THE COBORDISM OF MANIFOLDS WITH BOUNDARY, AND ITS APPLICATIONS TO SINGULARITY THEORY

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The BNR project on singularities and surgery I.

- Since 2011 have joined András Némethi (Budapest) and Maciej Borodzik (Warsaw) in a project on the topological properties of the singularities of complex hypersurfaces.
- The aim of the project is to study the topological properties of the **singularity spectrum**, defined using refinements of the eigenvalues of the monodromy of the Milnor fibre.
- We have posted 3 preprints on the Arxiv this year:

BNR1 http://arxiv.org/abs/1207.3066 Morse theory for manifolds with boundary

BNR2 http://arxiv.org/abs/1211.5964 Codimension 2 embeddings, algebraic surgery and Seifert forms

BNR3 http://arxiv.org/abs/1210.0798 On the semicontinuity of the mod 2 spectrum of hypersurface singularities

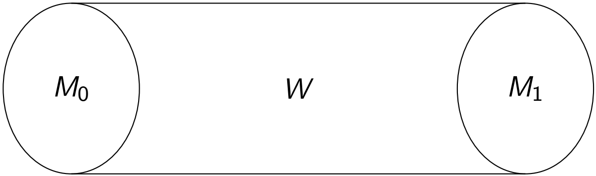
The BNR project on singularities and surgery II.

- The project combines singularity techniques with **algebraic surgery theory** to study the behaviour of the spectrum under deformations.
- Morse theory decomposes cobordisms of manifolds into elementary operations called surgeries.
- Algebraic surgery does the same for cobordisms of chain complexes with Poincaré duality – generalized quadratic forms.
- The applications to singularities need a Morse theory for the relative cobordisms of manifolds with boundary and their algebraic analogues.

Cobordism of closed manifolds

- Manifold = oriented differentiable manifold.
- An **(absolute)** $(m+1)$-**dimensional cobordism** $(W; M_0, M_1)$ consists of closed m-dimensional manifolds M_0, M_1 and an $(m+1)$-dimensional manifold W with boundary

$$\partial W \;=\; M_0 \sqcup -M_1\ .$$

- 

The cobordism of closed manifolds is nontrivial

- Cobordism is an equivalence relation.
- The equivalence classes constitute an abelian group Ω_m, with addition by disjoint union, and 0 the cobordism class of the empty manifold $\emptyset$.
- The cobordism groups Ω_m have been studied since the pioneering work of Thom in the 1950's.
- Low-dimensional examples:

$$\Omega_0 = \mathbb{Z} \, , \, \Omega_1 = \Omega_2 = \Omega_3 = 0 \, .$$

- The signature map

$$\sigma \, : \, \Omega_{4k} \to \mathbb{Z}$$

  is surjective for $k \geqslant 1$, and an isomorphism for $k = 1$, with

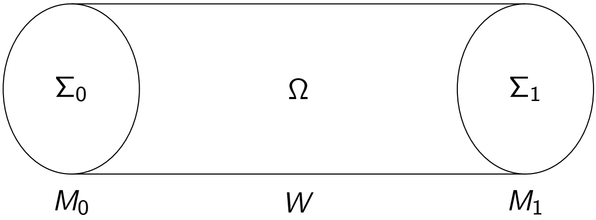
  $\sigma(M^{4k}) = \text{signature}(\text{intersection form } H^{2k}(M) \times H^{2k}(M) \to \mathbb{Z}) \in \mathbb{Z}\, .$

- The signature of a $4k$-dimensional manifold was first defined in 1923 by Hermann Weyl - in Spanish.

