

ON CUT-AND-PAST INVARIANCE OF KERVAIRE SEMI-CHARACTERISTIC

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ABSTRACT. In this note we study the relative Kervaire semi-characteristic and prove its invariance under cut-and-past operation. Our approach is analytic and follow very closely the method introduced by W. Zhang

1. INTRODUCTION

The purpose of this paper is to prove the Cut-and past invariance of Kervaire semi-characteristic for orientable manifolds of dimension $n = 4q + 1$. The method we use is analytic and is introduced by G. Zhang in [4]. In fact the main step is generalizing the theorem 1.3 of the previous reference to manifolds with boundary. We describe here briefly the content of the paper [4]. Let M be a n -dimensional *closed* manifold with vanishing Euler character, $\chi(M) = 0$. This last condition implies the existence of a nowhere-vanishing Vector field V on M . Let E be a vector bundle over M whose fiber at x is a complementary subspace for $V(x) \in T_x M$. If X is a transversal section of E then the set Z of its singularities is a one-dimensional closed sub-manifold of M . Therefore Z is a disjoint union of some closed curves γ 's. The vector field X is used to construct a real line bundle $\mathcal{L}$ over each closed curve γ and the index $\text{ind}_2(\gamma) \in \mathbb{Z}_2$ of the curve γ is defined as follows

$$\text{ind}_2(\gamma) = \begin{cases} 1 & \text{if } \mathcal{L} \text{ is trivial} \\ 0 & \text{if } \mathcal{L} \text{ otherwise} \end{cases}$$

The main results of [4] assert that the sum $\sum_{\gamma} \text{ind}_2(\gamma)$ is a topological invariant and the following equalities hold in $\mathbb{Z}_2$

$$(1.1) \quad \sum_{\gamma} \text{ind}_2(\gamma) = \begin{cases} 0 & \text{if } \dim M = 4q + 2 \text{ or } 4q + 3 \\ \text{sig}(M)/2 & \text{if } \dim M = 4q \\ \kappa(M) & \text{if } \dim M = 4q + 1 \end{cases}$$

Here for an oriented $4q$ -dimensional closed manifold $\text{sig}(M)$ denotes the signature of M while $\kappa(M) \in \mathbb{Z}_2$ stands for the Kervaire semi-characteristic of M which is defined by the following relation

$$\kappa(M) = \sum_j \dim H^{2j}(M) \in \mathbb{Z}_2.$$

In above as in the body of the paper we deal with the de Rham cohomology spaces. The reader should notice the analogy of the above equality with the Poincar-Hopf formula for the Euler characteristic. Since this formula holds only where the Euler Characteristic vanishes, it might be considered as a secondary Poincar-Hopf formula. The signature of the $4q$ -dimensional manifold is by definition the signature of the following symmetric non degenerate bilinear form

$$\beta : H^{2q}(M) \times H^{2q}(M) \rightarrow \mathbb{R}$$

$$\beta(\omega, \eta) = \int_M \omega \wedge \eta.$$

A remarkable property of the signature $\text{sig}(M)$ is expressed by the Hirzebruch signature theorem. This theorem asserts that $\text{sig}(M)$ is a characteristic number and, therefore, a cobordism invariant. In particular if M is the boundary of another manifold then $\text{sig}(M) = 0$.

2000 *Mathematics Subject Classification.* 58E05, 35J25.

Key words and phrases. Witten deformation, Kervaire invariant, Index Theory, Cut-and-Past operation.

This research was in part supported by a grant from IPM (NO. 89510130)

In this paper we prove that the Kervaire semi-characteristic of a closed orientable $(4q + 1)$ -dimensional manifold is a cut-and past invariant. More precisely let M be a closed manifold without boundary and let $N \subset M$ be a sub-manifold of co-dimension one that partitions M into two compact manifolds M_1 and M_2 with common boundary $\partial M_1 = \partial M_2 = N$. Given a diffeomorphism $\phi : N \rightarrow N$ the closed manifold $M_1 \sqcup_\phi M_2$ is defined by identifying points $x \in \partial M_1$ and $\phi(x) \in \partial M_2$. We prove in theorem 4.1 that in dimension $n = 4q + 1$ the equality $\kappa(M) = \kappa(M_1 \sqcup_\phi M_2)$ holds provided that $\chi(N) = 0$. To prove this theorem we state and prove a generalization of the counting Zhang formula for manifolds with boundary.

