COHN LOCALIZATION, GENERALIZED FREE PRODUCTS AND BOUNDARY LINKS

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- Given a ring A and a set Σ of elements, matrices, morphisms, ..., it is possible to construct a new ring $\Sigma^{-1}A$, the Cohn localization of A inverting all the elements in Σ. In general, A and $\Sigma^{-1}A$ are noncommutative.

- The Cohn localization of triangular matrix rings gives a new construction of generalized free products G (= amalgamated free product $G_1 *_H G_2$ and HNN extension $G *_H \{z\}$) and a new way of relating modules, chain complexes and quadratic forms over $\mathbb{Z}[G]$ to the components. For the application to μ-component boundary links $G = F_\mu = \{z_1, z_2, \ldots, z_\mu\}$.

Ore localization

- The Ore localization of a ring A
$$\Sigma^{-1}A = (A \times \Sigma)/\sim$$
is defined for a multiplicatively closed subset $\Sigma \subset A$ of elements $s \in A$ satisfying:

- Ore condition for all $a \in A$, $s \in \Sigma$ there exists $b \in A$, $t \in \Sigma$ such that $at = sb \in A$ (e.g. central, $as = sa$ for all $a \in A$, $s \in \Sigma$)

- The Ore localization is the ring of fractions
$$\Sigma^{-1}A = (A \times \Sigma)/\sim$$
with $\dfrac{a}{s} \in \Sigma^{-1}A$ the equivalence class
$(a,s) \sim (b,t)$ iff $atu = bsu \in A$ for some $u \in \Sigma$

- $\Sigma^{-1}A$ is a flat A-module, with $H_*(\Sigma^{-1}C) = \Sigma^{-1}H_*(C)$ for any A-module chain complex C.

Cohn localization

- A = ring, Σ = a set of morphisms $s : P \to Q$ of f.g. projective A-modules.

- A ring morphism $A \to B$ is <u>Σ-inverting</u> if each $1 \otimes s : B \otimes_A P \to B \otimes_A Q$ $(s \in \Sigma)$ is a B-module isomorphism.

- The <u>Cohn localization</u> $\Sigma^{-1}A$ is a ring with a Σ-inverting morphism $A \to \Sigma^{-1}A$ such that any Σ-inverting morphism $A \to B$ has a unique factorization $A \to \Sigma^{-1}A \to B$.

- $\Sigma^{-1}A$ exists, but could be 0. $\Sigma^{-1}A$ need not be a flat A-module, $H_*(\Sigma^{-1}C) \neq \Sigma^{-1}H_*(C)$.

- An element $fs^{-1}g \in \Sigma^{-1}A$ is an equivalence class of generalized fractions, triples $(s : P \to Q, f : P \to A, g : A \to Q)$ with $s \in \Sigma$ (Malcolmson).

The lifting problem for chain complexes

- A lift of a f.g. free $\Sigma^{-1}A$-module chain complex C is a f.g. projective A-module chain complex B with $\Sigma^{-1}B \simeq C$.

- Every n-dimensional f.g. free $\Sigma^{-1}A$-module chain complex C can be lifted if $n \leqslant 2$, or if $\Sigma^{-1}A$ is an Ore localization.

- For $n \geqslant 3$ there are lifting obstructions in $\mathrm{Tor}_i^A(\Sigma^{-1}A, \Sigma^{-1}A)$ for $i \geqslant 1$.

- Chain complex lifting = algebraic analogue of transversality. e-print AT.0304362

Stable flatness

- Definition A localization $\Sigma^{-1}A$ of a ring A inverting a set Σ of morphisms of f.g. projective A-modules is stably flat if

$$\mathrm{Tor}_i^A(\Sigma^{-1}A, \Sigma^{-1}A) = 0 \ (i \geqslant 1) .$$

- For stably flat $\Sigma^{-1}A$ have stable exactness:

$$H_*(\Sigma^{-1}C) = \varinjlim_{D} \Sigma^{-1}H_*(D)$$

with $C \to D$ such that $\Sigma^{-1}C \simeq \Sigma^{-1}D$.

