SOME NUMERICAL FUNCTIONS ASSOCIATED TO THE MASLOV INDEX

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The Many Facets of the Maslov Index

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11th April, 2014

Introduction

- Want to describe some numerical functions associated to the Maslov index (= nonadditivity invariant) of three lagrangians L_1, L_2, L_3 in a symplectic form (K, ϕ), particularly in the case

$$(K,\phi) \ = \ H_-(\mathbb{R}) \ = \ (\mathbb{R} \oplus \mathbb{R}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \ .$$

- There is a whole zoo of such functions in the literature:

$$\tau(x_1, x_2, x_3) \ , \ [x] \ , \ \{x\} \ , \ ((x)) \ , \ \mu(x) \ , \ \eta(x) \ , \ E(x) \ , \ \log z \ \ldots$$

related to Dedekind sums, Rademacher functions, ...

The space $\Lambda(1)$ of lagrangians in $H_-(\mathbb{R})$.

- The lagrangians of the symplectic form

$$(K,\phi) \;=\; H_-(\mathbb{R}) \;=\; (\mathbb{R}\oplus\mathbb{R}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$$

  are just the 1-dimensional subspaces

$$L(\theta) \;=\; \{(r\cos\theta, r\sin\theta)\,|\,r\in\mathbb{R}\} \subset K \;=\; \mathbb{R}\oplus\mathbb{R}$$

  for $\theta\in\mathbb{R}$, with

$$L(\theta) \;=\; L(\theta') \text{ if and only if } \theta'-\theta\in\pi\mathbb{Z}\subset\mathbb{R}\ .$$

- The function

$$S^1 \to \Lambda(1)\ ;\ z=e^{i\psi} \mapsto \sqrt{z} = L(\psi/2)$$

  is a homeomorphism.
- $\Lambda(1)$ may seem a very trivial example, but ...

From **Auguries of innocence**

To see a world in a grain of sand
And a heaven in a wild flower,
Hold infinity in the palm of your hand,
And eternity in an hour.

William Blake

Nonadditivity, jumps, and signs

- The **nonadditivity** of a function $f : \mathbb{R} \to \mathbb{R}$ is the function

$$\mathbb{R} \times \mathbb{R} \to \mathbb{R} \; ; \; (x, y) \mapsto f(x) + f(y) - f(x + y) \; .$$

- The **jump** of a function $f : \mathbb{R} \to \mathbb{R}$ at $x \in \mathbb{R}$ is

$$j(x) \; = \; \varinjlim_{\epsilon} (f(x + \epsilon) - f(x - \epsilon)) \in \mathbb{R} \; .$$

- The **sign** of $x \in \mathbb{R}$ is

$$\text{sgn}(x) \; = \; \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \; . \end{cases}$$

The whole and the part

- Given a real number $x \in \mathbb{R}$ let $[x] \in \mathbb{Z}$ be the **integer** part and let $\{x\} \in [0,1)$ be the **fractional** part, so that

$$x = [x] + \{x\} \in \mathbb{R} .$$

- Many interesting algebraic and number theoretic properties of the Maslov index can be traced to the jumps and nonadditivity of the functions

$$\mathbb{R} \to \mathbb{Z} \subset \mathbb{R} ;\ x \mapsto [x] ,\ \mathbb{R} \to [0,1) \subset \mathbb{R} ;\ x \mapsto \{x\} .$$

- First appeared in the context of algebraic topology of manifolds in the 1960's calculations by Hirzebruch of the signatures of manifolds bounding exotic spheres in general (Brieskorn varieties), torus knots in particular,

The nonadditivity of $[x]$ and $\{x\}$

- **Proposition** The functions

$$[\ ] \ : \ \mathbb{R} \to \mathbb{Z} \ ; \ x \mapsto [x] \ ,$$
$$\{\ \} \ : \ \mathbb{R} \to [0,1) \ ; \ x \mapsto \{x\} = x - [x]$$

have the following jump and nonadditive properties:

1. $\{\ \}$ is continuous on $\mathbb{R}\backslash\mathbb{Z}$, with a jump -1 at each $x \in \mathbb{Z}$.
2. $\{x\}+\{y\}-\{x+y\} \ = \ [x+y]-[x]-[y] \ = \ \begin{cases} 0 & \text{if } 0 \leqslant \{x\}+\{y\} < 1 \\ 1 & \text{if } 1 \leqslant \{x\}+\{y\} < 2. \end{cases}$
3. $\{x+1\} \ = \ \{x\}$.
4. $\{x\}+\{-x\} \ = \ \{x\}+\{1-x\} \ = \ \begin{cases} 1 & \text{if } x \in \mathbb{R}\backslash\mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$
5. $\{x+1/2\}-\{x\} \ = \ \begin{cases} 1/2 & \text{if } 0 \leqslant \{x\} < 1/2 \\ -1/2 & \text{if } 1/2 \leqslant \{x\} < 1. \end{cases}$

