ALGEBRAIC K-THEORY OVER THE INFINITE DIHEDRAL GROUP

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Algebraic K-theory

- The **algebraic K-groups** of an exact category $\mathbb{E}$ are
$$K_*(\mathbb{E}) \ = \ \pi_*(K(\mathbb{E})) \ \ (* \in \mathbb{Z}) \ ,$$
the homotopy groups of a spectrum $K(\mathbb{E})$ (Quillen, 1972).
- $K_0(\mathbb{E}) =$ **class group** of $\mathbb{E}$, with one generator $[P]$ for each object P in $\mathbb{E}$, and one relation $[P]-[Q]+[R]=0$ for each exact sequence in $\mathbb{E}$
$$0 \to P \to Q \to R \to 0 \ .$$
- $K_1(\mathbb{E}) =$ **torsion group** of $\mathbb{E}$, with one generator $\tau(f)$ for each automorphism $f: P \to P$ in $\mathbb{E}$, and relations
$$\tau(f \oplus f') \ = \ \tau(f) + \tau(f') \ , \ \tau(gf) \ = \ \tau(f) + \tau(g) \ .$$
- The algebraic K-groups of a ring R are
$$K_*(R) \ = \ K_*(\mathrm{Proj}(R))$$
with $\mathrm{Proj}(R) =$ exact category of f.g. projective R-modules.

Executive summary

- The 40-year old splitting theorems for the algebraic K-theory of polynomial extensions

  $$K_*(R[t]) \; = \; K_*(R) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R) \; ,$$

  $$K_*(R[t,t^{-1}]) \; = \; K_*(R) \oplus K_{*-1}(R) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R)$$

  have been recently extended to a splitting theorem for the algebraic K-theory of a dihedral extension

  $$K_*(R[D_\infty]) \; = \; K_*(R \to R[\mathbb{Z}_2] \times R[\mathbb{Z}_2]) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R)$$

  involving the same $\widetilde{\mathrm{Nil}}$-groups.
- Motivation from the algebraic K-theory obstructions to the codimension 1 splitting of homotopy equivalences of finite CW complexes.
- **Reference**

  J.Davis, Q.Khan, A.Ranicki, *Algebraic K-theory over D_∞*

  ArXiv/math.AT.0803.1639

Kernel modules

- The **kernels** of a map $f : M \to X$ of connected spaces are the relative homology $\mathbb{Z}[\pi_1(X)]$-modules

$$K_r(M) \ = \ H_{r+1}(\widetilde{f} : \widetilde{M} \to \widetilde{X})$$

  with $\widetilde{X}$ the universal cover of X, $\widetilde{M} = f^*\widetilde{X}$ the pullback cover of M, and $\widetilde{f} : \widetilde{M} \to \widetilde{X}$ a $\pi_1(X)$-equivariant lift of f.
- **Definition** A map $f : M \to X$ is π_1**-iso** if it induces isomorphisms $f_* : \pi_1(M) \cong \pi_1(X)$.
- **Theorem** (Hurewicz, Whitehead, 1930's) Let $f : M \to X$ be a π_1-iso map of connected CW complexes.
  - If $K_r(M) = 0$ for $r < n$ then the forgetful map

$$\pi_{n+1}(f : M \to X) \to K_n(M)$$

    is an isomorphism.
  - f is a homotopy equivalence if and only if $K_*(M) = 0$.

The Wall finiteness obstruction

- $K_0(\mathbb{Z}[\pi]) = K_0(\mathbb{Z}) \oplus \widetilde{K}_0(\mathbb{Z}[\pi])$ for any group π.
- A f.g. projective $\mathbb{Z}[\pi]$-module P is such that $[P] = 0 \in \widetilde{K}_0(\mathbb{Z}[\pi])$ if and only if P is stably f.g. free, i.e. such that $P \oplus F \cong G$ for f.g. free F, G.
- A CW complex X is **finitely dominated** if it is a homotopy retract of a finite CW complex, or equivalently if there exists π_1-iso map $f : M \to X$ from a finite CW complex M with $K_r(M) = 0$ for $r \neq n$, and $K_n(M)$ a f.g. projective $\mathbb{Z}[\pi_1(X)]$-module.
- **Finiteness obstruction** (Wall, 1965) A finitely dominated CW complex X has an algebraic K-theory invariant

  $$[X] \ = \ (-)^n[K_n(M)] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$$

  such that $[X]\widetilde{\ } = 0 \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ if and only if X is homotopy equivalent to a finite CW complex, if and only if $K_n(M)$ is a stably f.g. free $\mathbb{Z}[\pi_1(X)]$-module.