- (Neeman, R. and Schofield)
Examples of $\Sigma^{-1}A$ which are not stably flat, and $\Sigma^{-1}A$-module chain complexes which cannot be lifted.
Math. Proc. Camb. Phil. Soc. 2004, e-print RA.0205034

Theorem of Neeman + R.

If $A \to \Sigma^{-1}A$ is injective and stably flat then :

- have 'fibration sequence of exact categories'

  $$T(A,\Sigma) \to P(A) \to P(\Sigma^{-1}A)$$

  with $P(A)$ the category of f.g. projective A-modules and $T(A,\Sigma)$ the category of h.d. 1 Σ-torsion A-modules, and

- every finite f.g. free $\Sigma^{-1}A$-module chain complex can be lifted

- there are long exact sequences

  $$\cdots \to K_n(A) \to K_n(\Sigma^{-1}A) \to K_{n-1}(T(A,\Sigma)) \to K_{n-1}(A) \to \dots$$

  $$\cdots \to L_n(A) \to L_n(\Sigma^{-1}A) \to L_n(T(A,\Sigma)) \to L_{n-1}(A) \to \dots$$

  e-print RA.0109118

Group rings and Cohn localization

- Given a group G consider (commutative or Ore) localization of the integral group ring $\mathbb{Z}[G]$, e.g. $\mathbb{Q}[G] = (\mathbb{Z} - \{0\})^{-1}\mathbb{Z}[G]$. Localization is a "better" ring than $\mathbb{Z}[G]$, e.g. $\mathbb{Q}[G]$ is semisimple for finite G.

- The 'augmentation localization' $\Pi^{-1}\mathbb{Z}[F_\mu]$ inverts the set Π of square matrices in $\mathbb{Z}[F_\mu]$ which become invertible over $\mathbb{Z}$.

- If G is a generalized free product the matrix ring $M_k(\mathbb{Z}[G])$ for some $k \geqslant 1$ is a Cohn localization $\Pi^{-1}A$ of a $k \times k$ triangular matrix ring A. The localization map $A \to \Pi^{-1}A$ is an 'assembly' map. In the 'injective case' it is possible to describe the homological algebra of $\mathbb{Z}[G]$-modules and the algebraic K- and L-theory of $\mathbb{Z}[G]$ in terms of A and Π. In particular, this is the case for $G = F_\mu$ with $k = \mu + 1$.

Triangular matrix rings

Given rings A_1, A_2 and an (A_2, A_1)-bimodule B define the triangular matrix ring

$$A = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$$

with $P_1 = \begin{pmatrix} A_1 \\ B \end{pmatrix}$, $P_2 = \begin{pmatrix} 0 \\ A_2 \end{pmatrix}$ f.g. projective A-modules such that $A = P_1 \oplus P_2$.

<u>Proposition</u> (i) The category of A-modules is equivalent to the category of triples

$$M = (M_1, M_2, \mu : B \otimes_{A_1} M_1 \to M_2)$$

with M_1 an A_1-module, M_2 an A_2-module and μ an A_2-module morphism.
(ii) If $A \to C$ is a ring morphism such that there is a C-module isomorphism $C \otimes_A P_1 \cong C \otimes_A P_2$ then $C = M_2(D)$ with $D = \text{End}_C(C \otimes_A P_1)$,

$$\{A\text{-modules}\} \to \{C\text{-modules}\} \approx \{D\text{-modules}\};$$
$$M \mapsto (D\ D) \otimes_A M$$
$$= \text{coker}(D \otimes_{A_2} B \otimes_{A_1} M_1 \to D \otimes_{A_1} M_1 \oplus D \otimes_{A_2} M_2)$$

Generalized free products

- Theorem (Schofield, R.) Given group morphisms $H \to G_1$, $H \to G_2$ define

$$A = \begin{pmatrix} \mathbb{Z}[H] & 0 & 0 \\ \mathbb{Z}[G_1] & \mathbb{Z}[G_1] & 0 \\ \mathbb{Z}[G_2] & 0 & \mathbb{Z}[G_2] \end{pmatrix}$$

  and let $\Pi = \{P_2 \subset P_1, P_3 \subset P_1\}$ with P_i the ith column of A. Then

$$\Pi^{-1}A = M_3(\mathbb{Z}[G_1 *_H G_2]) \ .$$

  Stably flat for injective $H \to G_1$, $H \to G_2$.