Cobordism of manifolds with boundary

- An $(m+2)$-**dimensional (relative) cobordism** $(\Omega;\Sigma_0,\Sigma_1,W;M_0,M_1)$ consists of $(m+1)$-dimensional manifolds with boundary (Σ_0,M_0), (Σ_1,M_1), an absolute cobordism $(W;M_0,M_1)$, and an $(m+2)$-dimensional manifold Ω with boundary

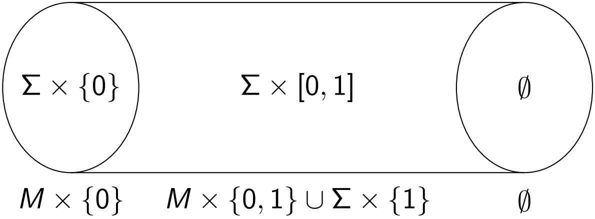
$$\partial\Omega \;=\; \Sigma_0 \cup_{M_0} W \cup_{M_1} -\Sigma_1 \ .$$

- 

The cobordism of manifolds with boundary is trivial

- **Proposition** Every manifold with boundary (Σ, M) is relatively cobordant to $(\emptyset, \emptyset)$ via the relative cobordism

$$\begin{aligned}&(\Omega; \Sigma_0, \Sigma_1, W; M_0, M_1)\\ =\ &(\Sigma \times [0,1]; \Sigma \times \{0\}, M \times [0,1] \cup \Sigma \times \{1\}; \emptyset, \emptyset)\end{aligned}$$

- 

- Relative cobordisms are interesting, all the same!

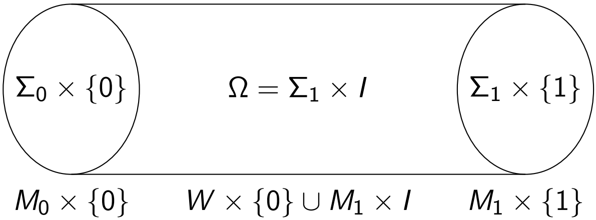
Right products

- A relative cobordism $(\Omega; \Sigma_0, \Sigma_1, W; M_0, M_1)$ is a **right product** if

$$
\begin{aligned}
&(\Omega; \Sigma_0, \Sigma_1, W; M_0, M_1) \\
&= (\Sigma_1 \times I; \Sigma_0 \times \{0\}, \Sigma_1 \times \{1\}, W \times \{0\} \cup M_1 \times I; \\
&\qquad\qquad M_0 \times \{0\}, M_1 \times \{1\})
\end{aligned}
$$

  with

$$\Sigma_1 = \Sigma_0 \cup_{M_0} W \ .$$

- 

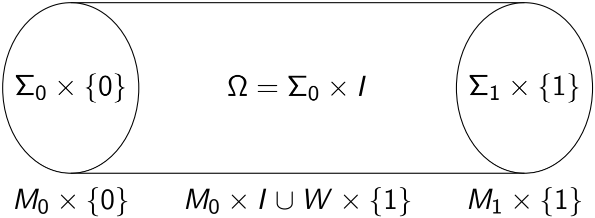
Left products

- A relative cobordism $(\Omega;\Sigma_0,\Sigma_1,W;M_0,M_1)$ is a **left product** if

$$
\begin{aligned}
&(\Omega;\Sigma_0,\Sigma_1,W;M_0,M_1)\\
&=\ (\Sigma_1\times I;\Sigma_0\times\{0\},\Sigma_1\times\{1\},W\times\{0\}\cup M_1\times I;\\
&\qquad\qquad\qquad\qquad\qquad M_0\times\{0\},M_1\times\{1\})
\end{aligned}
$$

  with

$$\Sigma_0\ =\ W\cup_{M_1}\Sigma_1\ .$$

- 

Geometric surgery

- Given an m-dimensional manifold M and an embedding

$$S^r \times D^{m-r} \subset M$$

  define the m-dimensional manifold obtained by an **index** $r+1$ **surgery**

$$M' \;=\; \mathrm{cl.}(M \backslash S^r \times D^{m-r}) \cup D^{r+1} \times S^{m-r-1} \;.$$

- The **trace** of the surgery is the $(m+1)$-dimensional cobordism $(W; M, M')$ obtained by attaching an **index** $(r+1)$ **handle** to $M \times I$

$$W \;=\; M \times I \cup_{S^r \times D^{m-r} \times \{1\}} D^{r+1} \times D^{m-r} \;.$$

- M is obtained from M' by surgery on $D^{r+1} \times S^{m-r-1} \subset M'$ of index $m-r$.