Let M be a compact manifold of dimension $n = 4q + 1$ with non-empty boundary ∂M . We assume that $\chi(\partial M) = 0$ which implies the existence of a nowhere vanishing vector field V on ∂M . Since the dimension of M is odd, this vector field has an extension to a nowhere vanishing vector field on M that we denote by the same symbol V . We define the bundle E as in the boundary-less case. Let X be a transverse section of E which is tangent to the boundary. The set of all singularities of X is the union of a finite number of closed curves γ 's. These closed curves are either subsets of M° ; the interior of M ; or subsets of the boundary. We will construct the real line bundles $\mathcal{L}$ over γ 's. The index $\text{ind}_2(\gamma) \in \mathbb{Z}_2$ is defined as before. Define the relative Kervaire semi-characteristic $\kappa(M, \partial M)$ by the following relation

$$(1.2) \quad \kappa(M, \partial M) = \sum_j \dim H^{2j}(M, \partial M) \in \mathbb{Z}_2.$$

We prove the following generalization of the Zhang formula in this context

$$\kappa(M, \partial M) = \sum_{\gamma \subset M^\circ} \text{ind}(\gamma).$$

As it is clear from this formula the singular curves in boundary have no contribution in calculating the relative Kervaire invariant.

Acknowledgement: The author would like to thank Mathematics Institute of the Georg-August university in Goettingen. A part of this paper is written when the author was a visitor there.

2. THE GEOMETRIC AND ANALYTIC SETTING

Let M be a compact manifold with boundary $\partial M \neq \emptyset$. Let $\Omega^p(M, \partial M)$ denote the set of differential p -forms ω such that $j^*\omega \in \Omega^p(\partial M)$ is exact. It is clear by this definition that the exterior derivative d maps $\Omega^p(M, \partial M)$ into $\Omega^{p+1}(M, \partial M)$. Therefore one has the following differential complex

$$\dots \rightarrow \Omega^{p-1}(M, \partial M) \xrightarrow{d} \Omega^p(M, \partial M) \xrightarrow{d} \Omega^{p+1}(M, \partial M) \rightarrow \dots$$

which define the relative de Rham cohomology groups

$$H^p(M, \partial M) := \frac{\ker d : \Omega^p(M, \partial M) \rightarrow \Omega^{p+1}(M, \partial M)}{\text{Im } d : \Omega^{p-1}(M, \partial M) \rightarrow \Omega^p(M, \partial M)}.$$

For giving a Hodge theoretic interpretation of these groups it is necessary to have a Riemannian metric g on M . We assume that this metric takes the following product form in a collar neighborhood $U = (-1, 0] \times \partial M$

$$(2.1) \quad g(u, y) = d^2u + g_0(y),$$

where g_0 is a Riemannian metric on ∂M . This Riemannian structure induces an inner product on $\Omega^*(M)$. The symbol $\delta : \Omega^*(M) \rightarrow \Omega^{p-1}(M)$ denotes the formal adjoint of the operator d with respect to this inner product. Each differential form ω takes the following form in the collar neighborhood U

$$\omega|_U = \omega_0(u, y) + du \wedge \omega_1(u, y).$$

The differential form ω satisfies the relative boundary condition if the following relations hold

$$(2.2) \quad \omega_0(0, y) = 0 \quad \text{and} \quad \frac{\partial \omega_1}{\partial t}(0, y) = 0$$

Let $\Omega^*(M, B) \subset \Omega^*(M)$ consist of all differential forms satisfying the relative boundary condition. For $\eta = \eta_0 + du \wedge \eta_1$ another differential form and $D = d + \delta$ one has the Green formula, c.f. [2, page 24]

$$(2.3) \quad \langle D\omega, \eta \rangle - \langle \omega, D\eta \rangle = \int_{\partial M} \langle \omega_0, \eta_1 \rangle - \int_{\partial M} \langle \omega_1, \eta_0 \rangle,$$