The triple signature in general

- Write the signature of a symmetric form (L, Φ) as

$$\sigma(L, \Phi) \in \mathbb{Z} \ .$$

- **Definition** (Wall, Leray, Kashiwara, ... 1970's)
  The **Maslov index** (aka **the triple signature**) of an ordered triple of lagrangians L_1, L_2, L_3 in a nonsingular symplectic form (K, ϕ) over $\mathbb{R}$ is the signature

$$\tau(L_1, L_2, L_3) \ = \ \sigma(L_1 \oplus L_2 \oplus L_3, \Phi_{123}) \in \mathbb{Z}$$

  of the symmetric form

$$\Phi_{123} \ = \ \begin{pmatrix} 0 & \phi_{12} & \phi_{13} \\ \phi_{21} & 0 & \phi_{23} \\ \phi_{31} & \phi_{32} & 0 \end{pmatrix}$$

  with

$$\phi_{ij} : L_i \times L_j \to \mathbb{R} \ ; \ (x_i, x_j) \mapsto \phi(x_i, x_j) \ .$$

The Maslov index $\tau(\theta_1, \theta_2, \theta_3)$ I.

- **Definition** The **Maslov index** of $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ is

$$\tau(\theta_1, \theta_2, \theta_3) \;=\; \tau(L(\theta_1), L(\theta_2), L(\theta_3)) \in \mathbb{Z} \;,$$

  the triple signature of the lagrangians $L(\theta_1), L(\theta_2), L(\theta_3)$ in $H_-(\mathbb{R})$.

- From the definition

$$\tau(\theta_1, \theta_2, \theta_3) \;=\; \sigma(\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}, \Phi_{123})$$

  with

$$\Phi_{123} \;=\; \begin{pmatrix} 0 & \sin(\theta_1 - \theta_2) & \sin(\theta_2 - \theta_3) \\ \sin(\theta_1 - \theta_2) & 0 & \sin(\theta_3 - \theta_1) \\ \sin(\theta_2 - \theta_3) & \sin(\theta_3 - \theta_1) & 0 \end{pmatrix}$$

The Maslov index $\tau(\theta_1, \theta_2, \theta_3)$ II.

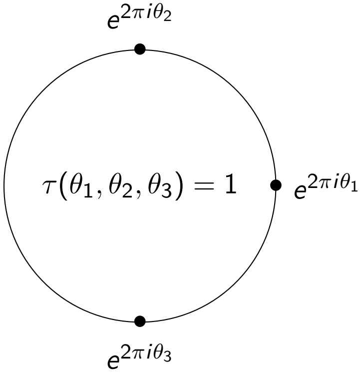
- The signature of a symmetric matrix is the number of changes of sign in the minors.
- The matrix Φ_{123} has minors

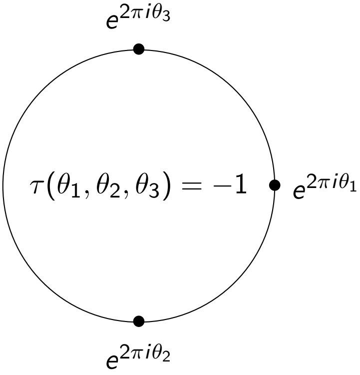
$$0 \ , \ \ -\sin^2(\theta_1 - \theta_2) \ , \ \sin(\theta_1 - \theta_2)\sin(\theta_2 - \theta_3)\sin(\theta_3 - \theta_1)$$

so that

$$\begin{aligned} \tau(\theta_1, \theta_2, \theta_3) &= \operatorname{sgn}(\sin(\theta_2 - \theta_1)\sin(\theta_3 - \theta_2)\sin(\theta_3 - \theta_1)) \\ &= \begin{cases} \operatorname{sgn}(\sigma) & \text{if } \{\theta_1/2\pi\}, \{\theta_2/2\pi\}, \{\theta_3/2\pi\} \in [0,1) \text{ are distinct} \\ & \text{with } \sigma \in \Sigma_3 \text{ the permutation such that} \\ & \{\theta_{\sigma(1)}/2\pi\} < \{\theta_{\sigma(2)}/2\pi\} < \{\theta_{\sigma(3)}/2\pi\} \\ 0 & \text{otherwise} \end{cases} \\ &\in \{-1, 0, 1\} \subset \mathbb{Z} \ . \end{aligned}$$

The Maslov index $\tau(\theta_1, \theta_2, \theta_3)$ III.

- Geometrically: 1 (resp. -1) if $e^{2\pi i\theta_1}$, $e^{2\pi i\theta_2}$, $e^{2\pi i\theta_3} \in S^1$ arranged clockwise (resp. counterclockwise) around S^1, and 0 if any coincidence.
- 



The Maslov index $\tau(\theta_1, \theta_2, \theta_3)$ IV.

- In view of the identity

$$\begin{aligned}\sin(\theta_2 - \theta_1)\sin(\theta_3 - \theta_2)\sin(\theta_3 - \theta_1) \\ = (\sin 2(\theta_2 - \theta_1) + \sin 2(\theta_3 - \theta_2) + \sin 2(\theta_1 - \theta_3))/4\end{aligned}$$

can also write

$$\tau(\theta_1, \theta_2, \theta_3) = \operatorname{sgn}(\sin 2(\theta_2 - \theta_1) + \sin 2(\theta_3 - \theta_2) + \sin 2(\theta_1 - \theta_3)) \ .$$

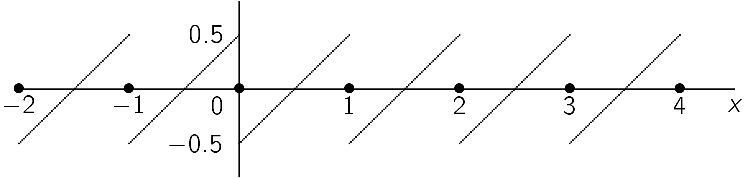
The sawtooth function $((x))$ I.