The handlebody decomposition theorem

- **Theorem** (Thom, Milnor 1961) Every absolute cobordism $(W; M, M')$ of closed m-dimensional manifolds has a handle decomposition, i.e. can be expressed as a union

$$(W; M, M') \;=\; \bigcup_{j=0}^{k}(W_j; M_j, M_{j+1}) \; (M_0 = M, M_{k+1} = M')$$

  of traces $(W_j; M_j, M_{j+1})$ of surgeries of non-decreasing index.
- Proved by Morse theory: there exists a Morse function

$$f \;:\; (W; M, M') \to (I; \{0\}, \{1\})$$

  with critical values in the gaps between
  $c_0 = 0 < c_1 < c_2 < \cdots < c_k < c_{k+1} = 1$ and

$$(W_j; M_j, M_{j+1}) \;=\; f^{-1}([c_j, c_{j+1}]; \{c_j\}, \{c_{j+1}\}) \;.$$

Half-surgeries

- Given an $(m+1)$-dimensional manifold with boundary (Σ_0, M_0) and an embedding $S^r \times D^{m-r} \subset M_0$ define the $(m+1)$-dimensional manifold with boundary obtained by an **index $r+1$ right half-surgery**

$$\begin{aligned}(\Sigma_1, M_1) \ &= \ (\Sigma_0 \cup_{S^r \times D^{m-r}} D^{r+1} \times D^{m-r}, \\ &\qquad \mathrm{cl.}(M_0 \backslash S^r \times D^{m-r}) \cup D^{r+1} \times S^{m-r-1}) \ .\end{aligned}$$

- Note that M_1 is the output of an index $r+1$ surgery on $S^r \times D^{m-r} \subset M_0$, and M_0 is the output of an index $m-r$ surgery on $D^{r+1} \times S^{m-r-1} \subset M_1$.
- There is an opposite notion of a **left half-surgery**, with input

$$(D^{r+1} \times D^{m-r}, D^{r+1} \times S^{m-r-1}) \subset (\Sigma_1, M_1)$$

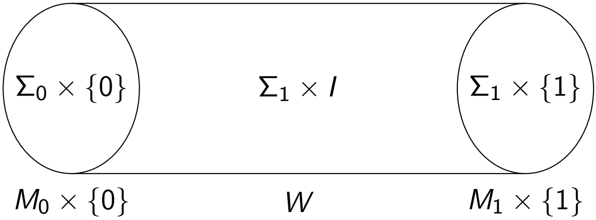
and output (Σ_0, M_0).

Half-handles

- The **trace** of the right half-surgery is the right product cobordism

$$(\Sigma_1 \times I; \Sigma_0 \times \{0\}, \Sigma_1 \times \{1\}, W; M_0, M_1)$$

with $W = M_0 \times I \cup D^{r+1} \times D^{m-r}$ the trace of the surgery on $S^r \times D^{m-r} \subset M_0$. (Σ_1, M_1) obtained from (Σ_0, M_0) by attaching an **index** $r+1$ **half-handle**.



The half-handlebody decomposition theorem

- **Theorem 1** (BNR1, 4.18) Every relative cobordism $(\Omega;\Sigma_0,\Sigma_1,W;M_0,M_1)$ consisting of non-empty connected manifolds is a union of right and left product cobordisms, namely the traces of right and left half-surgeries.
- Theorem 1 is proved by a relative version of the Morse theory proof of the Thom-Milnor handlebody decomposition theorem. Quite hard analysis!
- Theorem 1 has an algebraic analogue, for the relative cobordism of algebraic Poincaré pairs.
  Statement and proof in BNR2.