This implies that the operator D is formally self-adjoint on $\Omega^*(M, B)$. Therefore the Laplacian operator $\Delta = D^2$ is a formally positive second order elliptic operator on $\Omega^*(M, B)$ and its kernel consists of smooth closed differential forms satisfying the conditions (2.2) and

$$\ker \Delta = \ker D.$$

The following relation is the consequence of the Hodge theory in this relative boundary condition context

$$(2.4) \quad H^*(M, \partial M) \sim \ker \Delta$$

The spaces $\Omega^p(M, B)$; for $0 \leq p \leq n$; can be completed with respect to appropriate Sobolev norm to obtain the Sobolev spaces $W^p(M, B)$. The inclusion $\Omega^*(M, B) \hookrightarrow \Omega^*(M)$ induces an isometric embedding $W^p(M, B) \hookrightarrow W^p(M)$. Since the image of this embedding is closed, all classical theorems in the theory of the Sobolev spaces, e.g. Sobolev embedding theorem, Rellich's theorem and the elliptic estimate hold also in this context.

For next uses we need to introduce some tools from linear and Clifford algebra. For this purpose let E be a real vector space with an inner product $\langle, \rangle$. For $v \in E$ consider the following linear maps on $\Lambda^* E^*$

$$c(v) := v^* \wedge -i_v ; \quad \hat{c}(v) = v^* \wedge +i_v,$$

where $v^* \in E^*$ denotes the dual of the vector v with respect to the inner product. The inner product of E induces a natural inner product on $\Lambda^* E^*$ and these operators satisfy the following properties with respect to this inner product

$$(2.5) \quad c(v)^* = -c(v) ; \quad \hat{c}(v)^* = \hat{c}(v)^*.$$

Moreover it is easy to verify the following relations

$$(2.6) \quad \begin{aligned} c(v)c(w) + c(w)c(v) &= -2\langle v, w \rangle, \\ \hat{c}(v)\hat{c}(w) + \hat{c}(w)\hat{c}(v) &= 2\langle v, w \rangle, \\ c(v)\hat{c}(w) + \hat{c}(w)c(v) &= 0 \end{aligned}$$

Let $A : E \rightarrow E$ be an invertible self adjoint linear map whose matrix with respect to an orthonormal basis $\{e_i\}$ is a_{ij} . The following endomorphism on $\Lambda^* E^*$ is independent of the basis

$$\hat{A} = \sum_{ij} a_{ij} c(e_i) \hat{c}(e_j).$$

The following lemma gives a property of this operator that will be important in the next section.

Lemma 2.1. [5, lemma 4.8] *The following operator*

$$\begin{aligned} K : \Lambda^* E^* &\rightarrow \Lambda^* E^* \\ K &= \text{tr } |A| + \hat{A} \end{aligned}$$

is non-negative, its kernel is one dimensional and is generated by an element in $\Lambda^{\text{even}} E^$ if $\det A > 0$ and in $\Lambda^{\text{odd}} E^*$ if $\det A < 0$.*

In fact because A is symmetric it can be diagonalized in an orthonormal basis $\{e^j\}$. If the corresponding eigenvalues are positive for $j = 1, \dots, p$ and negative for $j = p+1 \dots n$ then the kernel of K is generated by $e^{p+1} \wedge e^{p+2} \wedge \dots \wedge e^n$.

3. COUNTING FORMULA

Before starting the main discussion we recall a fact from differential topology. It is true that if the Euler character of a manifold (with or without boundary) vanishes then this manifold carries a nowhere vanishing vector field which is outward on boundary points, c.f. [3, chapter 6]. Let $(M, \partial M)$ be an odd dimensional manifold with boundary. The Poincar duality implies the relation $\chi(M, \partial M) = -\chi(M)$. On the other hand the long exact sequence of the pair $(M, \partial M)$ in de Rham context reads

$$\cdots \rightarrow H^j(M, \partial M) \rightarrow H^j(M) \rightarrow H^j(\partial M) \rightarrow H^{j+1}(M, \partial M) \rightarrow \cdots$$

This implies, after summation over alternate traces, the relation

$$\chi(M, \partial M) - \chi(M) + \chi(\partial M) = 0.$$

From now-on we assume the following condition on the boundary manifold

$$(3.1) \quad \chi(\partial M) = 0.$$

This condition along the above discussion implies the vanishing of $\chi(M, \partial M)$ and $\chi(M)$. Therefore there is a nowhere vanishing vector field V' on M which is outward at boundary points. On the other hand, and with condition (3.1), there is a nowhere vanishing vector field V_0 on ∂M . We can extend this vector field in a collar neighborhood $U = (-1, 0) \times \partial M$ of ∂M such that its restriction to $\{t\} \times \partial M$ is V_0 with respect to the obvious identification $(t, x) \sim x$. We also denote this extension by V_0 . If η is a smooth function on M which coincides to t near boundary and is supported in a small neighborhood of the boundary then the vector field $V = \eta(t)V' + (1 - \eta(t))V_0$ is a nowhere vanishing vector field on M which is tangent to the boundary.