- Similarly for HNN extensions.

- See survey article Noncommutative localization in topology, e-print AT.0303046, for the connection with the Bass-Serre theory of groups acting on trees, and the algebraic K- and L-theory splitting theorems of Waldhausen and Cappell.

The codimension 2 placement problem

- For a connected space X with universal cover $\widetilde{X}$ and a $\mathbb{Z}[\pi_1(X)]$-module A

$$H_*(X;A) \;:=\; H_*(A \otimes_{\mathbb{Z}[\pi_1(X)]} C(\widetilde{X}))$$

- Let $X = M \backslash N$ be the complement of a codimension 2 embedding $N^n \subset M^{n+2}$. By Alexander duality

$$H_*(X) \;=\; H^{n+2-*}(M,N) \quad (* \neq 0, n+2)$$

depends only on the homotopy class of the inclusion $N \subset M$. However, $H_*(\widetilde{X})$ depends on the knotting of $N \subset M$.

- The applications of Cohn localization to boundary links $(M,N) = (S^{n+2}, \bigcup_{\mu} S^n)$ are a joint project with Des Sheiham.

Boundary links

- An <u>$(n+2)$-dimensional μ-component boundary link</u> is a locally flat embedding $\bigcup_{\mu} S^n \subset S^{n+2}$ with a μ-component Seifert surface

  $$(M^{n+1}, \partial M) = (\bigcup_{i=1}^{\mu} M_i, S^n) \subset S^{n+2}$$

  The $\mathbb{Z}$-homology equivalence to the trivial link complement

  $$f : X = S^{n+2} \backslash (\bigcup_{\mu} S^n) \to Y = \bigvee_{\mu-1} S^{n+1} \vee \bigvee_{\mu} S^1$$

  induces a surjection $\pi_1(X) \to \pi_1(Y) = F_\mu$.

- Can construct a Seifert surface M by taking f to be transverse at $*_1 \cup \cdots \cup *_\mu \subset Y$ and setting $M_i = f^{-1}(*_i)$.

The augmentation localization

- The augmentation $\mathbb{Z}[F_\mu] \to \mathbb{Z}$ factors through the Cohn localization $\Sigma^{-1}\mathbb{Z}[F_\mu]$ inverting the set Σ of square matrices in $\mathbb{Z}[F_\mu]$ which augment to invertible matrices in $\mathbb{Z}$. Stably flat (Farber and Vogel, 1992)

- A finite f.g. free $\mathbb{Z}[F_\mu]$-module chain complex C is such that $H_*(\Sigma^{-1}\mathbb{Z}[F_\mu]\otimes_{\mathbb{Z}[F_\mu]}C) = 0$ if and only if $H_*(\mathbb{Z} \otimes_{\mathbb{Z}[F_\mu]} C) = 0$.

- The localization map $\mathbb{Z}[F_\mu] \to \Sigma^{-1}\mathbb{Z}[F_\mu]$ detects knotting of a boundary link $\bigcup_\mu S^n \subset S^{n+2}$, in the sense that

  $$H_*(X;\mathbb{Z}[F_\mu]) = H_*(\widetilde{X})\ ,\ H_*(X;\Sigma^{-1}\mathbb{Z}[F_\mu]) = 0$$

  for $* \neq 0, 1, n+1$, with X the boundary link complement and $\widetilde{X}$ the cover of X induced from the universal cover $\widetilde{Y}$ of Y.