Fibred links

- A **link** is a codimension 2 submanifold $L^m \subset S^{m+2}$ with neighbourhood $L \times D^2 \subset S^{m+2}$.
- The **complement** of the link is the $(m+2)$-dimensional manifold with boundary

$$(C, \partial C) \;=\; (\mathrm{cl.}(S^{m+2} \backslash L \times D^2), L \times S^1)$$

such that

$$S^{m+2} \;=\; L \times D^2 \cup_{L \times S^1} C \;.$$

- The link is **fibred** if the projection $\partial C = L \times S^1 \to S^1$ can be extended to the projection of a fibre bundle $p : C \to S^1$, and there is given a particular choice of extension.
- The **monodromy** automorphism $(h, \partial h) : (F, \partial F) \to (F, \partial F)$ of a fibred link has $\partial h = \mathrm{id.} : \partial F = L \to L$ and

$$C \;=\; T(h) \;=\; F \times [0,1]/\{(y,0) \sim (h(y),1) \,|\, y \in F\} \;.$$

Every link has Seifert surfaces

- A **Seifert surface** for a link $L^m \subset S^{m+2}$ is a codimension 1 submanifold $F^{m+1} \subset S^{m+2}$ such that

$$\partial F \;=\; L \subset S^{m+2}$$

  with a trivial normal bundle $F \times D^1 \subset S^{m+2}$.
- Fact: every link $L \subset S^{m+2}$ admits a Seifert surface F.
  Proof: extend the projection $\partial C = L \times S^1 \to S^1$ to a map

$$p \;:\; C \;=\; \mathrm{cl.}(S^{m+2} \backslash L \times D^2) \to S^1$$

  representing $(1, 1, \ldots, 1) \in H^1(C) = \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \mathbb{Z}$ (one $\mathbb{Z}$ for each component of L) and let $F = p^{-1}(*) \subset S^{m+2}$ be the transverse inverse image of $* \in S^1$.
- In general, Seifert surfaces are not canonical. A fibred link has a canonical Seifert surface, namely the fibre F.

The link of a singularity

- Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be the germ of an analytic function such that the complex hypersurface

$$X \; = \; f^{-1}(0) \subset \mathbb{C}^{n+1}$$

has an isolated singularity at $x \in X$, with

$$\frac{\partial f}{\partial z_k}(x) \; = \; 0 \text{ for } k = 1, 2, \ldots, n+1 \; .$$

- For $\epsilon > 0$ let

$$\begin{aligned} D_\epsilon(x) \; &= \; \{y \in \mathbb{C}^{n+1} \,|\, \|y - x\| \leqslant \epsilon\} \; \cong \; D^{2n+2} \; , \\ S_\epsilon(x) \; &= \; \{y \in \mathbb{C}^{n+1} \,|\, \|y - x\| = \epsilon\} \; \cong \; S^{2n+1} \; . \end{aligned}$$

- For $\epsilon > 0$ sufficiently small, the subset

$$L(x)^{2n-1} \; = \; X \cap S_\epsilon(x) \subset S_\epsilon(x)^{2n+1}$$

is a closed $(2n-1)$-dimensional submanifold, the **link of the singularity** of f at x.

The link of singularity is fibred

- **Proposition** (Milnor, 1968) The link of an isolated hypersurface singularity is fibred.
- The complement $C(x)$ of $L(x) \subset S_\epsilon(x)^{2n+1}$ is such that

$$p \ : \ C(x) \to S^1 \ ; \ y \mapsto \frac{f(y)}{|f(y)|}$$

  is the projection of a fibre bundle.
- The **Milnor fibre** is a canonical Seifert surface

$$(F(x), \partial F(x)) \ = \ (p, \partial p)^{-1}(*) \subset (C(x), \partial C(x))$$

  with

$$\partial F(x) \ = \ L(x) \subset S(x)^{2n+1} \ .$$

- The fibre $F(x)$ is $(n-1)$-connected, and

$$F(x) \ \simeq \ \bigvee_{\mu} S^n \ , \ H_n(F(x)) \ = \ \mathbb{Z}^\mu$$

  with $\mu = b_n(F(x)) \geqslant 0$ the **Milnor number**.