- The **sawtooth function** $((x)) : \mathbb{R} \to [-1/2, 0)$ is defined by

$$((x)) \ = \ \begin{cases} \{x\} - 1/2 & \text{if } x \in \mathbb{R}\backslash\mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$

with $\{x\} \in [0, 1)$ the fractional part of $x \in \mathbb{R}$. Nonadditive:

$$((x)) + ((y)) - ((x+y)) \ = \ \begin{cases} -1/2 & \text{if } 0 < \{x\} + \{y\} < 1 \\ 1/2 & \text{if } 1 < \{x\} + \{y\} < 2 \\ 0 & \text{if } x \in \mathbb{Z} \text{ or } y \in \mathbb{Z} \text{ or } x + y \in \mathbb{Z}\ . \end{cases}$$



The origin of the sawtooth function $((x))$

- Used by Dedekind (1876) in his commentary on the Riemann Nachlass to count $\pm 2\pi i = \pm 4(\pi/2)i$ jumps in the imaginary part of the complex logarithm

$$\log(re^{i\theta}) \ = \ \log(r) + i(\theta + 2n\pi) \in \mathbb{C} \ \ (n \in \mathbb{Z}) \ .$$

- From Dedekind's commentary:

durchläuft, und der imaginäre Theil der Logarithmen zwischen $\pm \frac{\pi i}{2}$, also

$$= \pi i \left(\left(\frac{m t}{n} - \frac{1}{2} \right) \right)$$

zu nehmen ist, wenn der Deutlichkeit halber der von x um eine ganze Zahl abstehende, zwischen $\pm \frac{1}{2}$ liegende Werth nicht mit (x), sondern mit $((x))$ bezeichnet wird. Durch Anwendung der Transformation

Dedekind sums and signatures

- Eisenstein's formula for $x = p/q \in \mathbb{Q}$

$$((x)) \;=\; \frac{i}{2q}\sum_{j=1}^{q-1}\cot\frac{\pi j}{q}e^{2\pi ijx}\;.$$

- The **Dedekind sum** for $a, c \in \mathbb{Z}$ with $c \neq 0$ is

$$s(a,c) \;=\; \sum_{k=1}^{|c|-1}((\frac{k}{c}))((\frac{ka}{c})) \;=\; \frac{1}{4|c|}\sum_{k=1}^{|c|-1}\cot(\frac{k\pi}{c})\cot(\frac{ka\pi}{c}) \in \mathbb{Q}\;.$$

- Feature prominently in work of Hirzebruch and Zagier.
- Barge and Ghys, *Cocycles d'Euler et de Maslov* (1992) use $E(x)$ and Dedekind sums in the hyperbolic geometry interpretation of the Maslov index, related to the action of $SL_2(\mathbb{Z})$ on the upper half plane.
- Also Kirby and Melvin, *Dedekind sums, μ-invariants and the signature cocycle* (1994)

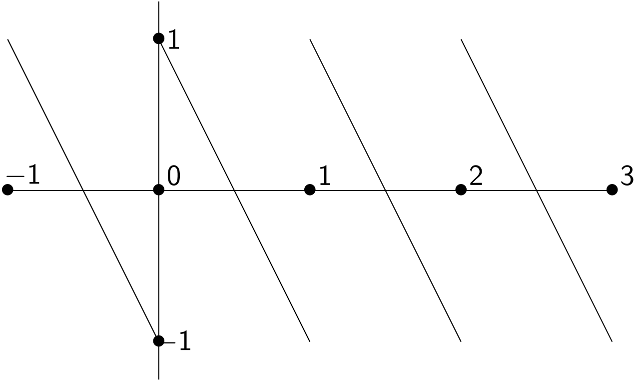
The sawtooth function $((x))$ II.

- **Proposition** The sawtooth function has the following jumping and nonadditivity properties:
  1. $((\ ))$ is continuous on $\mathbb{R}\backslash\mathbb{Z}$, with a jump -1 at each $x \in \mathbb{Z}$.
  2. $((0)) = ((1/2)) = 0$.
  3. $((x+1)) = ((x))$, $((-x)) = -((x))$.
  4. $((x)) = x + ([-x] - [x])/2 = (\{x\} - \{-x\})/2$.
  5. $$((x)) + ((y)) - ((x+y)) = \begin{cases} 0 & \text{if } x \in \mathbb{Z} \text{ or } y \in \mathbb{Z} \text{ or } x+y \in \mathbb{Z} \\ -1/2 & \text{if } 0 < \{x\} + \{y\} < 1 \\ 1/2 & \text{if } 1 < \{x\} + \{y\} < 2. \end{cases}$$
  6. $$((x+1/2)) = \begin{cases} \{x\} & \text{if } 0 \leqslant \{x\} < 1/2 \\ 0 & \text{if } \{x\} = 1/2 \\ \{x\} - 1 & \text{if } 1/2 < \{x\} < 1. \end{cases}$$

The reverse sawtooth function $\mu(x)$ I.

