2)$. From the complementary relations of (8.1.3), it follows that $\{q(t,0)\}_{0\le t\le 1}$ and $\{q(t,1)\}_{0\le t\le 1}$ are homotopic rel. $\mathscr{L}_0^2$ to the paths

$$\left\{\left(L^1_{A_t} + X_{t,0},\ L^2_{A_t} + *X_{t,0}\right)\right\}_{0\le t\le 1},$$

and

$$\left\{\left(L^1_{G_t} + X_{t,1},\ L^2_{G_t} + *X_{t,1}\right)\right\}_{0\le t\le 1},$$

respectively. This implies that the paths $\{(L^1_{A_t}, L^2_{A_t})\}_{0\le t\le 1}$ and $\{(L^1_{G_t}, L^2_{G_t})\}_{0\le t\le 1}$ are mutually homotopic rel. $\mathscr{L}_0^2$. Thus $\gamma(\{A_t\}) = \gamma(\{G_t\})$. This proves the proposition. q.e.d.

By Proposition 8.1, Figure 8.3 and (8.4), we see that the difference of the mod 8 degree $d([K_{j+1}]) - d([K_j])$ of the flat connections corresponding to two consecutive points on the curve E_k can be counted as 4, -2 or 2 according to whether the (oriented) arc in E_k connecting ρ_j and ρ_{j+1} is regularly homotopic rel. $\mathbf{L}$ to the arcs (i), (ii) or (iii) respectively in Figure 8.4.

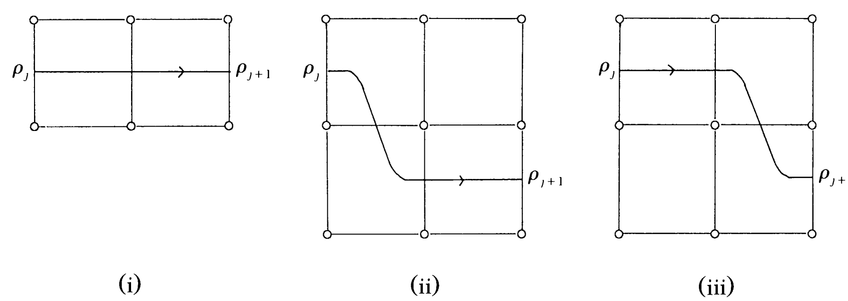

FIGURE 8.4

Here $\circ$ denotes those points with coordinates $(m\pi, n\pi)$ for m, n integers. The same conclusion holds for the flat connections corresponding to the points on E'_k. Consequently we have the following result:

(7.5) $\quad d\big([K_{j+1}]\big) - d\big([K_j]\big) = 4 \quad$ for $1/4 + k/2 < j \le -1$ and

$1 \le j < -3/4 - k/2;$

$d([K_1]) - d([K_0]) = -2$

and

$$d\big([K'_{j+1}]\big) - d\big([K'_j]\big) = 4 \qquad \text{for } 3/4 + k/2 < j \le -1 \quad \text{and}$$
$$1 \le j < -1/4 - k/2;$$

$$d([A'_1]) - d([A'_0]) = 2.$$

(II) *Computation of* $d([K_0])$ *and* $d([K'_1])$.

PROPOSITION 8.2. *Neither of the* mod 8 *degrees* $d([K_0])$ *and* $d([K'_1])$ *depends on a particular value of* $k = -1, -2, \dots$.

Proof. Let $N(Y)$ and Ψ: $N(Y) \to N_k$ both be as in the proof of Proposition 8.1. $N(Y)$ is the boundary connected sum of the 3-dimensional thickenings $N(Y_i)$ of Y_i $(i = 0, 1, 2)$, $N(Y) = N(Y_1) \natural N(Y_0) \natural N(Y_2)$. Let $U(Y_1)$ be a closed tubular neighborhood of Y_1 in $N(Y)$ and let $N_0(Y) = \overline{N(Y) - U(Y_1)}$ as before. We assume that $\Psi(U(Y_1))$ is the solid torus in N_k attached to N and $\Psi(N_0(Y)) = U(\nu \cup \partial N)$, where ν is the simple closed curve in N chosen in (I) and $U(\nu \cup \partial N)$ is a closed regular neighborhood of $\nu \cup \partial N$ in N.

