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# Universal Wu classes

Dedicated to Professor Masahiro Sugawara on his 60th birthday

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## § 1. Introduction

Let $BO$ be the space which classifies stable (real) vector bundles, and consider its mod 2 cohomology $H^*(BO; Z_2)$ (the coefficient $Z_2$ will be omitted often). Then, $H^*(BO)$ is the polynomial algebra over $Z_2$ on the universal Stiefel-Whitney classes $w_i \in H^i(BO)$ for $i \geqq 1$ [2, Th. 7.1]. Let $v_i \in H^i(BO)$ be the universal Wu classes (cf. [1, p. 225], [4, p. 315]) defined inductively by

(1) $$v_0 = 1 = w_0 \text{ and } w_i = \sum_{j=0}^{i} Sq^j v_{i-j}; \text{ i.e., } w = Sqv \text{ or } v = Sq^{-1}w$$

(Wu's formula, cf. [2, Th. 11.14]) for $w = \sum_i w_i$, $v = \sum_i v_i$ and the Steenrod squaring operator $Sq = \sum_i Sq^i$ with $Sq^{-1}$ given by $Sq^{-1}Sq = 1 = Sq\,Sq^{-1}$.

In this note, we prove a formula representing $v_i$ by $w_j$'s modulo

(2) the ideal $I^{(2)} = (w_1^2, w_2^2, \ldots)$ of $H^*(BO)$ generated by the squares $w_i^2$ for $i \geqq 1$:

THEOREM. (i) (Stong) $v_a \equiv v_{a_1} \cdots v_{a_l}$ mod $I^{(2)}$ *for any* $a \geqq 1$, *where* $a = a_1 + \cdots + a_l$ *is the dyadic expansion of* $a$.

(ii) $v_{2a} \equiv v_a^{(2)} + \sum_{i=0}^{a-1} w_i w_{2a-i}$ mod $I^{(2)}$ *for any power* $a$ *of* 2, *and* $v_1 = w_1$.
*Here, the notation* $x^{(2)}$ *for* $x \in H^a(BO)$ *is used in the following sence*:

(3) *If* $x \equiv \sum_{i=1}^{k} x_i \in H^a(BO)$ mod $I^{(2)}$ *with monomials* $x_i$ *on* $w_j$*'s, we have uniquely* $x^{(2)} \in H^{2a}(BO)$ mod $I^{(2)}$ *given by*

$$x^{(2)} (= (x^2 - \textstyle\sum_{i=1}^{k} x_i^2)/2) = \sum_{1 \leqq i < j \leqq k} x_i x_j \quad \text{and} \quad x^{(2)} = 0 \quad \text{if} \quad k \leqq 1;$$
$(x+y)^{(2)} \equiv x^{(2)} + y^{(2)} + xy$ *and* $(xy)^{(2)} \equiv 0$ mod $I^{(2)}$ *for any* $x$, $y$.

COROLLARY. $v \equiv 1 + \sum w_{i_1} \cdots w_{i_l}$ mod $I^{(2)}$,
*where* $\sum$ *is taken over all sequences* $1 \leqq i_1 < \cdots < i_l$ $(l \geqq 1)$ *satisfying*

(4) $\{i_1, \ldots, i_l\} = \{\alpha_1, \beta_1, \ldots, \alpha_m, \beta_m, \gamma_1, \ldots, \gamma_n\}$ $(l = 2m + n,\ m \geqq 0,\ n \geqq 0)$ *such that* $\alpha_j + \beta_j$ *and* $\gamma_j$ *are all powers of* 2.

A formula modulo the ideal generated by $w_i^2$ and $\prod_{j=1}^{4} w_{i_j}$ is previously known to the author. Theorem (i) is due to Professor Robert E. Stong, and the author is most grateful to his valuable advices during this work.

## § 2. Proof of Theorem (i)

Let $RP^k$ be the $k$-dimensional real projective space, and consider the $m$-fold product space $X_{n,m}=(RP^1\times RP^n)^m$ with the projections $p_i$: $X_{n,m}\to RP^1\times RP^n$ to the $i$th factor, $q_1$: $RP^1\times RP^n\to RP^1$ and $q_n$: $RP^1\times RP^n\to RP^n$ $(n\geqq 2)$. Moreover, let $\xi_k$ be the canonical line bundle over $RP^k$, and consider the vector bundle

$$\eta_{n,m} = \oplus_{i=1}^m (p_i^*q_n^*\xi_n\oplus\zeta_i^\perp),\ \zeta_i = p_i^*q_1^*\xi_1 \otimes p_i^*q_n^*\xi_n, \text{ over } X_{n,m},$$

where $\zeta^\perp$ is a bundle such that $\zeta\oplus\zeta^\perp$ is the trivial bundle.