Whitehead torsion

- The **torsion group** of a ring R is the abelian group

$$K_1(R) \;=\; K_1(\mathrm{Proj}(R)) \;=\; \varinjlim_n GL_n(R)^{ab} \;.$$

- An $n \times n$ invertible matrix M with entries in R has a **torsion** $\tau(M) \in K_1(R)$ such that $\tau(M) = 0$ if and only if M can be reduced to $1 \in R$ by stabilizations $M \mapsto M \oplus 1$, destabilizations and elementary row and column operations.
- (J.H.C. Whitehead, 1951) A homotopy equivalence $f : X \to Y$ of finite CW complexes has a **torsion**

$$\tau(f) \in Wh(\pi_1(X)) \;=\; K_1(\mathbb{Z}[\pi_1(X)])/\{\pm g \,|\, g \in \pi_1(X)\}$$

  such that $\tau(f) = 0$ if and only if f is **simple**, i.e. can be deformed to $1 : Y \to Y$ be elementary cell expansions and collapses.

The infinite dihedral group D_∞

- Free product of two cyclic groups of order 2

$$D_\infty \;=\; \mathbb{Z}_2 * \mathbb{Z}_2 \;=\; \{t_1, t_2 \,|\, (t_1)^2 = (t_2)^2 = 1\}$$

- Contains infinite cyclic subgroup of index 2

$$\{1\} \to \mathbb{Z} = \langle t_1 t_2 \rangle \to D_\infty \to \mathbb{Z}_2 \to 0 \ .$$

- D_∞ acts on $\mathbb{R}$ by

$$t_1 \;=\; \text{reflection in } 1/2 \;:\; \mathbb{R} \to \mathbb{R}\,;\; x \mapsto 1 - x \ ,$$
$$t_2 \;=\; \text{reflection in } -1/2 \;:\; \mathbb{R} \to \mathbb{R}\,;\; x \mapsto -1 - x$$

with

$$t_1 t_2 \;=\; \text{translation by } +2 \;:\; \mathbb{R} \to \mathbb{R}\,;\; x \mapsto 2 + x \ .$$

$\mathbf{Nil}_*(R, B)$

- Let R be a ring, and B an (R,R)-bimodule, P an R-module. An R-module morphism $\nu : P \to B \otimes_R P$ is **nilpotent** if for some $k \geqslant 1$
$$\nu^k \;=\; 0 \;:\; P \to B^k \otimes_R P \;,\; B^k \;=\; B \otimes_R B \otimes_R \cdots \otimes_R B \;.$$
- Let $\mathrm{Nil}(R,B)$ be the exact category of pairs (P,ν) with P a f.g. projective R-module and $\nu : P \to B \otimes_R P$ nilpotent,
$$\mathrm{Nil}_*(R,B) \;=\; K_*(\mathrm{Nil}(R,B)) \;.$$
- The composite of the exact functors
$$\mathrm{Proj}(R) \to \mathrm{Nil}(R,B) \;;\; P \mapsto (P,0)$$
$$\mathrm{Nil}(R,B) \to \mathrm{Proj}(R) \;;\; (P,\nu) \mapsto P$$
is the identity, so that
$$\mathrm{Nil}_*(R,B) \;=\; K_*(R) \oplus \widetilde{\mathrm{Nil}}_*(R,B)$$
with $\widetilde{\mathrm{Nil}}_*(R,B) = \ker(\mathrm{Nil}_*(R,B) \to K_*(R))$.