We adopt the splitting $N_k = M_1 \cup M_2$, where $M_1 = \Psi(N(Y))$, $M_2 = \overline{N_k - \Psi(N(Y))}$ and $M_1 \cap M_2 = \Sigma$ as before. We give a Riemannian metric on N_k such that, near Σ, it is isometric to $\Sigma \times [-1, 1]$, and we assume that the metrics on N and Σ are both independent of k.

Let $\Sigma_1 = \Sigma \cap N(Y_1)$ and $\Sigma_2 = \Sigma \cap N(Y_2)$. Let h_i: $\Sigma_i \to \Sigma$ and j_i: $\Sigma \to M_i$ be the inclusions $(i = 1, 2)$.

Introduce the vector spaces of real-valued harmonic 1-forms,

$$\mathscr{H}^1(\Sigma) = \{\eta \in \Omega^1(\Sigma) | d\eta = 0 \text{ and } d^*\eta = 0\},$$

$$\mathscr{H}^1(M_i) = \{\eta \in \Omega^1(M_i) | d\eta = 0, d^*\eta = 0 \text{ and } j_i^*(*\eta) = 0\},$$

where $*\eta$ denotes the Hodge star of η.

Then by de Rham's theorem, $\mathscr{H}^1(\Sigma)$ and $\mathscr{H}^1(M_i)$ are isomorphic to $H^1(\Sigma, \mathbf{R})$ and $H^1(M_i, \mathbf{R})$, respectively. For $\eta \in \mathscr{H}^1(\Sigma)$ (resp. $\in \mathscr{H}^1(M_i)$), we denote the corresponding cohomology class by $[\eta] \in H^1(\Sigma, \mathbf{R})$ (resp. $\in H^1(M_i, \mathbf{R})$). There is a natural isomorphism Υ: $\mathscr{H}^1(M_1) + \mathscr{H}^1(M_2) \to \mathscr{H}^1(\Sigma)$ which sends (η_1, η_2) to the projection of $(j_1^*\eta_1 - j_2^*\eta_2)$ onto the harmonic part of it. Also there is an isomorphism $H^1(\Sigma, \mathbf{R}) \to H^1(\Sigma_1, \mathbf{R}) + H^1(\Sigma_2, \mathbf{R})$ which sends $[\eta]$ to $(h_1^*[\eta], -h_2^*[\eta])$.

We can choose $\alpha_1 \in \Upsilon(\mathscr{H}^1(M_1) + 0)$ and $\alpha_2 \in \Upsilon(0 + \mathscr{H}^1(M_2))$ such that $h_1^*([\alpha_i]) = 0$ $(i = 1, 2)$ and $\langle [\alpha_1] \cup [\alpha_2], [\Sigma] \rangle < 0$, where $\cup$ denotes the cup product, $[\Sigma]$ is the fundamental class of Σ and $\langle\ ,\ \rangle$ is the Kronecker product. Let $\sigma_0 \in \mathrm{su}(2)$ be an element such that $-\mathrm{tr}(\sigma_0)^2 = 1$. We define $\omega_i \in \mathscr{H}^1(\Sigma) \otimes \mathrm{su}(2)$ by $\omega_i = \alpha_i \otimes \sigma_0$. Let β be an increasing C^∞ function on $[0, 1]$

such that $\beta = 0$ on $[0, 1/3]$ and $\beta = 1$ on $[2/3, 1]$. We define $a \in \Omega^1(N_k) \otimes$ su(2) by

$$(8.2.1)\quad \begin{aligned} a &= -*(d\beta/ds)(ds \wedge (\omega_1 - \omega_2)) && \text{on } \Sigma \times [0,1] \\ &= 0 && \text{on } N_k - \Sigma \times [0,1], \end{aligned}$$

where ω_i is regarded as a 1-form on $\Sigma \times [0, 1]$, constant with respect to $0 \leq s \leq 1$ $(i = 1, 2)$.