- **Definition** The **reverse sawtooth function** is

$$\mu \ : \ \mathbb{R} \to (-1,1] \ ; \ x \mapsto \mu(x) = 1 - 2\{x\}$$

- 

The reverse sawtooth function $\mu(x)$ II.

- **Proposition** The reverse sawtooth function has the following jumping and nonadditivity properties:
  1. $\mu(x) \ = \ \begin{cases} -2((x)) & \text{if } x \in \mathbb{R} \backslash \mathbb{Z} \\ 1 & \text{if } x \in \mathbb{Z} \ . \end{cases}$
  2. μ is continuous at $x \in \mathbb{R}\backslash\mathbb{Z}$, with a jump 2 at each $x \in \mathbb{Z}$.
  3. $\mu(x) + \mu(y) - \mu(x+y) = \begin{cases} +1 & \text{if } 0 \leqslant \{x\} + \{y\} < 1 \\ -1 & \text{if } 1 \leqslant \{x\} + \{y\} < 2. \end{cases}$
  4. $\mu(0) = 1$, $\mu(1/2) = 0$.
  5. $\mu(x+1) = \mu(x)$ for $x \in \mathbb{R}$.
  6. $\mu(x) + \mu(-x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Z} \\ 2 & \text{if } x \in \mathbb{Z}. \end{cases}$
  7. $\mu(x) - \mu(x+1/2) = 2\mu(x) - \mu(2x) = \begin{cases} +1 & \text{if } 0 \leqslant \{x\} < 1/2 \\ -1 & \text{if } 1/2 \leqslant \{x\} < 1. \end{cases}$

The function $E(x)$

- **Definition** (Barge and Ghys, Cocycles d'Euler et de Maslov, 1992)
  The E-function is

$$E \;:\; \mathbb{R} \to \mathbb{R}\,;\; x \mapsto x - ((x)) \;=\; ([x] - [-x])/2 \;=\; \begin{cases} [x] + 1/2 & \text{if } x \in \mathbb{R}\backslash\mathbb{Z} \\ x & \text{if } x \in \mathbb{Z} \end{cases}$$

- **Proposition** For any $x, y \in \mathbb{R}$

$$\begin{aligned} E(x+y) - E(x) - E(y) &= ((x)) + ((y)) - ((x+y)) \\ &= \begin{cases} 0 & \text{if } x \in \mathbb{Z} \text{ or } y \in \mathbb{Z} \text{ or } x+y \in \mathbb{Z} \\ -1/2 & \text{if } 0 < \{x\} + \{y\} < 1 \\ 1/2 & \text{if } 1 < \{x\} + \{y\} < 2. \end{cases} \end{aligned}$$

The Rademacher functions $\phi_n(x)$

- The **Rademacher functions**

$$\phi_n \ : \ \mathbb{R} \to \{-1,0,1\} \ ; \ x \mapsto \mathrm{sgn}(\sin 2^{n+1}\pi x) \ (n \geqslant 0)$$

are such that

(i) $\phi_0(x) \ = \ \mathrm{sgn}(\sin 2\pi x) \ = \ \begin{cases} +1 & \text{if } 0 < \{x\} < 1/2 \\ -1 & \text{if } 1/2 < \{x\} < 1 \\ 0 & \text{if } \{x\} = 0 \text{ or } 1/2 \ . \end{cases}$

(ii) $\phi_n(x) = \phi_0(2^n x) = 2((2^{n+1}x)) - 2^{n+2}((x))$.

(iii) $\phi_n(x+1) = \phi_n(x)$, $\phi_n(x+1/2) = \phi_n(-x) = -\phi_n(x)$.

(iv) $\phi_n(0) = \phi_n(1/2) = 0$.

(v) $\mu(x) - \mu(x+1/2) = 2\mu(x) - \mu(2x) = \phi_0(x)$ for $2x \in \mathbb{R}\backslash\mathbb{Z}$.

The Walsh functions

- The **Walsh functions** $\psi_n : \mathbb{R} \to \{-1, 0, 1\}$ ($n \geqslant 0$) are defined by

$$\psi_0(x) \;=\; 1 \;,\; \psi_n(x) \;=\; \phi_{n_1}(x)\phi_{n_2}(x)\ldots\phi_{n_k}(x)$$
$$(n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_k}) \;.$$

  In particular

$$\psi_{2^n}(x) \;=\; \phi_n(x) \;=\; \phi_0(2^n x) \;=\; \mathrm{sgn}(\sin 2^{n+1}\pi x) \;.$$

- The Walsh functions constitute a complete orthonormal set, behaving like trigonometric series on $[0, 2\pi]$:

$$\int_0^1 \psi_m(x)\psi_n(x)dx \;=\; \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \;. \end{cases}$$