The algebraic K-theory of the tensor algebra $T(B)$

- **Definition** The **tensor algebra** of an (R,R)-bimodule B is the ring
$$T(B) \;=\; R \oplus \bigoplus_{k=1}^{\infty} B^k \ .$$
- **Theorem** (Waldhausen, 1978) If B is flat as a right R-module and f.g. projective as an R-module then
$$K_*(T(B)) \;=\; K_*(R) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R,B)$$
with
$$\begin{aligned}&\widetilde{\mathrm{Nil}}_0(R,B) \to K_1(T(B)) \ ;\\ &[P,\nu] \mapsto \tau(1+\nu : T(B)\otimes_R P \to T(B)\otimes_R P) \ .\end{aligned}$$
- **Idea of proof** Higman linearization, an algebraic transversality technique which reduces all expressions involving B^k for $k \geqslant 1$ to the case $k=1$.

The algebraic K-theory of $R[t]$ and $R[\mathbb{Z}]$

- **Example** If $B = R$ then $T(B) = R[t]$ is the polynomial extension, and $\mathrm{Nil}(R, B) = \mathrm{Nil}(R)$ is the exact category of nilpotent endomorphisms $(P, \nu : P \to P)$. Linearization:

$$\begin{pmatrix} a_0 + a_1 t + a_2 t^2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_0 + a_1 t & t \\ -a_2 t & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_2 t & 1 \end{pmatrix}$$

- **Theorem** (Bass, 1968 and Quillen, 1972) For any ring R

$$\begin{aligned} K_*(R[t]) &= K_*(R) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R) , \\ K_*(R[\mathbb{Z}]) &= K_*(R) \oplus K_{*-1}(R) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R) \end{aligned}$$

with $R[\mathbb{Z}] = R[t, t^{-1}]$ and

$$\widetilde{\mathrm{Nil}}_0(R) \to K_1(R[t]) \; ; \; [P, \nu] \mapsto \tau(1 + \nu t : P[t] \to P[t]) \; .$$

$\mathbf{Nil}_*(R, B_1, B_2)$

- For any ring R

$$\mathrm{Proj}(R \times R) \;=\; \mathrm{Proj}(R) \times \mathrm{Proj}(R)\ .$$

- For any (R, R)-bimodules B_1, B_2

$$\mathrm{Nil}(R, B_1, B_2) \;=\; \mathrm{Nil}(R \times R, \begin{pmatrix} 0 & B_2 \\ B_1 & 0 \end{pmatrix})$$

is the exact category of quadruples

$$(\ P_1\ ,\ P_2\ ,\ \rho_1 : P_1 \to B_1 \otimes_R P_2\ ,\ \rho_2 : P_2 \to B_2 \otimes_R P_1\ )$$

with P_1, P_2 f.g. projective R-modules and ρ_1, ρ_2 R-module morphisms such that the composite

$$\rho_2\rho_1\ :\ P_1 \to B_1 \otimes_R B_2 \otimes_R P_1$$

is a nilpotent R-module morphism.

Generalized free products

- A group G is a **generalized free product** if it is
  - either an **amalgamated free product**

$$G = G_1 *_H G_2$$

with $i_1 : H \to G_1$, $i_2 : H \to G_2$ group morphisms, and

$$i_1(x) = i_2(x) \in G \ (x \in H) .$$

  - or an **HNN extension**

$$G = G_1 *_H \{t\}$$

with $i_1, i_2 : H \to G_1$ group morphisms, and

$$i_1(x)t = ti_2(x) \in G \ (x \in H) .$$

- A generalized free product is **injective** if i_1, i_2 are injections, in which case $H, G_1, G_2 \subset G$.

The Seifert-van Kampen and Mayer-Vietoris theorems

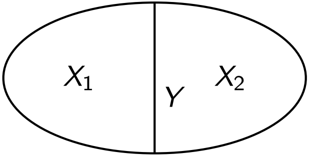
- A subspace $Y \subset X$ is **codimension 1** if Y has an open neighbourhood $Y \times \mathbb{R} \subset X$.
- **Theorem** (1930's) If X, Y are connected then:
  - (S-vK) the fundamental group of $\pi_1(X)$ is a generalized free product of $\pi_1(Y)$ and $\pi_1(X \backslash Y)$,
  - (M-V) the homology groups $H_*(X)$ fit into an exact sequence

$$\cdots \to H_n(Y) \to H_n(X \backslash Y) \to H_n(X) \to H_{n-1}(Y) \to \ldots$$

- For connected X, Y there are two cases:
  - (A) Y separates X, with $X \backslash Y$ disconnected
  - (B) Y does not separate X, with $X \backslash Y$ connected

The separating case (A)

- X is a union
$$X \;=\; X_1 \cup_Y X_2$$
with X_1, X_2 connected.