Then, the total Stiefel-Whitney (resp. Wu) class $w(\eta_{n,m})=\sum_i w_i(\eta_{n,m})$ (resp. $v(\eta_{n,m})=\sum_i v_i(\eta_{n,m})=Sq^{-1}w(\eta_{n,m})$) of $\eta_{n,m}$ is given by the following

LEMMA 2.1. *Put* $\alpha_i=w_1(p_i^*q_n^*\xi_n)$ *and* $\sigma_i=w_1(p_i^*q_1^*\xi_1)$. *Then*:

(i) $H^*(X_{n,m};Z_2)=Z_2[\sigma_1,\alpha_1,\dots,\sigma_m,\alpha_m]/(\sigma_1^2,\dots,\sigma_m^2,\alpha_1^{n+1},\dots,\alpha_m^{n+1})$.

(ii) $w(\eta_{n,m})=\prod_{i=1}^m\{1+\sigma_i(1+\alpha_i)^{-1}\}$; *i.e.*, $w_i(\eta_{n,m})=\sum(\prod_{k=1}^r\sigma_{i_k}\alpha_{i_k}^{s_k})$, *where the sum is taken over all* $1\leqq i_1<\cdots<i_r\leqq m$ *and* $s_k\geqq 0$ $(1\leqq k\leqq r)$ *with* $r+\sum_{k=1}^r s_k=i$.

(iii) $v(\eta_{n,m}) = \prod_{i=1}^m(1+\sum_{r\geqq 0}\sigma_i\alpha_i^{-1+2^r})$; *i.e.*, $v_i(\eta_{n,m}) = \sum(\prod_{k=1}^r\sigma_{i_k}\alpha_{i_k}^{-1+t_k})$, *where the sum is taken over all* $1\leqq i_1<\cdots<i_r\leqq m$ *and powers* $t_k$ *of* $2(1\leqq k\leqq r)$ *with* $\sum_{k=1}^r t_k=i$.

PROOF. (i) holds by the definition of $\xi_k$. $p_i^*q_k^*\xi_k$'s are line bundles, and the basic properties of the Stiefel-Whitney classes for line bundles imply that $w(p_i^*q_n^*\xi_n)=1+\alpha_i$, $w(\zeta_i^\perp)=(1+\sigma_i+\alpha_i)^{-1}$ and (ii), because $\sigma_i^2=0$ and so $(1+\alpha_i)(1+\sigma_i+\alpha_i)^{-1}=1+\sigma_i(1+\sigma_i+\alpha_i)^{-1}=1+\sigma_i(1+\alpha_i)^{-1}$. (ii) implies (iii), because the basic properties of $Sq$ (cf. [3]) show that $Sq\,\sigma_i=\sigma_i$, $Sq(\alpha_i^t)=\alpha_i^t+\alpha_i^{2t}$ for $t=2^r$, and

$$Sq(1+\textstyle\sum_{r\geqq 0}\sigma_i\alpha_i^{-1+2^r})=Sq\{1+\sigma_i(1+\sum_{r\geqq 0}\alpha_i^{2^r})^{-1}\}=1+\sigma_i(1+\alpha_i)^{-1}. \quad\square$$

LEMMA 2.2. *Put* $w_i=w_i(\eta_{n,m})\in H^i(X_{n,m};Z_2)$. *Then*:

(i) $w_i^2=0$ *for any* $i\geqq 1$, *and* $w_{i_1}\cdots w_{i_l}=0$ *for any* $i_k\geqq 1$ *and* $l>m$.

(ii) *In* $H^i(X_{n,m};Z_2)$ *with* $i\leqq n+1$, *the monomials* $w_{i_1}\cdots w_{i_l}$, *for* $1\leqq l\leqq m$, $1\leqq i_1<\cdots<i_l$ *and* $\sum_{k=1}^l i_k=i$, *are linearly independent.*

PROOF. Lemma 2.1 (i) and (ii) show the lemma, because
$w_{i_1}\cdots w_{i_l}=\sum_{1\leqq j_1,\dots,j_l\leqq m}(\prod_{k=1}^l\sigma_{j_k}\alpha_{j_k}^{-1+i_k})+\sum_{l'>l}(\prod_{k=1}^{l'}\sigma_{j_k}\alpha_{j_k}^{s_k})$. $\square$

LEMMA 2.3. *Let* $(t_1,\dots,t_r)$ *be a sequence of powers* $t_k$ *of* 2 *with* $\sum_{k=1}^r t_k=a$.

