# On the Cappell UNil groups

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## Manifold transversality

- Let $X$ be a manifold and let $Y \subset X$ a codimension $k$ submanifold, with a normal $k$-plane bundle

$$\nu_{Y \subset X} \;:\; Y \to BO(k) \;.$$

- **Transversality theorem** Every map $f : M \to X$ from a manifold is homotopic to a map which is transverse regular at $Y \subset X$, so that

$$N \;=\; f^{-1}(Y) \subset M$$

  is a codimension $k$ submanifold with normal bundle

$$\nu_{N \subset M} \;:\; N \xrightarrow{g = f|} Y \xrightarrow{\nu_{Y \subset X}} BO(k) \;.$$

## Splitting homotopy equivalences

- **In general, the restrictions of homotopy equivalences are not homotopy equivalences.**
- **Definition** A homotopy equivalence $f : M \to X$ of $m$-dimensional manifolds **splits** at a submanifold $Y \subset X$ if $f$ is homotopic to a map (also denoted by $f$) with the restrictions

$$g = f| : N = f^{-1}(Y) \to Y \ , \ h = f| : M \backslash N \to X \backslash Y$$

  also homotopy equivalences.
- **Example** If $f$ is homotopic to a diffeomorphism then $f$ splits at every submanifold $Y \subset X$.
- **Borel rigidity conjecture** If $X$ is aspherical then every homotopy equivalence is homotopic to a homeomorphism, and splits at every submanifold $Y \subset X$. Verified in many cases, starting with $m = 2$.
- **Contrapositive** If a homotopy equivalence $f$ does not split at a submanifold $Y \subset X$ then $f$ is not homotopic to a diffeomorphism.

## Surgery obstruction theory

- The Browder-Novikov-Sullivan-Wall surgery theory was developed in the 1960's to study the homotopy types of manifolds. It builds on the *h*- and *s*-cobordism theorems, and like them only works for dimensions $\geqslant 5$.
- **Main theorem of surgery** (1970) An $n$-dimensional normal map $g : N \to Y$ has a **surgery obstruction**

$$\sigma_*(g) \in L_n(\mathbb{Z}[\pi_1(Y)])$$

such that $\sigma_*(g) = 0$ if (and for $n \geqslant 5$ only if) $g$ is normal bordant to a homotopy equivalence rel $\partial N$. Moreover, every element $x \in L_n(\mathbb{Z}[\pi])$ is such a rel $\partial N$ surgery obstruction for some $g : N \to Y$ with $\pi_1(Y) = \pi$.

## Three ways of defining the Wall groups $L_*$

- 1. **Geometry**. For a finitely presented group $\pi$ $L_n(\mathbb{Z}[\pi])$ is a bordism group of normal maps $g : N \to Y$ of $n$-dimensional manifolds with boundary with $\pi_1(Y) = \pi$ and $\partial g : \partial N \to \partial Y$ a homotopy equivalence.
- 2. **Forms**. For a ring with involution $A$ $L_{2i}(A)$ is the Witt group of $(-)^i$-quadratic forms on f.g. free $A$-modules. $L_{2i+1}(A)$ the stable automorphism group of such forms, or equivalently formations. 4-periodic: $L_* = L_{*+4}$.
- 3. **Chain complexes**. $L_n(A)$ is the cobordism group of chain complexes $C$ of f.g. free $A$-modules with an $n$-dimensional quadratic Poincaré structure.
- Definition 2 = Definition 3 always.
- Definition 1 = Definition 2 for $n \geqslant 5$, with $A = \mathbb{Z}[\pi]$.

## The codimension $k$ splitting obstruction groups

- For a morphism of rings with involution $A \to B$ there are **relative $L$-groups** $L_*(A \to B)$, defined to fit into an exact sequence

$$\cdots \to L_n(A) \to L_n(B) \to L_n(A \to B) \to L_{n-1}(A) \to \ldots .$$

- **Definition** (Wall, 1970) The **codimension $k$ splitting obstruction groups** $LS_*(X,Y)$ are defined geometrically for any manifold $X$ and codimension $k$ submanifold $Y \subset X$, to fit into the exact sequence

$$\begin{aligned}\cdots &\to L_{n+k+1}(\mathbb{Z}[\pi_1(X \backslash Y)] \to \mathbb{Z}[\pi_1(X)]) \to LS_n(X,Y)\\ &\to L_n(\mathbb{Z}[\pi_1(Y)]) \to L_{n+k}(\mathbb{Z}[\pi_1(X \backslash Y)] \to \mathbb{Z}[\pi_1(X)]) \to \ldots\end{aligned}$$

- Can also be defined algebraically (R.) but using horribly big objects.