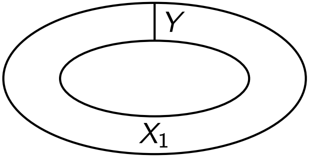
- The fundamental group is an amalgamated free product
$$\pi_1(X) \;=\; \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2)$$
- The homology groups are related by a Mayer-Vietoris exact sequence
$$\cdots \to H_n(Y) \to H_n(X_1) \oplus H_n(X_2) \to H_n(X) \to H_{n-1}(Y) \to \dots .$$

The non-separating case (B)

- X is a union

$$X = X_1 \cup_{Y \times \{0,1\}} Y \times [0,1]$$

with X_1 connected.



- The fundamental group is an *HNN* extension

$$\pi_1(X) = \pi_1(X_1) *_{\pi_1(Y)} \{t\} \ .$$

- The homology groups are related by a Mayer-Vietoris exact sequence

$$\cdots \to H_n(Y) \xrightarrow{i_1 - i_2} H_n(X_1) \to H_n(X) \to H_{n-1}(Y) \to \dots \ .$$

The algebraic K-theory of generalized free products

- **Theorem** (Waldhausen, 1978) (A) For an injective amalgamated free product $G = G_1 *_H G_2$

  $$K_n(R[G]) \ = \ K_n(R[H] \to R[G_1] \times R[G_2]) \oplus \widetilde{\mathrm{Nil}}_{n-1}(R[H], B_1, B_2)$$

  with $B_j = R[G_j \backslash H]$ ($j = 1, 2$), and an almost-Mayer-Vietoris exact sequence

  $$\cdots \to K_n(R[H]) \oplus \widetilde{\mathrm{Nil}}_n(R[H], B_1, B_2) \to K_n(R[G_1]) \oplus K_n(R[G_2])$$

  $$\to K_n(R[G]) \xrightarrow{\partial} K_{n-1}(R[H]) \oplus \widetilde{\mathrm{Nil}}_{n-1}(R[H], B_1, B_2) \to \ldots$$

- (B) For an injective *HNN* extension $G = G_1 *_H \{t\}$

  $$K_n(R[G]) \ = \ K_n(i_1 - i_2 : R[H] \to R[G_1]) \oplus \widetilde{\mathrm{Nil}}_{n-1}$$

  with an almost-Mayer-Vietoris exact sequence

  $$\cdots \to K_n(R[H]) \oplus \widetilde{\mathrm{Nil}}_n \to K_n(R[G_1])$$

  $$\to K_n(R[G]) \xrightarrow{\partial} K_{n-1}(R[H]) \oplus \widetilde{\mathrm{Nil}}_{n-1} \to \ldots \ .$$

The algebraic K-theory of $R[D_\infty]$

- For $R[D_\infty] = R[\mathbb{Z}_2 * \mathbb{Z}_2]$ take $B_1 = B_2 = R$, so that
$$K_*(R[D_\infty]) \;=\; K_*(R \to R[\mathbb{Z}_2] \times R[\mathbb{Z}_2]) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R,R,R)\ .$$
- The exact category
$$\mathrm{Nil}(R,R,R) \;=\; \mathrm{Nil}(R \times R, \begin{pmatrix} 0 & R \\ R & 0 \end{pmatrix})$$
has objects quadruples $(P_1, P_2, \rho_1 : P_1 \to P_2, \rho_2 : P_2 \to P_1)$ with P_1, P_2 f.g. projective R-modules and $\rho_2\rho_1 : P_1 \to P_1$ nilpotent, and
$$K_*(\mathrm{Nil}(R,R,R)) \;=\; K_*(R) \oplus K_*(R) \oplus \widetilde{\mathrm{Nil}}_*(R,R,R)\ .$$
- An element $x_1 \in P_1$ can be "killed" by an "algebraic cell exchange"
$$(P_1, P_2, \rho_1, \rho_2) \to (P_1/\langle x_1\rangle, P_2, [\rho_1], [\rho_2])$$
if and only if $\rho_1(x_1) = 0 \in P_2$. Similarly for $x_2 \in P_2$.