Let M be equipped with a Riemannian metric g which takes the product form (2.1) in the collar neighborhood U of the boundary. From now on we assume the condition (3.1) is satisfied, therefore there is a nowhere vanishing vector field V on M such that $V(x) \in T_x \partial M$ for $x \in \partial M$. We can and will assume that $\|V\| = 1$. For each $x \in M$ let $E_x \subset T_x M$ denote the orthogonal complement of $V(x) \subset T_x M$ and denote the set of all E_x 's by E which is a vector bundle of rank $4q$ over M . Let X be a transverse section of E . In the boundary points we assume $X|_{\partial M}$ be tangent to the boundary. Let $X = (X_1, X_2, \dots, X_n)$ with respect to collar coordinates (2.1). With above conditions in boundary points $\partial X_1 / \partial u \neq 0$. Therefore we may assume that

$$(3.2) \quad \frac{\partial X_1}{\partial u} = \pm 1$$

The zero set Z of X is a compact 1-dimensional sub-manifold of M whose connected components, i.e. topological circles, are either disjoint from or included in the boundary. In the sequel we denote these circle generically by γ .

By transversality of X , the vector field V is no-where perpendicular to γ . In fact $\ker T_x X = T_x \gamma$ and by transversality $T_x M = \ker T_x X \oplus E_x$ for $x \in \gamma$, so $T_x M = T_x \gamma \oplus E_x$. This implies, since V is everywhere orthogonal to E , that V is no-where orthogonal to γ . Consequently one can deform continuously the vector field V to a unit vector field which is tangent to curves γ 's. We denote this new vector field by V and deform the Riemannian metric in neighborhoods of curves so that the vector field V be orthogonal to E . We work under these assumptions because the quantities that we are interested in are homotopy invariant.

Let x be a singular point of X . By transversality of X , the map $T_x X : E_x \rightarrow E_x$ is a linear isomorphism that we denote by $A(x)$. If x is a boundary point then $T_x X : E_{0x} \rightarrow E_{0x}$ is also an isomorphism and we denote this isomorphism by $A_0(x)$. Let $\{e_j\}_{j=2}^n$ be an orthonormal basis of E_x and $A_{ij}(x)$ be the matrix of $A(x)$ with respect to this basis. The following operator is indeed basis independent

$$\begin{aligned} \hat{A}(x) : \Lambda^* E_x^* &\rightarrow \Lambda^* E_x^* \\ \hat{A}(x) &= \sum_{ij=2}^n A_{ji}(x) c(e_i) \hat{c}(e_j). \end{aligned}$$

If x is a boundary points then we assume that $e_2 = \frac{\partial}{\partial u}$. In this case the following operator is basis independent

$$\begin{aligned}\hat{A}_0(x) &: \Lambda^* E_{0x}^* \rightarrow \Lambda^* E_{0x}^* \\ \hat{A}_0(x) &= \sum_{i,j=3}^n A_{0ji}(x) c(e_i) \hat{c}(e_j).\end{aligned}$$

Therefore, given a circle γ , have a field A (or A_0 if γ is in the boundary) of invertible matrices of order $(n-1) \times (n-1)$ (or of order $(n-2) \times (n-2)$) along γ as well as a field of Clifford morphisms $\hat{A}$ (or $\hat{A}_0$) acting on sections of the bundle $\Lambda^* E^*$ (or $\Lambda^* E_0^*$) over γ . From Lemma 2.1 for each $x \in \gamma$ the operators

$$(3.3) \quad K(x) := \text{tr} |A(x)| + \hat{A}(x) : \Lambda^* E_x^* \rightarrow \Lambda^* E_x^*$$