- Every Lebesgue integrable function $f : I \to \mathbb{R}$ has a Walsh-Fourier expansion

$$F(x) \;=\; \sum_{n=0}^{\infty} c_n\psi_n(x) \text{ with } c_n \;=\; \int_0^1 f(x)\psi_n(x)dx \in \mathbb{R}$$

The Fourier-Walsh expansion of $\mu(x)$

- **Proposition** The Fourier-Walsh expansion of the reverse sawtooth function $\mu(x)$ is

$$\mu(x) = \sum_{k=0}^{\infty} \psi_{2^k}(x)/2^{k+1} = \sum_{k=0}^{\infty} \phi_0(2^k x)/2^{k+1} ,$$

with

$$\int_0^1 \mu(x)\psi_n(x)dx = \begin{cases} 1/2^{k+1} & \text{if } n = 2^k \\ 0 & \text{if } n \neq 2^k. \end{cases}$$

The function $\eta(x)$ I.

- **Definition** The η-**invariant** function is

$$\eta \ : \ \mathbb{R} \to (-1,1] \ ;$$

$$\theta \mapsto \eta(\theta) \ = -2((\theta/\pi)) \ = \ \begin{cases} \mu(\theta/\pi) = 1 - 2\{\theta/\pi\} & \text{if } \theta/\pi \in \mathbb{R}\backslash\mathbb{Z} \\ 0 & \text{if } \theta/\pi \in \mathbb{Z} \ . \end{cases}$$

- First appeared in Atiyah, Patodi and Singer, *Spectral asymmetry and Riemannian geometry* (1974) as a spectral invariant η-invariant.
- Key ingredient in *On the Maslov index* Cappell, Lee and Miller (1994)

The function $\eta(x)$ II.

- The η-invariant function $\eta : \mathbb{R} \to (-1, 1]$ has the following properties:
  (i) η is continuous at $\theta \in \mathbb{R} \backslash \pi\mathbb{Z}$, jumping by 2 at $\theta \in \pi\mathbb{Z}$.
  (ii) $\eta(\pi n/2) = 0$ ($n \in \mathbb{Z}$).
  (iii) $\eta(\theta + \pi) = \eta(\theta)$, $\eta(-\theta) = -\eta(\theta)$,
  (iv) $2\eta(\theta) - \eta(2\theta) = \eta(\theta) + \eta(\pi/2 - \theta) = \mathrm{sgn}(\sin 2\theta)$.
  (v)

$$\begin{aligned}\eta(\theta) + \eta(\phi) - \eta(\theta + \phi) &= \mathrm{sgn}(\sin(\theta)\sin(\phi)\sin(\theta + \phi)) \\ &= \begin{cases} 1 & \text{if } \theta/\pi, \phi/\pi, (\theta+\phi)/\pi \in \mathbb{R}\backslash\mathbb{Z} \text{ and } 0 \leqslant \{\theta/\pi\} + \{\phi/\pi\} < 1 , \\ -1 & \text{if } \theta/\pi, \phi/\pi, (\theta+\phi)/\pi \in \mathbb{R}\backslash\mathbb{Z} \text{ and } 1 \leqslant \{\theta/\pi\} + \{\phi/\pi\} < 2 , \\ 0 & \text{otherwise} . \end{cases}\end{aligned}$$

The function $\eta(x)$ III.

(vi) In view of the identity

$$\sin(2\theta) + \sin(2\phi) - \sin(2(\theta+\phi)) = 4\sin(\theta)\sin(\phi)\sin(\theta+\phi)$$

also have

$$\eta(\theta)+\eta(\phi)-\eta(\theta+\phi) = \mathrm{sgn}\big(\sin(2\theta)+\sin(2\phi)-\sin(2(\theta+\phi))\big) \in \{-1,0,1\}\ .$$

(vii)

$$\begin{aligned} \eta(\theta) &= -2((\theta/\pi)) = -2\theta/\pi + ([\theta/\pi] - [-\theta/\pi]) \\ &= 2E(\theta/\pi) - 2\theta/\pi = \begin{cases} 1 - 2\{\theta/\pi\} & \text{if } \theta/\pi \in \mathbb{R}\backslash\mathbb{Z} \\ 0 & \text{if } \theta/\pi \in \mathbb{Z}\ . \end{cases} \end{aligned}$$

The multiplicativity of the exponential and nonadditivity of the logarithm I.

- The **exponential** function

$$\exp\ :\ \mathbb{C} \to \mathbb{C}\backslash\{0\}\ ;\ z \mapsto e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!} \in \mathbb{C}\backslash\{0\}$$

  is such that
  (i) $z \mapsto e^z$ is continuous
  (ii) $e^0 = 1$, $e^{z+w} = e^z e^w \in \mathbb{C}$
  (iii) $e^z = e^w \in \mathbb{C}\backslash\{0\}$ if and only if $z - w = 2\pi i k$ for some $k \in \mathbb{Z}$.
- The **principal logarithm** function

$$\log\ :\ \mathbb{C}\backslash\{0\} \to \mathbb{R} + i(-\pi, \pi] \subset \mathbb{C}$$

  is defined as usual by

$$\log(z)\ =\ \log(|z|) + i\arg(z)\ \ (\arg(z) \in (-\pi, \pi])\ .$$

The multiplicativity of the exponential and nonadditivity of the logarithm II.