*Then, for any* $b \geqq 1$, *the number of all subsequences* $(t_{j_1}, \dots, t_{j_s})$ $(1 \leqq j_1 < \cdots < j_s \leqq r)$ *with* $\sum_{k=1}^{s} t_{j_k} = b$ *is congruent to* $\binom{a}{b}$ mod 2.

PROOF. If $t_k = 1$ for all $k$, then the lemma is trivial. Assume $t_k \geqq 2$ for some $k$; and consider $T = (t_1, \dots, t_{k-1}, u, v, t_{k+1}, \dots, t_r)$ with $u = v = t_k/2$, and its subsequences $S \subset T$. Then, $\#\{S \mid S \ni u, S \not\ni v\} = \#\{S \mid S \not\ni u, S \ni v\}$ and $\#\{S \mid S \ni u, v,$ or $S \not\ni u, v\} = \#\{$all subsequences of $(t_1, \dots, t_r)\}$, where $\#$ denotes the number of elements. Thus the lemma holds by induction. □

PROPOSITION 2.4. $v_a(\eta_{n,m}) = \prod_{i=1}^{l} v_{a_i}(\eta_{n,m})$, *where* $a = a_1 + \cdots + a_l$ *is the dyadic expansion of* $a \geqq 1$ *(i.e.,* $a_1 > \cdots > a_l$ *and they are powers of* 2).

PROOF. Compare the both sides by Lemma 2.1 (iii), by noticing that $\sigma_i^2 = 0$. Then the equality follows from Lemma 2.3, since $\binom{a}{a_i} \equiv 1$ mod 2. □

PROOF OF THEOREM (i). Take $n$ and $m$ to satisfy $n+1 \geqq a$ and $(m+1)(m+2) > 2a$, and let $\tilde{\eta}_{n,m}: X_{n,m} \to BO$ be the classifying map of the bundle $\eta_{n,m}$ over $X_{n,m}$. Then, $\tilde{\eta}_{n,m}^*(v_a) = v_a(\eta_{n,m}) = \prod_{i=1}^{l} v_{a_i}(\eta_{n,m}) = \tilde{\eta}_{n,m}^*(\prod_{i=1}^{l} v_{a_i})$ by Proposition 2.4; hence $v_a - \prod_{i=1}^{l} v_{a_i}$ is in $I^{(2)}$ by Lemma 2.2. □

## §3. Proof of Theorem (ii)

LEMMA 3.1. $\sum_{i=0}^{a-1} Sq^i(xv_{a-i}) = \sum_{i=0}^{a-1} (Sq^i x) w_{a-i}$ *for any* $x \in H^*(BO: Z_2)$.

PROOF. $\sum_{i=0}^{a} Sq^i(xv_{a-i}) = \sum_{i=0}^{a} \sum_{j=0}^{i} [= \sum_{j=0}^{a} \sum_{i=j}^{a}] (Sq^j x)(Sq^{i-j} v_{a-i}) = \sum_{j=0}^{a} (Sq^j x) w_{a-j}$ by (1), and the lemma holds since $v_0 = 1 = w_0$. □

LEMMA 3.2. *Let a be a power of* 2, *and* $0 \leqq b < 2a$. *Then,*

$$w_{2a+b} + Sq^b v_{2a} + \sum_{i=0}^{b-1} w_{b-i} Sq^i v_{2a} \equiv \sum_{i=0}^{a-1} w_{a+b-i} Sq^i v_a \quad \textit{if} \quad b < a,$$

$$\equiv \sum_{i=b-a+1}^{a-1} \Big(w_{a+b-i} + \sum_{j=0}^{b-a} w_{b-i-j} Sq^j v_a\Big) Sq^i v_a \quad \textit{if} \quad b \geqq a, \quad \text{mod } I^{(2)}.$$