Let θ be the trivial connection on $N_k \times \mathrm{SU}(2)$. For a small $\varepsilon > 0$, we define a smooth connection $\theta(\varepsilon)$ on $N_k \times \mathrm{SU}(2)$ by

$$\theta(\varepsilon) = \theta + \varepsilon a.$$

Then, in [T], Taubes showed that, for small $\varepsilon > 0$, the 0-eigenvalue of D_θ with multiplicity 3 splits into three small eigenvalues $\{\varepsilon^2\lambda_1, \varepsilon^2\lambda_2, \varepsilon^2\lambda_3\}$ of $D_{\theta(\varepsilon)}$ up to order ε^3, where $\{\lambda_1, \lambda_2, \lambda_3\}$ are the eigenvalues of the symmetric bilinear form on su(2) defined by

$$(8.2.2)\qquad \tau_a(\sigma_1, \sigma_2) = -2\,\mathrm{tr}\int_\Sigma [\sigma_1, \omega_1] \wedge [\sigma_2, \omega_2]$$

(see Sections 2 and 7, (7.7) in [T]). As in the choice of ω_1 and ω_2, the bilinear form (8.2.2) is positive definite. Hence if we set $\theta' = \theta(\varepsilon)$ as in (2.2) in Section 2, then $p(\theta') = 3$.

Now $\mathscr{H}^1(\Sigma) \otimes \mathrm{su}(2)$ is a non-degenerate symplectic vector space with the symplectic pairing,

$$(8.2.3)\qquad \langle \omega, \omega' \rangle = -\mathrm{tr}\int_\Sigma \omega \wedge \omega'$$

for $\omega, \omega' \in \mathscr{H}^1(\Sigma) \otimes \mathrm{su}(2)$.

Let $L(\omega_1, \omega_2)$ be the su(2)-subspace of $\mathscr{H}^1(\Sigma) \otimes \mathrm{su}(2)$ spanned by the two vectors $(\omega_1 - \omega_2)$ and $*(\omega_1 - \omega_2)$. We call $L(\omega_1, \omega_2)^\perp$ the orthogonal complement of $L(\omega_1, \omega_2)$ with respect to the symplectic pairing of (8.2.3). Then $\dim_{\mathbf{R}} L(\omega_1, \omega_2) = \dim_{\mathbf{R}} L(\omega_1, \omega_2)^\perp = 6$.

Now, for $k = -1, -2, \ldots,$ the curve E_k in Figure 8.1 contains the point $e_0 = (\pi/2, 0)$. It corresponds to the representation $e_0\colon \pi_1(N) \to \mathrm{SU}(2)$ such that $e_0(\pi_1(N))$ is the binary dihedral group in SU(2) and it does not depend on a particular value of k. Note that $e_0(m) \neq \pm 1$ and $e_0(\ell) = 1$.

Having assumed that $\Psi(U(Y_1))$ is the solid torus in N_k attached to N, $\Sigma \subset \mathrm{Int}\, N$, we can take two simple closed curves on $\Sigma_1 \subset \Sigma$ isotopic to m and ℓ in N. We denote these two curves on Σ_1 by the same letters. Then the restriction of e_0 on $\pi_1(\Sigma)$, $\xi_0 = e_0|\pi_1(\Sigma)$, is an irreducible representation of $\pi_1(\Sigma)$ into SU(2) such that $\xi_0(\ell) = 1$.

There is a smooth path $\{\xi^t\}_{0 \le t \le 1/2}$ in the space of all the representations of $\pi_1(\Sigma)$ into SU(2) such that (1) ξ^0 = the trivial representation and $\xi^{1/2} = \xi_0$, (2) ξ^t is an irreducible representation for $0 < t \le 1/2$, and (3) $\xi^t(\ell) = 1$ for $0 \le t \le 1/2$.

Let $\{B_t\}_{0 \le t \le 1/2}$ be the smooth path of the flat connections on $\Sigma \times \mathrm{SU}(2)$ associated to the path of the representations $\{\xi^t\}_{0 \le t \le 1/2}$. Let $\mathscr{H}_{B_t}$ be the space of the su(2)-valued harmonic 1-forms on Σ with respect to B_t for $0 < t \le 1/2$. Then $\mathscr{H}_{B_t}$ converges to a subspace of $\mathscr{H}^1(\Sigma) \otimes \mathrm{su}(2)$ as $t \to 0$. We can choose the above $\{\xi^t\}_{0 < t \le 1/2}$, and hence $\{B_t\}_{0 < t \le 1/2}$, so that the following condition holds:

$$\lim_{t \to 0} \mathscr{H}_{B_t} = L(\omega_1, \omega_2). \tag{8.2.4}$$

Let $\{A_t\}_{0 \le t \le 1}$ be a smooth one-parameter family of smooth connections on $N_k \times \mathrm{SU}(2)$ satisfying the following conditions:

(1) $A_0 = \theta(\varepsilon)$ and $A_1 = K_0$.