The intersection form

- Let $(F, \partial F)$ be a $2n$-dimensional manifold with boundary, such as a Seifert surface. Denote $H_n(F)$/torsion by $H_n(F)$.
- The **intersection form** is the $(-1)^n$-symmetric bilinear pairing

$$b \ : \ H_n(F) \times H_n(F) \to \mathbb{Z} \ ; \ (y, z) \mapsto \langle y^* \cup z^*, [F] \rangle$$

  with $y^*, z^* \in H^n(F, \partial F)$ the Poincaré-Lefschetz duals of $y, z \in H_n(F)$ and $[F] \in H_{2n}(F, \partial F)$ the fundamental class.
- The intersection pairing is $(-1)^n$-symmetric

$$b(y, z) \ = \ (-1)^n b(z, y) \in \mathbb{Z} \ .$$

- The adjoint $\mathbb{Z}$-module morphism

$$b = (-1)^n b^* : H_n(F) \to H_n(F)^* = \mathrm{Hom}_{\mathbb{Z}}(H_n(F), \mathbb{Z}) \ ;$$
$$y \mapsto (z \mapsto b(y, z)) \ .$$

  is an isomorphism if ∂F and F have the same number of components.

The monodromy theorem

- The monodromy induces an automorphism of the intersection form

$$h_* \ : \ (H_n(F), b) \to (H_n(F), b) \ ,$$

or equivalently $h^* : (H^n(F), b^{-1}) \to (H^n(F), b^{-1})$.

- **Monodromy theorem** (Brieskorn, 1970)
For the fibred link $L \subset S^{2n+1}$ of a singularity the $\mu = b_n(F)$ eigenvalues of the monodromy automorphism

$$h^* : H^n(F; \mathbb{C}) = \mathbb{C}^\mu \to H^n(F; \mathbb{C}) = \mathbb{C}^\mu$$

are roots of 1

$$\lambda_k \ = \ e^{2\pi i \alpha_k} \in S^1 \subset \mathbb{C} \ (1 \leqslant k \leqslant \mu)$$

for some $\{\alpha_1, \alpha_2, \ldots, \alpha_\mu\} \in \mathbb{Q}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z}$. Furthermore, h^* is such that for some $N \geqslant 1$

$$((h^*)^N - \mathrm{id.})^{n+1} \ = \ 0 \ : \ H^n(F; \mathbb{C}) \to H^n(F; \mathbb{C}) \ .$$

The spectrum of a singularity

- Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ have an isolated singularity at $x \in f^{-1}(0)$, with Milnor fibre $F^{2n} = F(x)$ and Milnor number $\mu = b_n(F)$.
- Steenbrink (1976) used analysis to construct a mixed Hodge structure on $H^n(F;\mathbb{C})$, with both a Hodge and a weight filtration. Invariant under h^* and polarized by b. Each $\alpha_k \in \mathbb{Q}/\mathbb{Z}$ has a lift to $\widetilde{\alpha}_k \in \mathbb{Q}$.
- The **spectrum** of f at x is

$$\mathrm{Sp}(f) \;=\; \sum_{k=1}^{\mu} \widetilde{\alpha}_k \in \mathbb{N}[\mathbb{Q}]$$

- **Arnold semicontinuity conjecture** (1981)
  The spectrum is semicontinuous: if (f, x) is adjacent to (f', x') with $\mu' < \mu$ then $\widetilde{\alpha}_k \leqslant \widetilde{\alpha}'_k$ for $k = 1, 2, \ldots, \mu'$.
- Varchenko (1983) and Steenbrink (1985) proved the conjecture using Hodge theoretic methods.

The mod 2 spectrum

- The real Seifert form and the spectral pairs of isolated hypersurface singularities (Némethi, Comp. Math. 1995) Introduced the **mod 2 spectrum** of f at an isolated hypersurface singularity

$$\mathrm{Sp}_2(f) \;=\; \sum_{k=1}^{\mu} \widetilde{\alpha}_k \in \mathbb{N}[\mathbb{Q}/2\mathbb{Z}]$$

  and related it to the real Seifert form.
- The spectrum is an analytic invariant, and the semicontinuity is analytic. How much of it is purely topological?

The BNR programme

- Borodzik+Némethi The spectrum of plane curves via knot theory (Journal LMS, 2012) applied the cobordism theory of links, **Murasugi-type inequalities** for the **Tristram-Levine signatures** to give a topological proof of the semicontinuity of the mod 2 spectrum of the links of isolated singularities of $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$.
- Ranicki High-dimensional knot theory (Springer, 1998) Algebraic surgery in codimension 2.
- BNR1+BNR2+BNR3 (2012) use relative Morse theory and algebraic surgery to prove more general Murasugi-type inequalities, giving a topological proof for semicontinuity of the mod 2 spectrum of the links of isolated singularities of $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ for all $n \geqslant 1$.