- The principal logarithm is such that
  - (i) $z \mapsto \log(z)$ is continuous on $\mathbb{C}\backslash\{(-\infty,0]\}$.
  - (ii) $\log(1)=0$, $\log(-1)=\pi i$, $\log(\pm i)=\pm\pi i/2$.
  - (iii) If $z=re^{i\theta}\in\mathbb{C}\backslash\{0\}$ for $r>0$ $\theta\in\mathbb{R}$ then

$$
\begin{aligned}
\log(z) &= \log(r)+\pi i\mu(\frac{\pi-\theta}{2\pi}) \\
&= \log(r)+\pi i(1-2\{\frac{\pi-\theta}{2\pi}\}) \\
&= \begin{cases} \log(r)+2\pi i((\dfrac{\theta+\pi}{2\pi})) & \text{for } \theta/\pi\in\mathbb{R}\backslash(2\mathbb{Z}+1), \\ & \qquad \text{with } 2\pi((\dfrac{\theta+\pi}{2\pi}))\in(-\pi,\pi) \\ \log(r)+\pi i & \text{for } \theta/\pi\in 2\mathbb{Z}+1, \text{ with } z=-r \end{cases} \\
&= \begin{cases} \log(r)-\pi i\eta(\dfrac{\theta+\pi}{2}) & \text{for } \theta/\pi\in\mathbb{R}\backslash(2\mathbb{Z}+1), \\ \log(r)+\pi i & \text{for } \theta/\pi\in 2\mathbb{Z}+1, \text{ with } z=-r. \end{cases}
\end{aligned}
$$

The multiplicativity of the exponential and nonadditivity of the logarithm III.

- **Proposition** The exponential and principal logarithm functions have the following properties:
  (i) $e^{\log(z)} = z \in \mathbb{C}\backslash\{0\}$ for all $z \in \mathbb{C}\backslash\{0\}$.
  (ii) For $z = x + iy \in \mathbb{C}$

$$\log(e^z) \; = \; z - 2\pi ik \in \mathbb{C} \text{ for } (2k-1)\pi < y \leqslant (2k+1)\pi \; ,$$

  that is

$$\log(e^{x+iy}) \; = \; x + \pi i(1 - 2\{\frac{\pi - y}{2\pi}\}) \in \mathbb{C}\backslash\{0\} \; .$$

  (iii) If $z \in \mathbb{C}\backslash\{(-\infty, 0]\}$ then

$$\log(z) \; = \; \int_{-\infty}^{0} (\frac{1}{x-z} - \frac{1}{x-1})dx \in \mathbb{C} \; .$$

The multiplicativity of the exponential and nonadditivity of the logarithm IV.

(iv) For $z_1, z_2 \in \mathbb{C}\backslash\{0\}$

$$\log(z_1 z_2) - \log(z_1) - \log(z_2) \;=\; i(\arg(z_1 z_2) - \arg(z_1) - \arg(z_2))$$

$$= \begin{cases} 2\pi i & \text{if } -2\pi < \arg(z_1) + \arg(z_2) \leqslant -\pi \\ 0 & \text{if } -\pi < \arg(z_1) + \arg(z_2) \leqslant \pi \\ -2\pi i & \text{if } \pi < \arg(z_1) + \arg(z_2) \leqslant 2\pi \;. \end{cases}$$

(v) For $\theta_1, \theta_2 \in \mathbb{R}$

$$\log(e^{i(\theta_1+\theta_2)}) - \log(e^{i\theta_1}) - \log(e^{i\theta_2})$$

$$= \begin{cases} -2\pi i & \text{if } 0 \leqslant \{\frac{\pi-\theta_1}{2\pi}\} + \{\frac{\pi-\theta_2}{2\pi}\} < 1/2 \\ 0 & \text{if } 1/2 \leqslant \{\frac{\pi-\theta_1}{2\pi}\} + \{\frac{\pi-\theta_2}{2\pi}\} < 3/2 \\ 2\pi i & \text{if } 3/2 \leqslant \{\frac{\pi-\theta_1}{2\pi}\} + \{\frac{\pi-\theta_2}{2\pi}\} < 2 \;. \end{cases}$$

The expressions of $\eta(\theta)$ in terms of $((\ ))$, log and $\{\ \}$

- For any $\theta \in \mathbb{R}$

$$
\begin{aligned}
\eta(\theta) &= -2((\theta/\pi)) \\
&= \begin{cases} \dfrac{1}{\pi i}\log(-e^{-2i\theta}) = 1 - 2\{\dfrac{\theta}{\pi}\} & \text{if } e^{i\theta} \neq \pm 1 \\ 0 & \text{if } e^{i\theta} = \pm 1\ . \end{cases}
\end{aligned}
$$

Properties of the Maslov index $\tau(\theta_1, \theta_2, \theta_3)$ I.

- The triple signature function

$$\tau \ : \ \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \{-1, 0, 1\} \ ; \ (\theta_1, \theta_2, \theta_3) \mapsto$$
$$\tau(L(\theta_1), L(\theta_2), L(\theta_3)) \ = \ \mathrm{sgn}(\sin(\theta_2 - \theta_1)\sin(\theta_3 - \theta_2)\sin(\theta_3 - \theta_1))$$

has the following properties.