The Nil$-$Nil theorem

- The construction of the groups $\widetilde{\mathrm{Nil}}_*(R,R,R)$ is more complicated than that of $\widetilde{\mathrm{Nil}}_*(R)$. However:
- **Nil$-$Nil Theorem** (DKR, 2008) For any ring R the functors of exact categories

$$i \ : \ \mathrm{Nil}(R) \to \mathrm{Nil}(R,R,R) \ ; \ (P,\nu) \mapsto (P,P,\nu,1) \ ,$$
$$j \ : \ \mathrm{Nil}(R,R,R) \to \mathrm{Nil}(R) \ ; \ (P_1,P_2,\rho_1,\rho_2) \mapsto (P_1,\rho_2\rho_1)$$

  induce inverse isomorphisms

$$i \ : \ \widetilde{\mathrm{Nil}}_*(R) \ \cong \ \widetilde{\mathrm{Nil}}_*(R,R,R) \ ,$$
$$j \ : \ \widetilde{\mathrm{Nil}}_*(R,R,R) \ \cong \ \widetilde{\mathrm{Nil}}_*(R) \ .$$

- **Idea of proof** For any (P_1,P_2,ρ_1,ρ_2) it is possible to make ρ_2 an isomorphism by elementary algebraic cell exchanges. Details in DKR.

The transfer theorem

- **Theorem** (DKR, 2008) The transfer for the index 2 subgroup $\mathbb{Z} \subset D_\infty$

$$\begin{aligned} K_*(R[D_\infty]) &= K_*(R \to R[\mathbb{Z}_2] \times R[\mathbb{Z}_2]) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R,R,R) \\ \to K_*(R[\mathbb{Z}]) &= K_*(R) \oplus K_{*-1}(R) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R) \end{aligned}$$

is the diagonal embedding on the $\widetilde{\mathrm{Nil}}$-groups

$$\begin{aligned} \widetilde{\mathrm{Nil}}_{*-1}(R,R,R) &\cong \widetilde{\mathrm{Nil}}_{*-1}(R) \\ &\to \widetilde{\mathrm{Nil}}_{*-1}(R) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R) ~;~ x \mapsto (x,x) ~. \end{aligned}$$

Some non-zero $\widetilde{\text{Nil}}$-groups

- **Example** (Bass, 1968) For $R = \mathbb{Z}[\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}]$

$$\widetilde{\text{Nil}}_0(R) \ = \ \ker(K_1(R[t]) \to K_1(R))$$

is an infinitely generated group of exponent a power of two.

- **Corollary** (DKR, 2008) For $R = \mathbb{Z}[\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}]$

$$\widetilde{\text{Nil}}_0(R, R, R) \ = \ \widetilde{\text{Nil}}_0(R)$$

is an infinitely generated group of exponent a power of two.

- This is the first nontrivial computation of the type (A) codimension 1 splitting obstruction groups $\widetilde{\text{Nil}}_*(R, B_1, B_2)$.

Codimension 1 transversality and splitting

- A map of finite CW complexes $f : M \to X$ is **transverse** at a codimension 1 subcomplex $Y \subset X$ if f is cellular and $N = f^{-1}(Y) \subset M$ is a codimension 1 subcomplex.
- **Proposition** Every map is simple homotopic to a transverse map.
- A homotopy equivalence of finite CW complexes $f : M \to X$ is **split** at a codimension 1 subcomplex $Y \subset X$ if it is transverse and the restrictions

$$g \;=\; f| \;:\; N \to Y \;,\; h \;=\; f| \;:\; M \backslash N \to X \backslash Y$$

  are also homotopy equivalences.
- The Whitehead torsion $\tau(f) \in Wh(\pi_1(X))$ of a split f is

$$\tau(f) \;=\; \tau(h) - \tau(g) \in \mathrm{im}(Wh(\pi_1(X \backslash Y)) \to Wh(\pi_1(X))) \;.$$

  So $[\tau(f)] \in \mathrm{coker}(Wh(\pi_1(X \backslash Y)) \to Wh(\pi_1(X)))$ is an obstruction to codimension 1 splitting.