PROOF. Hereafter, 'mod $I^{(2)}$' is often omitted. We notice that

(5) $Sq^i(I^{(2)}) \subset I^{(2)}$, and $Sq^i v_k \equiv 0$ if $i \geqq k \geqq 1$ (e.g., $k = 2a + b - i \leqq a$),

by the definition of $I^{(2)}$ in (2) and the dimensional reason. Hence

$$\sum_{i=0}^{b} Sq^i v_{2a+b-i} \equiv \sum_{i=0}^{b} Sq^i(v_{2a} v_{b-i}) = \sum_{i=0}^{b} (Sq^i v_{2a}) w_{b-i} = A, \quad \text{and}$$

$$w_{2a+b} + A \equiv \sum_{i=b+1}^{c-1} Sq^i v_{a+c-i} \equiv \sum_{i=b+1}^{c-1} Sq^i(v_a v_{c-i}) \quad (c = a+b)$$

$$= \sum_{i=b+1}^{c-1} \sum_{j=0}^{i} \Big[= \sum_{j=0}^{b} \sum_{i=b+1}^{c-1} + \sum_{j=b+1}^{c-1} \sum_{i=j}^{c-1}\Big] (Sq^j v_a)(Sq^{i-j} v_{c-i}) = B,$$

by (1), Theorem (i) and Lemma 3.1. Moreover, if $0 \leqq b < a$, then

$$B \equiv \sum_{j=0}^{b} (Sq^j v_a)(w_{c-j}+C) + \sum_{j=b+1}^{a-1} (Sq^j v_a) w_{c-j}, \quad \text{where}$$

$$C = \sum_{i=0}^{b-j} Sq^i v_{c-j-i} \equiv \sum_{i=0}^{b-j} Sq^i (v_a v_{b-j-i}) = \sum_{i=0}^{b-j} (Sq^i v_a) w_{b-j-i};$$

hence $\sum_{j=0}^{b} (Sq^j v_a) C \equiv 0$. If $a \leqq b < 2a$, then

$$B \equiv \sum_{j=b-a+1}^{a-1} (Sq^j v_a)(w_{c-j}+C) \quad \text{and} \quad C \equiv \sum_{i=0}^{b-a} (Sq^i v_a) w_{b-j-i}. \quad \square$$

Now, since $v_1 = w_1$, Theorem (ii) follows from the following

PROPOSITION 3.3. *Let $a$ be a power of 2. Then,*

$$v_{2a} \equiv w_{2a} + \sum_{i=0}^{a-1} w_{a-i} Sq^i v_a \quad \textit{and} \quad v_{4a} \equiv v_{2a}^{(2)} + \sum_{i=0}^{2a-1} w_i w_{4a-i} \bmod I^{(2)}.$$

PROOF. Lemma 3.2 implies the first congruence by taking $b=0$, and the second one by (3) as follows: $w_{4a}+v_{4a}+w_{2a}v_{2a} \equiv \sum_{i=1}^{2a-1} w_{2a-i} Sq^i v_{2a} \equiv \sum_{i=1}^{4} A_i$, where

$$A_1 = \sum_{i=1}^{2a-1} w_{2a-i} w_{2a+i},\ A_2 = \sum_{i=1}^{2a-1} \sum_{j=0}^{i-1} [= \sum_{j=0}^{2a-1} \sum_{i=j+1}^{2a-1}] w_{2a-i} w_{i-j} Sq^j v_{2a}$$
$$\equiv 0,$$

$$A_3 = (\sum_{i=1}^{a-1} \sum_{j=0}^{a-1} + \sum_{i=a}^{2a-1} \sum_{j=i-a+1}^{a-1}) [= \sum_{j=0}^{a-1} \sum_{i=1}^{a+j-1}] w_{2a-i} w_{a+i-j} Sq^j v_a$$
$$\equiv 0,$$

$$A_4 = \sum_{i=a}^{2a-1} \sum_{j=i-a+1}^{a-1} \sum_{k=0}^{i-a} [= \sum_{0 \leqq k < j < a} \sum_{i=a+k}^{a+j-1}] w_{2a-i} w_{i-j-k} (Sq^j v_a)(Sq^k v_a)$$
$$\equiv \sum_{0 \leqq k < j < a} w_{a-k} w_{a-j} (Sq^j v_a)(Sq^k v_a). \quad \square$$

## § 4. Proof of Corollary

For the set $\boldsymbol{N}$ of all positive integers, denote by $\boldsymbol{N}_2 \subset \boldsymbol{N}$ the subset of all powers of 2, and consider the collection $\mathfrak{S}$ of all finite subsets $S \subset \boldsymbol{N}$ satisfying

(6) $S = \{t_1 - r_1, r_1, \ldots, t_l - r_l, r_l, t_{l+1}, \ldots, t_m\}$, $\sharp S = m + l \geqq 1$ and $0 \leqq l \leqq m$, for $t_i \in \boldsymbol{N}_2$ $(1 \leqq i \leqq m)$ and $r_i \in \boldsymbol{N}$ with $r_i < t_i/2$ $(1 \leqq i \leqq l)$, (see (4)).