(2) For $0 \le t \le 1/2$, A_t restricts to the product $B_t \times 1$ on $\Sigma \times [-1/4, 1/4] \times \mathrm{SU}(2)$, where $\{B_t\}_{0 \le t \le 1/2}$ is the above path of flat connections on $\Sigma \times \mathrm{SU}(2)$.

(3) For $1/2 \le t \le 1$, A_t restricts to the flat connection on $N \times \mathrm{SU}(2)$ corresponding to ρ^t, where $\{\rho^t\}_{1/2 \le t \le 1}$ is the arc in E_k connecting $\rho^{1/2} = e_0$ and $\rho^1 = \rho_0$.

By the definition of $d([K_0])$ in (2.2) in Section 2, and by the above remark on the small eigenvalues of $D_{\theta(\varepsilon)}$, we have

$$d([K_0]) = 3 + \text{the spectral flow of } \{D_{A_t}\}_{0 \le t \le 1}.$$

For $0 < t \le 1$, the operators $\mathscr{E}^i_{A_t}, \mathscr{E}^{i*}_{A_t}$ are defined. By Proposition 5.1, perturbing A_t if necessary, we may assume that $\operatorname{Ker} \mathscr{E}^{i*}_{A_t} = 0$ $(i = 1, 2)$. Then the continuous paths of the Lagrangian pair of $\mathbf{V}$, (L^1_t, L^2_t) are defined.

Since N_k is a homology 3-sphere, $j_1^*(H^1(M_1, \mathbf{R}))$ and $j_2^*(H^1(M_2, \mathbf{R}))$ are mutually transversal in $H^1(\Sigma, \mathbf{R})$. It follows that, for any small $\varepsilon > 0$, we can choose $\delta > 0$ such that, for $0 < t \le \delta$, $L^1_t \cap L^2_t = 0$. Thus we can define the invariant $\gamma(\{A_t\})$ by taking the limit of $\gamma(\{A_t\}_{\delta \le t \le 1})$ as $\delta \to 0$.

For $r \ge 0$, let $N_k(r)$ be the elongated manifold defined at the beginning of Section 7. As was mentioned there, A_t defines naturally the connection $A_t(r)$ on $N_k(r) \times \mathrm{SU}(2)$ and hence the operator $D_{A_t(r)}$ for $0 \le t \le 1$. By the construction of $\theta(\varepsilon)$ and (8.2.4), there is $\delta_0 > 0$ such that $\operatorname{Ker} D_{A_t(r)} = 0$ for $r \ge 0$ and $0 < t < \delta_0$. In the proof of Theorem 1.1 in Section 7, we need the fact that $\operatorname{Ker} D_{A_0(r)} = \operatorname{Ker} D_{A_1(r)} = 0$ for $r \ge 0$, and this is the only point at which we used the irreducible flatness conditions on A_0 and A_1. Hence we can adopt essentially the same argument as in Section 7 in this case, and we have