The codimension 1 splitting theorem

- **Theorem** (Farrell-Hsiang (1968), Waldhausen (1969/78)) Let (X, Y) be a codimension 1 finite CW pair with X, Y connected and $\pi_1(Y) \to \pi_1(X)$ injective.
- (i) The Whitehead group of X fits into an exact sequence

$$Wh(\pi_1(Y)) \xrightarrow{i_1 - i_2} Wh(\pi_1(X \backslash Y)) \to Wh(\pi_1(X))$$

$$\xrightarrow{\partial} \widetilde{K}_0(\mathbb{Z}[\pi_1(Y)]) \oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi_1(Y)], B_1, B_2) \to \widetilde{K}_0(\mathbb{Z}[\pi_1(X \backslash Y)])$$

  with $B_j = \mathbb{Z}[\pi_1(X_j) \backslash \pi_1(Y)]$ $(j = 1, 2)$ in case (A).
- (ii) A homotopy equivalence $f : M \to X$ from a finite CW complex M splits at $Y \subset X$ if and only if

$$\begin{aligned}\tau(f) &\in \mathrm{im}(Wh(\pi_1(X \backslash Y)) \to Wh(\pi_1(X))) \\ &= \ker(\partial : Wh(\pi_1(X)) \to \widetilde{K}_0(\mathbb{Z}[\pi_1(Y)]) \oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi_1(Y)], B_1, B_2))\ .\end{aligned}$$

The universal cover of a union $X = X_1 \cup_Y X_2$ I.

- The Bass-Serre tree T of an injective amalgamated free product $G = G_1 *_H G_2$ has

$$T^{(0)} \;=\; [G : G_1] \cup [G : G_2] \;,\;\; T^{(1)} \;=\; [G : H] \;.$$

- Let $X = X_1 \cup_Y X_2$ be a finite CW complex with a type (A) codimension 1 subcomplex $Y \subset X$ such that the morphisms

$$i_j \;:\; \pi_1(Y) \to \pi_1(X_j) \; (j = 1, 2)$$

are injective, and $\pi_1(X) = \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2)$.

- The universal cover of X is

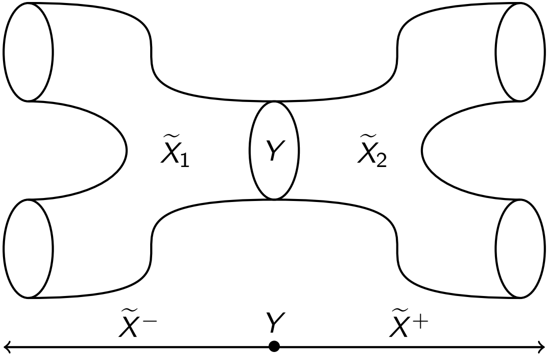
$$\widetilde{X} \;=\; \bigcup_{g_1 \in [\pi_1(X):\pi_1(X_1)]} g_1\widetilde{X}_1 \cup \bigcup_{h \in [\pi_1(X):\pi_1(Y)]} h\widetilde{Y} \bigcup_{g_2 \in [\pi_1(X):\pi_1(X_2)]} g_2\widetilde{X}_2$$

with $\widetilde{X}_1, \widetilde{X}_2, \widetilde{Y}$ the universal covers of X_1, X_2, Y.

The universal cover of a union $X = X_1 \cup_Y X_2$ II.

$$\widetilde{X}^+ = \widetilde{X}_2 \cup \bigcup_{h_2 \in [G_2:H]\backslash\{H\}} h_2 \widetilde{X}_1 \cup \bigcup_{h_1 \in [G_1:H]\backslash\{H\}, h_2 \in [G_2:H]\backslash\{H\}} h_2 h_1 \widetilde{X}_2 \cup \dots$$

$$\widetilde{X}^- = \widetilde{X}_1 \cup \bigcup_{h_1 \in [G_1:H]\backslash\{H\}} h_1 \widetilde{X}_2 \cup \bigcup_{h_1 \in [G_1:H]\backslash\{H\}, h_2 \in [G_2:H]\backslash\{H\}} h_1 h_2 \widetilde{X}_1 \cup \dots$$



$$\widetilde{X} = \widetilde{X}^- \cup_{\widetilde{Y}} \widetilde{X}^+$$

The type (A) codimension 1 splitting obstruction I.