Seifert forms

- For any link $L^{2n-1} \subset S^{2n+1}$ and Seifert surface $F^{2n} \subset S^{2n+1}$ the intersection form has a **Seifert form** refinement

$$S \ : \ H_n(F) \times H_n(F) \to \mathbb{Z}$$

such that

$$b(y, z) \ = \ S(y, z) + (-1)^n S(z, y) \in \mathbb{Z} \ .$$

- Seifert (for $n = 1$, 1934) and Kervaire (for $n \geqslant 2$, 1965) defined S geometrically using the linking of n-cycles in $L, L' \subset S^{2n+1}$, with L' a copy of L pushed away.
- In terms of adjoints

$$b \ = \ S + (-1)^n S^* \ : \ H_n(F) \to H^n(F) \ = \ H_n(F)^* \ .$$

The variation map of a fibred link

- The **variation map** of a fibred link $L^{2n-1} \subset S^{2n+1}$ is an isomorphism
$$V \ : \ H_n(F, \partial F) \to H_n(F)$$
satisfying the **Picard-Lefschetz** relation
$$h - \mathrm{id.} \ = \ V \circ b \ : \ H_n(F) \to H_n(F) \ .$$

- The Seifert form of a fibred link $L^{2n-1} \subset S^{2n+1}$ with respect to the fibre Seifert surface $F^{2n} \subset S^{2n+1}$ is an isomorphism
$$S \ = \ V^{-1} \circ b \ : \ H_n(F) \to H^n(F) \ \cong \ H_n(F)^* \ .$$

The cobordism of links

- A **cobordism of links** is a codimension 2 submanifold

$$(K^{2n}; L_0, L_1) \subset S^{2n+1} \times ([0,1]; \{0\}, \{1\})$$

with trivial normal bundle $K \times D^2 \subset S^{2n+1} \times [0,1]$.

- An *h*-**cobordism** of links is a cobordism such that the inclusions $L_0, L_1 \subset K$ are homotopy equivalences, e.g. if

$$(K; L_0, L_1) \ \cong \ L_0 \times ([0,1]; \{0\}, \{1\}) \ .$$

- The *h*-cobordism theory of knots was initiated by Milnor (with Fox) in the 1950's. In the last 50 years the *h*-cobordism theory of knots and links has been much studied by topologists, both for its own sake and for the applications to singularity theory.

The cobordism of links of singularities I.

- Suppose that $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ has only isolated singularities $x_1, x_2, \dots, x_k \in X = f^{-1}(0)$ with $\|x_j\| < 1$. Let $B_j \subset D^{2n+2}$ be small balls around the x_j's, with links

$$L(x_j) \;=\; X \cap \partial B_j \subset \partial B_j \;\cong\; S^{2n+1} \;.$$

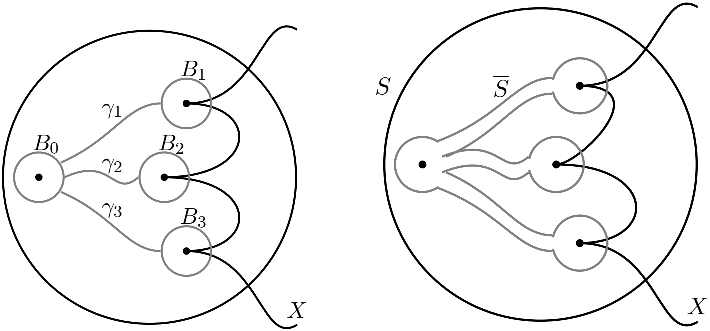
- Assume that $S = S^{2n+1}$ is transverse to X, with $L = X \cap S \subset S$ the **link at infinity**.
- Choose disjoint ball $B_0 \subset B$, and paths γ_j inside D^{2n+2} from ∂B_0 to ∂B_j, with neighbourhoods U_j. The union

$$U \;=\; B_0 \cup \bigcup_{j=1}^{k} (B_j \cup U_j)$$

is diffeomorphic to D^{2n+2}. Will construct cobordism between the links

$$L \;,\; \overline{L} \;=\; \coprod_{j=1}^{k} L(x_j) \subset \partial U = \overline{S} \cong S^{2n+1} \;.$$

The cobordism of links of singularities II.The boleadoras trick



The cobordism of links of singularities III.