$$(3.4) \quad K_0(x) := \text{tr} |A_0(x)| + \hat{A}_0(x) : \Lambda^* E_{0x}^* \rightarrow \Lambda^* E_{0x}^*$$

are non-negative with one-dimensional kernels. Moreover, by the discussion after lemma 2.1 and special form of the vector field X in the collar neighborhood U (which is expressed by (3.2)), for $x \in \partial M$ one of the relations $\ker K_0(x) = j^* \ker K(x)$ or $\ker K_0(x) = i_{e_2} \ker K(x)$ hold depending on the sign in (3.2) (here $j : \partial M \rightarrow M$ is the inclusion map). According to above discussion the kernel of the operators K (and K_0) form a real line bundle $\mathcal{L}$ over γ . In [4] the index $\text{ind}_2(\gamma)$ is defined to be 1 if this bundle is trivial and 0 if not. Moreover the relations (1.1) are proved.

Remark 3.1. *If $\gamma \subset \partial M$ then the line bundles made of $\ker K$ and $\ker K_0$ are naturally isomorphic. Therefore the index $\text{ind}_2(\gamma)$ is the same if we define it through the boundary data.*

In particular for orientable closed manifolds of dimension $n = 4q+1$ the Kervaire semi-characteristic is proved to be given by the sum of the indices of the singular circles γ 's. We are going to prove the similar theorem for the case where the boundary is not empty.

Theorem 3.2. *Let M be a $n = 4q+1$ dimensional orientable manifold with boundary ∂M which satisfies the relation $\chi(\partial M) = 0$. The following formula holds in $\mathbb{Z}_2$*

$$\kappa(M, \partial M) = \sum_{\gamma \in M^\circ} \text{ind}_2(\gamma)$$

Here the summation is taken over those circles which are included in M° ; the interior of M ; and $\kappa(M, \partial M)$ is given by (1.2)

In particular the boundary has no contribution in calculating the relative Kervaire semi-characteristic. The rest of this section is devoted to prove this theorem. We follow very closely the method introduced by Zhang in [4]. We recall that $\Omega^*(M, B)$ consists of the differential forms on M which satisfy the relative boundary condition (2.2). Since the vector field V is tangent to the boundary, the Clifford actions $c(V)$ and $\hat{c}(V)$ on $\Omega(M)$ respect this boundary condition and define actions on $\Omega(M, B)$. Therefore the following operator acts on $\Omega^{\text{even}}(M, B)$

$$D_V = \frac{1}{2}(\hat{c}(V)(d + \delta) - (d + \delta)\hat{c}(V))$$

This is indeed a first order elliptic differential operator which is, thank to relations (2.3) and (2.5), skew-adjoint. So its Kernel is finite dimensional and its dimension modulo 2 is invariant with respect to continuous deformation of D_V , c.f. [1]. This invariance is denoted by $\text{ind}_2(D_V)$. The relevance of this operator is through the following formula

$$\text{ind}_2(D_V) = \kappa(M, \partial M).$$

The proof is completely similar to the proof of theorem 1.2 in [4] once one the relation (2.4).

Let s be a parameter varying in the set of non-negative real numbers. The vector field X can be used to define the following deformation of the operator D_V

$$(3.5) \quad D_s = D_V + s\hat{c}(V)\hat{c}(X) : \Omega^{\text{even}}(M, B) \rightarrow \Omega^{\text{even}}(M, B)$$

Because V and X are orthogonal, It is clear from relations (2.6) that $\hat{c}(V)\hat{c}(X)$ is a skew-adjoint operator on $\Omega^*(M, B)$ of order zero. Therefore $\text{ind}_2(D_s)$ is well defined and independent of s , i.e.

$$(3.6) \quad \text{ind}_2(D_s) = \text{ind}_2(D_V) = \kappa(M, \partial M)$$

Let $\{e_j\}$ for $1 \leq j \leq n$ be a local oriented orthonormal basis for TM and denote by ∇ the Riemannian connection on TM , as well as its extension to Λ^*T^*M . The following formula is a direct result of relations (2.6)

$$(3.7) \quad D_s^2 = D_V^2 - s \sum_{j=0}^{4q} (c(e_j)\hat{c}(\nabla_{e_j}X) - \langle \nabla_{e_j}X, V \rangle c(e_j)\hat{c}(V)) - s^2|X|$$

Let Z be the set of the singular points of X and W an open neighborhood of Z . The following lemma can be proved using the classical theorem of Sobolev spaces. For a proof in boundary-less case we refer to [5, proposition]. This proof goes on literary in our boundary case.