- Given a transverse map

$$f = f_1 \cup_g f_2 : M = M_1 \cup_N M_2 \to X = X_1 \cup_Y X_2$$

the kernel modules fit into an M-V exact sequence

$$\begin{aligned} \cdots \to \mathbb{Z}[\pi_1(X)] \otimes_{\mathbb{Z}[\pi_1(Y)]} K_r(N) \to \\ \mathbb{Z}[\pi_1(X)] \otimes_{\mathbb{Z}[\pi_1(X_1)]} K_r(M_1) \oplus \mathbb{Z}[\pi_1(X)] \otimes_{\mathbb{Z}[\pi_1(X_2)]} K_r(M_2) \\ \to K_r(M) \to \mathbb{Z}[\pi_1(X)] \otimes_{\mathbb{Z}[\pi_1(Y)]} K_{r-1}(N) \to \dots \end{aligned}$$

- f is a homotopy equivalence iff π_1-iso and $K_*(M) = 0$.
- f is a split homotopy equivalence iff f_1, f_2, g are π_1-iso and $K_*(M_1) = K_*(M_2) = K_*(N) = 0$.
- Let $\overline{X} = \widetilde{X}/\pi_1(Y)$, $\overline{X}_j = \widetilde{X}_j/\pi_1(Y)$ so that $\overline{X} = \overline{X}^- \cup_Y \overline{X}^+$ with $\overline{X}_1 \subset \overline{X}^-$, $\overline{X}_2 \subset \overline{X}^+$. Similarly for $\overline{M} = \overline{M}^- \cup_Y \overline{M}^+$.

The type (A) codimension 1 splitting obstruction II.

- **Lemma** (Waldhausen) (i) For $n \geqslant 2$ a homotopy equivalence

$$f \;=\; f_1 \cup_g f_2 \;:\; M \;=\; M_1 \cup_N M_2 \to X \;=\; X_1 \cup_Y X_2$$

is simple homotopic to one concentrated in dimension n, with

$$K_r(M_1) \;=\; K_r(M_2) \;=\; K_r(N) \;=\; 0 \text{ for } r \neq n \;.$$

- (ii) The codimension 1 splitting obstruction of f is

$$\partial[\tau(f)] \;=\; ([P_1],[P_1,P_2,\rho_1,\rho_2]) \in \widetilde{K}_0(R) \oplus \widetilde{\mathrm{Nil}}_0(R,B_1,B_2)$$

with $R = \mathbb{Z}[\pi_1(Y)]$, $B_j = \mathbb{Z}[\pi_1(X_j)\backslash i_j\pi_1(Y)]$ $(j = 1,2)$, and

$$P_1 \;=\; K_{n+1}(\overline{M}^+, N) \;,\; P_2 \;=\; K_{n+1}(\overline{M}^-, N)$$

f.g. projective R-modules such that $P_1 \oplus P_2 = K_n(N)$ is f.g. free and

$$[P_1] + [P_2] \;=\; [K_n(N)] \;=\; 0 \in \widetilde{K}_0(R) \;.$$

Semi-splitting

- Let $f : M \to X$ is a homotopy equivalence of finite CW complexes, and let $Y \subset X = X_1 \cup_Y X_2$ be a type (A) codimension 1 subcomplex. A homotopy equivalence

$$f \ = \ f_1 \cup_g f_2 \ : \ M \ = \ M_1 \cup_N M_2 \to X \ = \ X_1 \cup_Y X_2$$

is **semi-split** if

$$K_*(\overline{M}_2, N) \ = \ 0 \ .$$

- If f is concentrated in dimension n it follows from the exact sequence

$$\begin{aligned} 0 \to K_{n+1}(\overline{M}_2, N) \to P_2 = K_{n+1}(\overline{M}^+, N) \xrightarrow{\rho_2} \\ B_2 \otimes_R P_1 = K_{n+1}(\overline{M}^+, \overline{M}_2) \to K_n(\overline{M}_2, N) \to 0 \end{aligned}$$

that f is semi-split if and only if $\rho_2 : P_2 \to B_2 \otimes_R P_1$ is an isomorphism.