(i) $\tau(0, \theta, \pi/2) = \mathrm{sgn}(\sin\theta\cos\theta) = \mathrm{sgn}(\sin 2\theta)$.

(ii) The η-function is the average triple signature

$$\begin{aligned} \eta(\theta) &= \int_{\ell \in \Lambda(1)} \tau(\ell, L(0), L(\theta))d\ell \\ &= \int_{z \in S^1} \tau(\sqrt{z}, L(0), L(\theta))dz \\ &= \frac{1}{2\pi}\int_0^{2\pi} \tau(\psi/2, 0, \theta)d\psi \in \mathbb{R} \ . \end{aligned}$$

Properties of the Maslov index $\tau(\theta_1,\theta_2,\theta_3)$ II.

(iii)

$$\begin{aligned}\tau(\theta_1,\theta_2,\theta_3) &= \tau(\theta_1+\psi,\theta_2+\psi,\theta_3+\psi)\\ &= \eta(\theta_2-\theta_1)+\eta(\theta_3-\theta_2)+\eta(\theta_1-\theta_3)\\ &= 2(E((\theta_1-\theta_2)/\pi)+E((\theta_2-\theta_3)/\pi)+E((\theta_3-\theta_1)/\pi))\\ &= -2\big((((\theta_1-\theta_2)/\pi))+(((\theta_2-\theta_3)/\pi))+(((\theta_3-\theta_1)/\pi)\\ &\in \{-1,0,1\}\subset \mathbb{R}\ .\end{aligned}$$

(iv)

$$\eta(\theta_1)+\eta(\theta_2)-\eta(\theta_1+\theta_2) \;=\; \tau(0,\theta_1,-\theta_2) \;=\; \tau(0,\theta_2,-\theta_1)$$

$$= \begin{cases} +1 & \text{if } 0<\{\theta_1/\pi\}+\{\theta_2/\pi\}<1,\ \theta_1/\pi,\theta_2/\pi\in\mathbb{R}\backslash\mathbb{Z}\\ -1 & \text{if } 1<\{\theta_1/\pi\}+\{\theta_2/\pi\}<2,\ \theta_1/\pi,\theta_2/\pi\in\mathbb{R}\backslash\mathbb{Z} \in \{-1,0,1\} \\ 0 & \text{otherwise}\end{cases}$$

Properties of the Maslov index $\tau(\theta_1,\theta_2,\theta_3)$ III.

(v) For $\theta_1=\theta_2=\theta$

$$\begin{aligned} 2\eta(\theta)-\eta(2\theta) &= \tau(0,\theta,-\theta) = \phi_0(\theta/\pi) \\ &= \text{sign}(\sin 2\theta) = \begin{cases} +1 & \text{if } 0<\{\theta/\pi\}<1/2 \\ -1 & \text{if } 1/2<\{\theta/\pi\}<1 \in \{-1,0,1\}\subset \mathbb{R} \ . \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(vi)

$$\begin{aligned} &\tau(\theta_1,\theta_2,\theta_1+\theta_2) = \eta(\theta_1)-\eta(\theta_2)+\eta(\theta_2-\theta_1) = \tau(0,\theta_2-\theta_1,\theta_2) \\ &= \begin{cases} +1 & \text{if } 0<\{(\theta_2-\theta_1)/\pi\}+\{-\theta_2/\pi\}<1,\ (\theta_2-\theta_1)/\pi,\theta_2/\pi\in\mathbb{R}\backslash \\ -1 & \text{if } 1<\{(\theta_2-\theta_1)/\pi\}+\{-\theta_2/\pi\}<2,\ (\theta_2-\theta_1)/\pi,\theta_2/\pi\in\mathbb{R}\backslash \\ 0 & \text{otherwise}\ . \end{cases} \end{aligned}$$

Properties of the Maslov index $\tau(\theta_1, \theta_2, \theta_3)$ IV.

(vii) $\tau(\theta_{\sigma(1)}, \theta_{\sigma(2)}, \theta_{\sigma(3)}) = \mathrm{sgn}(\sigma)\tau(\theta_1, \theta_2, \theta_3)$ for any $\sigma \in \Sigma_3$.

(viii) $\tau(-\theta_1, -\theta_2, -\theta_3) = -\tau(\theta_1, \theta_2, \theta_3)$.

(ix) $\tau(\theta_1, \theta_2, \theta_3) = 0$ if $\theta_1 = \theta_2$.

(x) $\tau(\theta_1, \theta_2, \theta_3) = \tau(0, \theta_2 - \theta_1, \theta_3 - \theta_1)$.

Properties of the Maslov index $\tau(\theta_1, \theta_2, \theta_3)$ V.