Lemma 3.3. *There exist positive constant a and b such that for any $s \geq 1$ and $\omega \in \Omega^*(M, B)$ the following inequality holds*

$$\|D_s\|_0^2 \geq a(\|\omega\|_1^2 + (s-b)\|\omega\|_0^2)$$

provided that $\text{supp}(\omega) \cap W = \emptyset$.

This lemma suggests that to compute the dimension of $\ker D_s$, when s gets larger and larger, it suffices to compute the dimension of the kernel of the restriction of D_s to arbitrary small open neighborhoods of Z . Even more, to compute the dimension of the kernel of this operator when s goes toward infinity, it is enough to compute the dimension of the kernel of the corresponding linearized operator along closed curves γ 's. Fix a circle $\gamma \subset Z$. Since M is oriented the normal bundle $E \rightarrow \gamma$ is oriented and trivial. so $E \simeq \gamma \times \mathbb{R}^{4q}$. Let $(t, y_2, y_3, \dots, y_n)$ denote the Euclidean coordinates of $\gamma \times \mathbb{R}^n$. If γ is included in the boundary we let $y_2 = u$ be the collar parameter of 2.1. Using these coordinates we have the following Bochner type formula (3.7) where D_s stands for the linearization of D_s .

$$D_s^2 = -\frac{\partial^2}{\partial t^2} + \left\{ -\sum_{i=2}^n \frac{\partial^2}{\partial y_i^2} + s\hat{A}(t) + s^2\langle |A|y, |A|y \rangle \right\}.$$

Using (3.3) the above expression can be rewritten as follows

$$(3.8) \quad D_s^2 = -\frac{\partial^2}{\partial t^2} + \left\{ -\sum_{i=2}^n \frac{\partial^2}{\partial y_i^2} - s \text{tr} |A(t)| + s^2\langle |A|y, |A|y \rangle \right\} + sK(t).$$

In above expression the operator in the braces is harmonic oscillator operator whose kernels is one dimensional generated by the function $\exp(-\frac{s}{2}\langle |A|y, |A|y \rangle)$. The operator $K(t)$ is an algebraic operator and by lemma 2.1 its kernel is one-dimensional real subspace of $\Lambda^{\text{even}} E_t^*$ if $\det A(t) > 0$ and of $\Lambda^{\text{odd}} E_t^*$ if $\det A(t) < 0$. In each case these subspaces put together form a real line bundle $\mathcal{L}$ over γ . Now we assume the sign positive in (3.2) is occurred (the negative sign is similar). In this case the kernel of the operator (3.8) has the following form if $\det A > 0$

$$(3.9) \quad \exp\left(-\frac{s}{2}\langle |A|y, |A|y \rangle\right)\omega,$$

while it takes the following form if $\det A < 0$

$$(3.10) \quad \exp\left(-\frac{s}{2}\langle |A|y, |A|y \rangle\right)dt \wedge \omega,$$

ω in (3.9) is a parallel section of $\mathcal{L}$ of even degree while in (3.10) its degree is odd (this is why we have put dt in the later case). So, independent of the sign of $\det A$, as long as the line bundle $\mathcal{L}$ is trivial and γ is included in the interior of M , the kernel of D_s , as an operator on sections of $E \rightarrow \gamma$

is one-dimensional. On the other hand, in the boundary points the operator D_s^2 takes the following form

$$D_s^2 = -\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial u^2} - \left\{ \sum_{i=3}^n \frac{\partial^2}{\partial y_i^2} - s \operatorname{tr} |A(t)| + s^2 \langle |A|y, |A|y \rangle \right\} + sK(t)$$

If γ is a boundary curve then by our assumption on the sign in (3.2) the differential form ω in (3.9) does not contain the term du . Therefore $dt \wedge du \wedge \omega$ satisfies the boundary condition (2.2) and generates the kernel of D_s . If $\det A < 0$ then ω in relation (3.10) is a parallel section of $E \rightarrow \gamma$ with odd degree. In this case $du \wedge \omega$ satisfies the boundary condition (2.2) and generates the kernel of D_s provided that $\mathcal{L}$ is trivial. Summarizing the dimension of $\ker D_s$ equals the number of closed curves γ 's whose associated line bundles $\mathcal{L}$ are trivial, regardless this curve is in the interior or on the boundary of M , i.e.