LEMMA 4.1. *In (6), $m$, $l$, $t_i$ and $r_i$ are unique for $S$, by ordering elements to satisfy $t_i > t_{i+1}$, or $t_i = t_{i+1}$ and $r_i < r_{i+1}$ for $i < l$, and $t_j > t_{j+1}$ for $j > l$.*

PROOF. We note that $t_i/2 < t_i - r_i \notin \boldsymbol{N}_2$ for $i \leqq l$ in (6). Hence, if $S \subset \boldsymbol{N}_2$, then $l = 0$ and so $m = \sharp S$ and the lemma holds. Let $S \not\subset \boldsymbol{N}_2$. Then $l \geqq 1$ and $s_1 = \max(S - \boldsymbol{N}_2) = t_1 - r_1$ by the above order. Here, $t_1/2 < t_1 - r_1 = s_1 < t_1$; hence $t_1 \in \boldsymbol{N}_2$ is unique, and so is $r_1$. Since $S - \{s_1, r_1\} \in \mathfrak{S}$, the lemma is proved by induction. $\square$

For any $a \in \boldsymbol{N}$, put $\mathfrak{S}(a) = \{S \in \mathfrak{S} \mid \sum_{s \in S} s = a\}$. Then, we have the following

LEMMA 4.2. *Assume that $a = a_1 + a_2$ for $a_1 \in \boldsymbol{N}_2$ and $a_2 \in \boldsymbol{N}$ with $a_1 \geqq a_2$. Then:*

(i) $S_1 \cup S_2 \in \mathfrak{S}(a)$ *for* $S_k \in \mathfrak{S}(a_k)$ *with* $S_1 \cap S_2 = \phi$; *and* $\sharp(S_1 \cup S_2) \geqq 3$ *if* $a_1 = a_2$.

(ii) *Conversely, for any* $S \in \mathfrak{S}(a)$ *with* $\sharp S \geqq 3$ *if* $a_1 = a_2$, *there are an odd number of unordered pairs* $\{S_1, S_2\}$ *of* $S_k \in \mathfrak{S}(a_k)$ *with* $S_1 \cap S_2 = \phi$ *and* $S_1 \cup S_2 = S$.

PROOF. (i) is clear by definition. For $S = \{t_1 - r_1, r_1, \dots, t_l - r_l, r_l, t_{l+1}, \dots, t_m\} \in \mathfrak{S}(a)$ and any $S_k \in \mathfrak{S}(a_k)$ in (ii), Lemma 4.1 means that if $t_i - r_i \in S_k$ $(i \leqq l)$, then $r_i \in S_k$. Thus, the number of all such $\{S_1, S_2\}$ is equal to that of all subsequences $(t_{i_1}, \dots, t_{i_n})$ of $(t_1, \dots, t_m)$ satisfying $\sum_{j=1}^{n} t_{i_j} = a_1$ (resp. $i_1 = 1$, in addition, if $a_1 = a_2$). Now, the latter is congruent to $\binom{a}{a_1}\left(\text{resp.} \binom{a-t_1}{a_1-t_1} \text{ if } a_1 = a_2\right)$ mod 2 by Lemma 2.3, which is odd by assumption. Thus (ii) is proved. $\square$

Now, according to this lemma, the Theorem implies immediately that $v_a \equiv \sum_{S \in \mathfrak{S}(a)} \prod_{s \in S} w_s \bmod I^{(2)}$ by induction, which is the Corollary by definition.

## References


[1] J. W. Milnor, On the Stiefel-Whitney numbers of complex manifolds and of spin manifolds, Topology **3** (1965), 223–230.

[2] J. W. Milnor and J. D. Stasheff, Characteristic classes, Ann. of Math. Studies **76**, Princeton Univ. Press, Princeton, 1974.

[3] N. E. Steenrod and D. B. A. Epstein, Cohomology operations, Ann. of Math. Studies **50**, Princeton Univ. Press, Princeton, 1962.

[4] R. E. Stong, Notes on cobordism theory, Princeton Math. Notes, Princeton Univ. Press, Princeton, 1968.



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