(xi) For any $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ define a loop in $\Lambda(1)$ from $L(0) = \mathbb{R} \oplus 0$ through $L(\pi\eta(\theta_2 - \theta_1))$ and $L(\pi\eta(\theta_2 - \theta_1) + \pi\eta(\theta_3 - \theta_2))$ and then back to $L(0)$

$$\omega(\theta_1, \theta_2, \theta_3) \ : \ S^1 \to \Lambda(1) \ ;$$

$$e^{2\pi i t} \mapsto \begin{cases} L(3\pi t\eta(\theta_2 - \theta_1)) & \text{if } 0 \leqslant t \\ L(\pi\eta(\theta_2 - \theta_1) + (3t-1)\pi\eta(\theta_3 - \theta_2)) & \text{if } 1/3 \leqslant \\ L(\pi\eta(\theta_2 - \theta_1) + \pi\eta(\theta_3 - \theta_2) + (3t-2)\pi\eta(\theta_1 - \theta_3)) & \text{if } 2/3 \leqslant \end{cases}$$

with lift

$$\widetilde{\omega}(\theta_1, \theta_2, \theta_3) \ : \ I = [0,1] \to \widetilde{\Lambda(1)} = \mathbb{R} \ ;$$

$$e^{2\pi i t} \mapsto \begin{cases} 3t\eta(\theta_2 - \theta_1) & \text{if } 0 \leqslant t \leqslant 1/3 \\ \eta(\theta_2 - \theta_1) + (3t-1)\eta(\theta_3 - \theta_2) & \text{if } 1/3 \leqslant t \leqslant 2/ \\ \eta(\theta_2 - \theta_1) + \eta(\theta_3 - \theta_2) + (3t-2)\eta(\theta_1 - \theta_3) & \text{if } 2/3 \leqslant t \leqslant 1 \end{cases}$$

Properties of the Maslov index $\tau(\theta_1,\theta_2,\theta_3)$ VI.

(xii) The degree of $\omega(\theta_1,\theta_2,\theta_3)$ is the triple signature Maslov index

$$\begin{aligned}\deg(\omega(\theta_1,\theta_2,\theta_3)) &= \eta(\theta_2-\theta_1)+\eta(\theta_3-\theta_2)+\eta(\theta_1-\theta_3)\\ &= \tau(\theta_1,\theta_2,\theta_3)\in\mathbb{Z}\end{aligned}$$

(xiii) $\tau(\theta_1,\theta_2,\theta_3)=\tau(L(\theta_1),L(\theta_2),L(\theta_3))$.

(xiv) (Bunke, *On the glueing problem for the η-invariant*,1997)

$$\frac{1}{\pi}\int_{\theta_1=0}^{\pi}\tau(L(\theta_1),L(\theta_2),L(\theta_3))d\theta_1 \;=\; \mu(\frac{\theta_3-\theta_2}{\pi}) \text{ if } 0<\theta_2,\theta_3<\pi\ .$$

Properties of the Maslov index $\tau(\theta_1, \theta_2, \theta_3)$ VII.

(xv) (W.Meyer *Die Signatur von lokalen Koeffizientensystemen und Faserbündeln* 1972, Atiyah *The logarithm of the Dedekind η-function*,1987 The surface with 3 boundary components

$$(X, \partial X) \;=\; (\mathrm{cl.}(S^2 \backslash \bigcup_3 D^2), \bigcup_3 S^1)$$

has $\pi_1(X) = F_2 = \{g_1, g_2\}$ the free group on 2 generators g_1, g_2. Let E be the local coefficient system over X of flat hermitian vector spaces classified by the group morphism

$$\pi_1(X) = F_2 \to U(1) = S^1 \;;\; g_j \mapsto e^{i\theta_j} \;(j = 1, 2)\;.$$

The index of a first-order elliptic operator $\overline{\partial}$ coupled to E is the signature of $(\mathbb{C}, i\phi)$, with $(H^1(X, \partial X; E) = \mathbb{C}, \phi)$ the skew-hermitian form over $\mathbb{C}$ defined by the cup-product and the hermitian form on E, and

Properties of the Maslov index $\tau(\theta_1, \theta_2, \theta_3)$ VIII.

xvi)

$$\begin{aligned}\sigma(\mathbb{C}, i\phi) &= 2(((\theta_1/2\pi)) + ((\theta_2/2\pi)) - (((\theta_1+\theta_2)/2\pi))) \\ &= \eta((\theta_1+\theta_2)/2) - \eta(\theta_1/2) - \eta(\theta_2/2) \\ &= -\tau(0, \theta_1/2, -\theta_2/2) \\ &= \begin{cases} -1 & \text{if } 0 < \{\theta_1/2\pi\} + \{\theta_2/2\pi\} < 1 \\ +1 & \text{if } 1 < \{\theta_1/2\pi\} + \{\theta_2/2\pi\} < 2 \\ 0 & \text{otherwise .}\end{cases}\end{aligned}$$

The discontinuous measurable function

$U(1) \times U(1) \to \mathbb{R}$;

$(e^{i\theta_1}, e^{i\theta_2}) \mapsto \sigma(\mathbb{C}, i\phi)/2 = ((\theta_1/2\pi)) + ((\theta_2/2\pi)) - (((\theta_1+\theta_2)/2\pi))$

is a bounded cocycle representing a generator of $H^2(U(1)) = \mathbb{Z}$, corresponding to the universal cover regarded as a central group extension

$$\mathbb{Z} \to \mathbb{R} \to U(1) = S^1 \ .$$