- The $2n$-dimensional submanifold

$$\begin{aligned} K^{2n} &= X \cap \mathrm{cl.}(D^{2n+2} \backslash \bigcup_{j=1}^{k} B_j) \\ &\subset \mathrm{cl.}(D^{2n+2} \backslash U) \cong S^{2n+1} \times [0,1] \end{aligned}$$

defines a cobordism of links

$$(K; L, \overline{L}) \subset S^{2n+1} \times ([0,1]; \{0\}, \{1\}) .$$

- The Milnor fibres $F, \overline{F}$ for the links $L, \overline{L}$ are such that

$$F \cup_L K \cup_{\overline{L}} \overline{F} \cong F \cup_L X'$$

with $X' \subset D^{2n}$ the smoothing of X inside D^{2n+2} such that $X' \cap B_j = F(x_j)$ is a push-in of the Milnor fibre of $L(x_j)$, and $\overline{F} = F(x_1) \cup \cdots \cup F(x_k)$.
- $(K; L, \overline{L})$ is not an h-cobordism of links in general.

The Tristram-Levine signatures $\sigma_\xi(F)$

- **Definition** (1969) The **Tristram-Levine signatures** of a link $L^{2n-1} \subset S^{2n+1}$ with respect to a Seifert surface F and $\xi \in S^1$

  $$\sigma_\xi(F) \ = \ \text{signature}(H_n(F;\mathbb{C}), (1-\xi)S+(-1)^{n+1}(1-\bar{\xi})S^*) \in \mathbb{Z}\,.$$

- The $(-1)^{n+1}$-hermitian form related to the complement cl.$(D^{2n+2} \backslash F' \times D^2)$ of push-in $F' \subset D^{2n+2}$.
- Tristram and Levine studied how $\sigma_\xi(F)$ behave under
  1. change of Seifert surface,
  2. change of ξ,
  3. the h-cobordism of links.
- **Theorem** (Levine, 1970) For $n > 1$ the signatures $\sigma_\xi(F) \in \mathbb{Z}$ determine the h-cobordism class of a knot $S^{2n-1} \subset S^{2n+1}$ modulo torsion.
- For the BNR project need to also consider how $\sigma_\xi(F)$ behaves under
  4. the cobordism of links.

The relation between $\mathbf{Sp}_2(f)$ and $\sigma_\xi(F(x))$

- Borodzik+Némethi Hodge-type structures as link invariants (2012, Ann. Inst. Fourier).
- Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ have isolated singularity at $x \in f^{-1}(0)$ with link $L(x) \subset S^{2n+1}$ and the mod 2 spectrum $\mathrm{Sp}_2(f)$, where $|\mathrm{Sp}_2(f)| = \mu = b_n(F(x))$.
- If $\alpha \in [0,1)$ is such that $\xi = e^{2\pi i\alpha}$ is not an eigenvalue of the monodromy

$$h^* \ : \ H^n(F(x);\mathbb{C}) = \mathbb{C}^\mu \to H^n(F(x);\mathbb{C}) = \mathbb{C}^\mu$$

then

$$\begin{aligned} |\mathrm{Sp}_2(f) \cap (\alpha, \alpha+1)| \ &= \ \big(\, \mu - \sigma_\xi(F(x))\,\big)/2 \ , \\ |\mathrm{Sp}_2(f) \backslash (\alpha, \alpha+1)| \ &= \ \big(\, \mu + \sigma_\xi(F(x))\,\big)/2 \ . \end{aligned}$$