$$\dim \ker D_s = \sum_{\gamma \subset M^\circ} \operatorname{ind}_2(\gamma) + \sum_{\gamma \subset \partial M} \operatorname{ind}_2(\gamma)$$

By remark 3.1 for evaluating the boundary contribution in this formula we can just use the boundary date. By the formula (1.1) the boundary contribution in above formula is equal to $\operatorname{sig}(\partial M)/2$ which vanishes because of the cobordism invariance of the signature. Therefore using (3.6) we get the following formula for relative Kervaire semi-Characteristic

$$(3.11) \quad \kappa(M, \partial M) = \sum_{\gamma \subset M^\circ} \operatorname{ind}_2(\gamma)$$

and this complete the proof of the theorem 3.2.

4. APPLICATION

4.1. Cut-and-past invariance of Kervaire semi-Characteristic. In this section we use the main theorem 3.2 to prove the cut-and-past invariance of the Kervaire semi-characteristic. Let M be a closed manifold without boundary and let $N \subset M$ be a sub-manifold of codimension one that partitions M into two compact manifolds M_1 and M_2 with common boundary $\partial M_1 = \partial M_2 = N$. Given a diffeomorphism $\phi : N \rightarrow N$ the closed manifold $M_1 \sqcup_\phi M_2$ is defined by identifying points $x \in \partial M_1$ and $\phi(x) \in \partial M_2$. It is clear that $M_1 \sqcup_\phi M_2$ is produced by cutting M along N and then pasting through ϕ , hence the nomination.

Theorem 4.1. *Let M be a $(4k+1)$ -dimensional closed manifold. Kervaire semi-characteristic is invariant with respect to cut-and-past operation $\kappa(M) = \kappa(\partial M_1 \sqcup_\phi \partial M_2)$ provided that $\chi(N) = 0$.*

proof Since $\chi(N) = 0$ and considering the dimensions, there is a nowhere vanishing vector field V on N that has a nowhere vanishing extension to M_1 . We denote this extension by V_1 . Let E_1 be the bundle over M_1 whose fiber at x is orthogonal to $V_1(x)$. Let X_1 be a transversal section of E which is tangent to N . The zero set of X consists some circles γ in M_1 which are either disjoint from, or included in N . Moreover each of these circle carries a real line bundle $\mathcal{L}$. Now consider the push-forward vector field $V_2 = \phi_*(V_1)$ and extent it to a nowhere vanishing vector field V_2 on M_2 . Let E_2 be the vector bundle on M_2 whose fiber are orthogonal to V_2 . Let X_2 be a transversal section of E_2 such that on the boundary N it satisfies $X_2 = \phi_*(X_1)$. These geometric data fit together to build up corresponding geometric data on the whole manifold $\partial M_1 \sqcup_\phi \partial M_2$. This way we get vector fields V and X on, and vector bundle E over $\partial M_1 \sqcup_\phi \partial M_2$. Applying the Zhang counting formula (1.1) to these data we get the following relation

$$\kappa(\partial M_1 \sqcup_\phi \partial M_2) = \sum_{\gamma \in M_1^\circ} \operatorname{ind}_2(\gamma) + \sum_{\gamma \in M_2^\circ} \operatorname{ind}_2(\gamma) + \sum_{\gamma \in N} \operatorname{ind}_2(\gamma).$$

Using the theorem 3.2, the first and second terms in the right side of the above expression are respectively equal to relative Kervaire semi-characteristics $\kappa(M_1, \partial M_1)$ and $\kappa(M_2, \partial M_2)$. Concerning the last term, it vanishes because of the cobordism invariance of signature. Therefore

$$(4.1) \quad \kappa(\partial M_1 \sqcup_\phi \partial M_2) = \kappa(M_1, \partial M_1) + \kappa(M_2, \partial M_2).$$

The right side of this relation has no reference to the pasting diffeomorphism ϕ so the left side is indeed independent of ϕ . $\square$

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