Γ-groups

- Theorem (Cappell-Shaneson, 1980)
  For $n \geqslant 4$ the concordance group $C_n(F_\mu)$ of μ-component $(n+2)$-dimensional boundary links (with F_μ-structure) is the relative Γ-group

$$C_n(F_\mu) \;=\; \Gamma_{n+3}\left(\begin{array}{ccc} \mathbb{Z}[F_\mu] & \longrightarrow & \mathbb{Z}[F_\mu] \\ \big\downarrow & \Phi & \big\downarrow \\ \mathbb{Z}[F_\mu] & \longrightarrow & \mathbb{Z} \end{array}\right)$$

  in the exact sequence

$$\begin{aligned} \cdots \to L_{n+3}(\mathbb{Z}[F_\mu]) \to \Gamma_{n+3}(\mathbb{Z}[F_\mu] \to \mathbb{Z}) \\ \to \Gamma_{n+3}(\Phi) \to L_{n+2}(\mathbb{Z}[F_\mu]) \to \dots \ . \end{aligned}$$

- In particular, $C_{2q}(F_\mu) = 0$ for $q \geqslant 2$.

Seifert, Blanchfield, computation

- (Levine for $\mu = 1$ 1969, Ko, Mio, 1987) The expression of $C_{2q-1}(F_\mu)$ in terms of Seifert matrices.

- (Kearton for $\mu = 1$ 1973, Duval, 1986) The expression of $C_{2q-1}(F_\mu)$ for $q \geqslant 2$ in terms of Blanchfield forms.

- (Levine for $\mu = 1$ 1970, Sheiham, 2002) The computation of $C_{2q-1}(F_\mu)$ (infinitely generated) for $q \geqslant 2$, using Seifert forms.

The L-theory localization sequence

- **Theorem** (R., 2003) The Cappell-Shaneson exact sequence is the noncommutative L-theory localization exact sequence

$$\cdots \to L_{n+3}(\mathbb{Z}[F_\mu]) \to L_{n+3}(\Sigma^{-1}\mathbb{Z}[F_\mu]) \to L_{n+3}(T(\mathbb{Z}[F_\mu],\Sigma)) \to L_{n+2}(\mathbb{Z}[F_\mu]) \to \ldots$$

with $\Gamma_{n+3}(\Phi) = L_{n+3}(T(\mathbb{Z}[F_\mu],\Sigma))$ the cobordism group of $(n+2)$-dimensional $\mathbb{Z}$-contractible quadratic Poincaré complexes over $\mathbb{Z}[F_\mu]$. The F_μ-link concordance class of a boundary link $\bigcup_\mu S^n \subset S^{n+2}$ is the cobordism class of the complex $(\mathcal{C}(\widetilde{f})_{*+1},\psi)$ with $\widetilde{f}: C(\widetilde{X}) \to C(\widetilde{Y})$ the canonical $\mathbb{Z}$-coefficient chain equivalence.

- Can recover the middle dimensional Blanchfield-Duval form for $n = 2q-1$.

The Cayley localization (I)

- For $\mu \geqslant 1$ and any ring R let A be the $(\mu+1)\times(\mu+1)$ triangular matrix ring

$$A = \begin{pmatrix} R & R\oplus R & R\oplus R & \dots & R\oplus R \\ 0 & R & 0 & \dots & 0 \\ 0 & 0 & R & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & R \end{pmatrix}$$

- An A-module V consists of R-modules $V_0, V_1, \dots, V_\mu$ and R-module morphisms $g_{i,1}, g_{i,2} : V_i \to V_0$, labelled by Cayley graph of F_μ.

- Let $Q_0, Q_1, \dots, Q_\mu$ be the f.g. projective A-module defined by the columns of A, and

$$\Pi = \{\sigma_{i,j} : Q_i \to Q_0 | i = 1, 2, \dots, \mu, j = 1, 2\}$$

with $\sigma_{i,j}$ the projection of the jth factor.

The Cayley localization (II)

- Theorem (Schofield, R.) The Cohn localization of A inverting Π is
$$\Pi^{-1}A \;=\; M_{\mu+1}(R[F_\mu])$$
with the endomorphism ring of $\Pi^{-1}Q_0$ freely generated by the automorphisms $z_i = \sigma_{i,1}(\sigma_{i,2})^{-1}$.