The relative cobordism of Seifert forms

- For every cobordism of links
$$(K^{m+1}; L_0, L_1) \subset S^{m+2} \times ([0,1]; \{0\}, \{1\})$$
there exists a relative cobordism of the Seifert surfaces
$$(E^{m+2}; F_0, F_1; K; L_0, L_1) \subset S^{m+2} \times ([0,1]; \{0\}, \{1\}) \ .$$
- **Definition** An **enlargement** of a Seifert form (H, S) is a Seifert form of the type
$$(H', S') \ = \ (H \oplus A \oplus B, \begin{pmatrix} S & 0 & T \\ 0 & 0 & U \\ V & W & X \end{pmatrix})$$
- **Theorem 2** (BNR2) If $m = 2n - 1$ the Seifert form $(H_n(F_1), S_1)$ is obtained from the Seifert form $(H_n(F_0), S_0)$ by a sequence of enlargements and their formal inverses.
- Proved by Levine (1970) for h-cobordisms of knots $S^{2n-1} \subset S^{2n+1}$, with $S + (-)^n S^*$ and $U + (-)^n W^*$ invertible.

The behaviour of the Tristram-Levine signatures under relative cobordism

- Conventional surgery and Morse theory used to describe the behaviour of the signature under cobordism.
- The BNR project required the further development of surgery and Morse theory for manifolds with boundary, in order to describe the behaviour of the Tristram-Levine signatures under the relative cobordism of Seifert surfaces of links.

The Murasugi-type inequality

- **Theorem 3** (BNR2, BNR3) Suppose given a cobordism of $(2n-1)$-dimensional links

$$(K; L_0, L_1) \subset S^{2n+1} \times ([0,1]; \{0\}, \{1\})$$

and Seifert surfaces $F_0, F_1 \subset S^{2n+1}$ for $L_0, L_1 \subset S^{2n+1}$. Then for any $\xi \neq 1 \in S^1$

$$\begin{aligned} & |\sigma_\xi(L_0) - \sigma_\xi(L_1)| \\ & \leqslant b_n(F_0 \cup_{L_0} K \cup_{L_1} F_1) - b_n(F_0) - b_n(F_1) + n_0(\xi) + n_1(\xi) \end{aligned}$$

with b_n the nth Betti number and

$$n_j(\xi) \ = \ \text{nullity}((1-\xi)S_j + (-1)^{n+1}(1-\bar{\xi})S_j^*) \ (j = 0, 1) \ .$$

- Proved by applying Theorem 1 to express the relative cobordism as a union of elementary right and left product cobordisms, and working out the effect on σ_ξ.

The semicontinuity of the mod 2 spectrum

- **Theorem 4** (BNR3) Let $f_t : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ ($t \in \mathbb{C}$) be a family of germs of analytic maps such that $x_0 \in (f_0)^{-1}(0)$ is an isolated singularity. For a small $\epsilon > 0$, $\|t\| > 0$ let $x_1, x_2, \ldots, x_k \in (f_t)^{-1}(0) \cap B_\epsilon(0)$ be all the singularities of f_t in $B_\epsilon(0)$. Let $\alpha \in [0,1]$ be such that $\xi = e^{2\pi i \alpha}$ is not an eigenvalue of the monodromy h_0 of x_0. Then

$$|\mathrm{Sp}_{2,0}(f_0) \cap (\alpha, \alpha+1)| \geqslant \sum_{j=1}^{k} |\mathrm{Sp}_{2,j}(f_t) \cap (\alpha, \alpha+1)| \ ,$$

$$|\mathrm{Sp}_{2,0}(f_0) \backslash [\alpha, \alpha+1]| \geqslant \sum_{j=1}^{k} |\mathrm{Sp}_{2,j}(f_t) \backslash [\alpha, \alpha+1]|$$

  where $\mathrm{Sp}_{2,0}(f_0)$, $\mathrm{Sp}_{2,j}(f_t)$ are the mod 2 spectra of x_0, x_j.
- Proved topologically using Theorems 2 and 3. Apply the Murasugi-type inequality to the singularity construction of the relative cobordism of Seifert surfaces between $F(x_0)$ and $\overline{F} = \coprod_{j=1}^{k} F(x_j)$.