The topological semi-splitting theorem

- **Theorem** Let $X = X_1 \cup_Y X_2$ be a finite CW complex with a type (A) decomposition. If the group morphisms

$$i_1 \ : \ \pi_1(Y) \to \pi_1(X_1) \ , \ i_2 \ : \ \pi_1(Y) \to \pi_1(X_2)$$

  are injective then every homotopy equivalence $f : M \to X$ of finite CW complexes is simple homotopic to one which semi-splits.
- **Idea of proof** As for the Nil-Nil theorem, realizing algebraic cell exchanges by geometric cell exchanges.

Groups over D_∞

- **Definition** A **group over** D_∞ is a group G with a surjection
$$p\ :\ G \to D_\infty\ =\ \mathbb{Z}_2 * \mathbb{Z}_2\ =\ \langle t_1\rangle * \langle t_2\rangle\ .$$
- **Lemma 1** G is an amalgamated free product
$$G\ =\ G_1 *_H G_2$$
with $G_i = p^{-1}(\langle t_i\rangle)$ for $(i = 1, 2)$, and
$$H\ =\ G_1 \cap G_2\ =\ \ker(p)\ ,\ [G_i : H]\ =\ 2\ .$$
- **Lemma 2** G has an index 2 subgroup $\overline{G} = p^{-1}(\langle t_1 t_2\rangle)$ which is an HNN extension
$$\overline{G}\ =\ H \times_\alpha \mathbb{Z}\ ,\ ht = t\alpha(h)$$
with $\mathbb{Z} = \langle t\rangle$ for any $t \in p^{-1}\langle t_1 t_2\rangle$ and
$$\alpha\ :\ H \to H\ ;\ h \mapsto t^{-1}ht\ .$$

Further results from the DKR paper I.

- The extension of the Nil-Nil-theorem to an arbitrary group $G = G_1 *_H G_2$ over $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$

$$\begin{aligned} K_n(R[G]) &= K_n(i_1 \times i_2) \oplus \widetilde{\mathrm{Nil}}_{n-1}(R[H], B_1, B_2) \\ &= K_n(i_1 \times i_2) \oplus \widetilde{\mathrm{Nil}}_{n-1}(R[H], B_1 \otimes_{R[H]} B_2) \end{aligned}$$

  with $i_1 \times i_2 : R[H] \to R[G_1] \times \mathbb{R}[G_2]$ and $B_j = R[G_j \backslash H]$.
- The extension of the transfer theorem to the index 2 subgroup $\overline{G} = H \times_\alpha \mathbb{Z} \subset G$. The transfer

$$\begin{aligned} K_*(R[G]) &= K_*(i_1 \times i_2) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R[H], B_1, B_2) \\ \to K_*(R[\overline{G}]) &= K_*(1 - \alpha : R[H] \to R[H]) \\ &\quad \oplus \widetilde{\mathrm{Nil}}_{*-1}(R[H], B_1 \otimes_{R[H]} B_2) \oplus \widetilde{\mathrm{Nil}}_{*-1}(R[H], B_1 \otimes_{R[H]} B_2) \end{aligned}$$

  is the diagonal embedding on the $\widetilde{\mathrm{Nil}}$-groups.

Further results from the DKR paper II.

- The extension of the Nil-Nil-theorem to (R,R)-bimodules B_1, B_2 such that B_2 is f.g. projective R-module and flat as a right R-module. The functors of exact categories

$$\begin{aligned} i \ : \ & \mathrm{Nil}(R, B_1 \otimes_R B_2) \to \mathrm{Nil}(R, B_1, B_2) \ ; \\ & \qquad (P, \nu) \mapsto (P, B_2 \otimes_R P, \nu, 1) \ , \\ j \ : \ & \mathrm{Nil}(R, B_1, B_2) \to \mathrm{Nil}(R, B_1 \otimes_R B_2) \ ; \\ & \qquad (P_1, P_2, \rho_1, \rho_2) \mapsto (P_1, \rho_2\rho_1) \end{aligned}$$

  induce isomorphisms

$$\widetilde{\mathrm{Nil}}_*(R, B_1 \otimes_R B_2) \ \cong \ \widetilde{\mathrm{Nil}}_*(R, B_1, B_2) \ .$$

- The reduction of the Farrell–Jones isomorphism conjecture in algebraic K-theory to the family of finite-by-cyclic groups, avoiding the need for virtually cyclic groups of infinite dihedral type.