$$d([K_0]) = 3 + \gamma(\{A_t\}). \tag{8.2.5}$$

We compare (8.2.5) for $k \neq \hat{k}$ negative integers. Let $\{\hat{A}_t\}_{0 \le t \le 1}$ be the above path of connections on $M \times \mathrm{SU}(2)$ corresponding to $\hat{k}$. Let Ψ and $\hat{\Psi}$ be the embeddings of $N(Y)$ to N_k and $N_{\hat{k}}$ respectively. Let $\mathscr{g} = (\hat{\Psi}|\Sigma)^{-1}(\Psi|\Sigma)$ which is a diffeomorphism of Σ which is the identity on Σ_2. The irreducible flat connection B_t on $\Sigma \times \mathrm{SU}(2)$ is independent of k and $\hat{k}$ for $0 < t \le 1$; $\mathscr{g}$ induces the isomorphisms $\mathscr{g}_*$ of the symplectic vector spaces $\mathscr{H}_{B_t}$. By choosing a suitable trivialization Θ: $\mathscr{H}_{B_t} \to \mathbf{V}$, we may assume that $\mathscr{g}_*$ is identified with an element $g \in \mathrm{Sp}(3, \mathbf{R})$ for $0 < t \le 1$. Let $\{(L^1_{A_t}, L^2_{A_t})\}_{0<t\le 1}$ and $\{(L^1_{\hat{A}_t}, L^2_{\hat{A}_t})\}_{0<t\le 1}$ be the paths of the Lagrangian pairs for k and $\hat{k}$, respectively. Then we have the relations, $L^1_{\hat{A}_t} = L^1_{A_t}$ and $L^2_{\hat{A}_t} = gL^2_{A_t}$ for $0 < t \le 1$. Since g is the identity on a 2-dimensional subspace in $\mathbf{V}$ and it is a transvection on the complementary 2-dimensional subspace, we can choose a path $\{g_u\}_{0 \le u \le 1}$ in $\mathrm{Sp}(3, \mathbf{R})$ such that $g_0 = g$, $g_1 = 1$ (the identity) and $L^1_{A_0} \cap g_u L^2_{A_0} = 0$, $L^1_{A_1} \cap g_u L^2_{A_1} = 0$ for $0 \le u \le 1$. The paths of Lagrangian pairs of $\mathbf{V}$, $\{(L^1_t, g_u L^2_t)\}_{0<t\le 1}$ $(0 \le u \le 1)$, give a homotopy connecting the paths of the Lagrangian pairs for k and $\hat{k}$. Therefore $\gamma(\{A_t\}_{0<t\le 1})$ is independent of k.

This completes the proof of the independence of $d([K_0])$ on k. Essentially the same argument shows that $d([K_1'])$ is independent of k, and we omit the details.

This completes the proof of Proposition 8.2. q.e.d.

It is well-known that N_{-1} is orientation preservingly diffeomorphic to the Brieskorn homology 3-sphere $\Sigma(2, 3, 7)$. In [F-S], $I_* = I_*(\Sigma(2, 3, 7))$ was calculated. The result is that $I_{\text{odd}} = I_0 = I_4 = 0$ and I_2 and I_6 are both free abelian groups of rank 1. Hence $d([K_0])$ and $d([K_1'])$ are 2 and 6 respectively. Combining this with (8.5) above, we obtain Theorem 1.2.

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References

[A-B] M. F. Atiyah and R. Bott, The Yang-Mills equations over Riemann surfaces, Phil. Trans. R. Soc. Lond. A **308** (1982), 523–615.

[A-P-S 1] M. F. Atiyah, V. K. Patodi, and I. M. Singer, Spectral asymmetry and Riemannian geometry I, Math. Proc. Camb. Philos. Soc. **77** (1975), 43–69.

[A-P-S 2] ———, Spectral asymmetry and Riemannian geometry II, Math. Proc. Camb. Philos. Soc. **78** (1975), 405–443.

[A-P-S 3] ———, Spectral asymmetry and Riemannian geometry III, Math. Proc. Camb. Philos. Soc. **79** (1976), 71–99.

[B] G. Burde, SU(2)-representation spaces for two-bridge knot groups, preprint, J. W. Goethe-Universität, Frankfurt am Main, 1988.

[E-N] D. Eisenbud and W. Neumann, *Three-Dimensional Link Theory and Invariants of Plane Curve Singularities*, Annals of Math. Studies **11**, Princeton, 1985.

[F] A. Floer, An instanton-invariant for 3-manifolds, Commun. Math. Phys. **118** (1988), 215–240.

[F2] ———, A relative Morse index for the symplectic action, Comm. Pure Appl. Math. **41** (1988), 393–407.

[F-S] R. Fintushel and J. R. Stern, Instanton homology of Seifert–fibered homology three spheres, to appear in Proc. London Math. Soc.

[Ko] U. Koschorke, Infinite dimensional *K*-theory and characteristic classes of Fredholm bundle maps, Proc. Sympos. Pure Math. Vol. 15, A.M.S., Providence, RI, 1970.

[Pa] T. Parker, Gauge theories on four dimensional Riemannian manifolds, Commun. Math. Phys. **85** (1982), 199.

[T] C. H. Taubes, Casson's invariant and gauge theory, J. Diff. Geom. **31** (1990), 547–599.

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