- Example Let X be a manifold (e.g. boundary link exterior) with a map $f : X \to \bigvee_\mu S^1$ transverse at $*_1 \cup \cdots \cup *_2$. Let X_0 be obtained from X by cutting out neighbourhoods of $X_i = f^{-1}(*_i)$ $(i = 1, 2, \ldots, \mu)$. The construction of the induced F_μ-cover $\widetilde{X}$ from $X_0, X_1, \ldots, X_\mu$ and the Cayley graph gives a lifting of $\mathbb{Z}[F_\mu]$-module chain complex $C(\widetilde{X})$ to an A-module chain complex $D(\widetilde{X})$ such that $\Pi^{-1}D(\widetilde{X}) = C(\widetilde{X})$ (with $R = \mathbb{Z}$).

Blanchfield and Seifert modules (I)

- A Seifert module (V, s) is a f.g. projective R-module V together with an endomorphism $s : V \to V$ and a direct sum decomposition $V = V_1 \oplus V_2 \oplus \cdots \oplus V_\mu$.

- A Blanchfield (= F_μ-link) module B is a homological dimension 1 $R[F_\mu]$-module such that

  $$\bigoplus(1-z_i) : \bigoplus_{i=1}^{\mu} B \to B; (b_1, b_2, \ldots, b_\mu) \mapsto \sum_{i=1}^{\mu}(1-z_i)b_i$$

  is an R-module isomorphism, $F_\mu = \langle z_1, z_2, \ldots, z_\mu \rangle$.

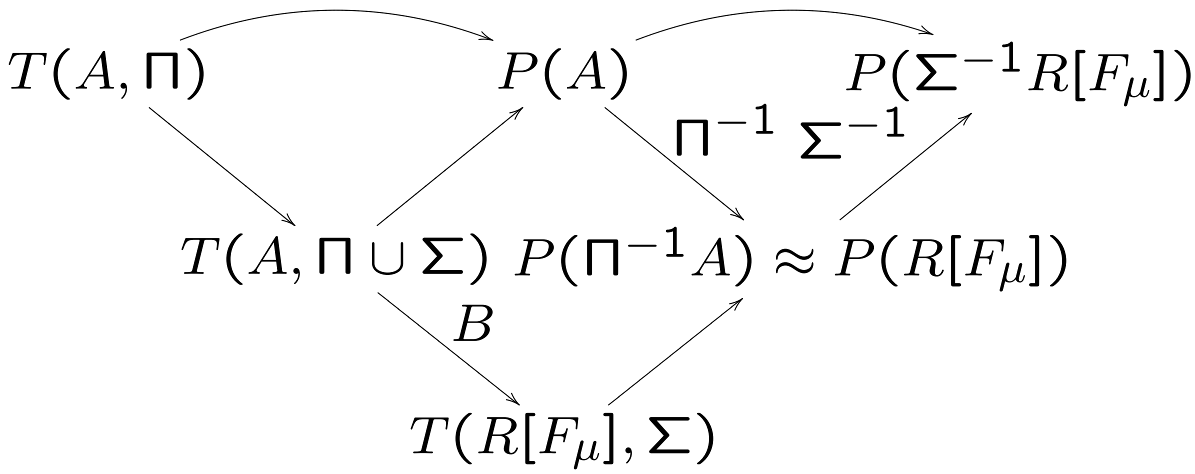
- The covering of a Seifert module (V, s) is the Blanchfield module

  $$B(V, s) = \text{coker}(1 - s + sz : V[F_\mu] \to V[F_\mu])$$

  with $z = \sum_{i=1}^{\mu} \pi_i z_i : V[F_\mu] \to V[F_\mu]$ and $\pi_i : V \to V_i \to V$.

Blanchfield and Seifert modules (II)

- For any ring with involution R there is defined a commutative braid of exact categories with chain duality and functors



  with $\Sigma^{-1}R[F_\mu]$ the augmentation Cohn localization, $\Pi^{-1}A$ the Cayley Cohn localization and

  $T(A, \Pi) = \{\text{Seifert modules } (V, s) \text{ with } B(V, s) = 0\}$,
  $T(A, \Pi \cup \Sigma) = \{\text{Seifert modules}\}$,
  $T(R[F_\mu], \Sigma) = \{\text{Blanchfield modules}\}$

- Theorem (R.+Sheiham) The braid induces a commutative braid of exact sequences in algebraic K and L-theory for $R = \mathbb{Z}$.