## The codimension $k$ splitting obstruction theorem

- **Theorem** (Wall, 1970) Given a homotopy equivalence $f : M \to X$ of $m$-dimensional manifolds and a codimension $k$ submanifold $Y \subset X$ there is defined a **splitting obstruction**
$$s(f) \in LS_{m-k}(X, Y)$$
such that $f$ splits at $Y \subset X$ if (and for $m - k \geqslant 5$ only if) $s(f) = 0$.
- $s(f)$ has image $\sigma(g) \in L_{m-k}(\mathbb{Z}[\pi_1(Y)])$, the surgery obstruction of the transverse normal map $g = f| : N = f^{-1}(Y) \to Y$.
- For $k \geqslant 3$ the splitting obstruction is just the surgery obstruction
$$\begin{aligned} \pi_1(X \backslash Y) \ &= \ \pi_1(X) \ , \\ s(f) \ &= \ \sigma(g) \in LS_{m-k}(X, Y) \ = \ L_{m-k}(\mathbb{Z}[\pi_1(Y)]) \ . \end{aligned}$$
- The $LS$-groups for $k = 1, 2$ differ from the $L$-groups, and are harder to compute!

## Codimension 1 with trivial $\nu$

- If $(X, Y)$ is a codimension 1 pair with $\nu_{Y\subset X} : Y \to BO(1)$ trivial and $X, Y$ connected there are two cases, depending on how $Y$ separates $X$.
- **Case (A)** $X\backslash Y$ is disconnected, in which case there are 2 components $X_1, X_2$ and

$$\pi_1(X) \ = \ \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2)$$

  is an amalgamated free product.
- **Case (B)** $X\backslash Y$ is connected, so that

$$\pi_1(X) \ = \ \pi_1(X\backslash Y) *_{\pi_1(Y)} \{t\}$$

  is an *HNN* extension.
- The codimension 1 splitting properties of 3-manifolds $X$ at surfaces $Y \subset X$ with $\pi_1(Y) \to \pi_1(X)$ injective and $\nu$ trivial were much studied by Grushko, Stallings, Haken, Waldhausen, ...


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THE UGLY
THE GOOD
AND THE BAD

## The UNil-groups: the good, the bad and the ugly I.

- In **Splitting obstructions of hermitian forms and manifolds with $\mathbb{Z}_2 \subset \pi_1$** (Bull. A.M.S. 79, 1973) Cappell used the algebraic and geometric properties of the Wall $L$-groups to construct homotopy equivalences
$$f \ : \ M \to X = \mathbb{R}\mathbb{P}^{4k+1} \# \mathbb{R}\mathbb{P}^{4k+1} \ (k \geqslant 1)$$
which do not split at $Y = S^{4k} \subset X$.
- In **Manifolds with fundamental group a generalized free product** and **Unitary nilpotent groups and hermitian $K$-theory I.** (Bull. A.M.S. 80, 1974) Cappell defined $\mathrm{UNil}_{2*}$ algebraically using forms, by hermitian analogy with the Waldhausen Nil-groups. Stated many vanishing results. The non-splitting examples interpreted as non-trivial UNil elements. $\mathrm{UNil}_{2*+1}$ defined geometrically, using the geometric Shaneson splitting
$$L^s_*(\mathbb{Z}[\pi \times \mathbb{Z}]) \ = \ L^s_*(\mathbb{Z}[\pi]) \oplus L_{*-1}(\mathbb{Z}[\pi]) \ .$$

**The UNil-groups: the good, the bad and the ugly II.**

- **Theorem** (C., 1974) The codimension 1 splitting groups $LS_*(X,Y)$ in the case of trivial $\nu_{Y\subset X}: Y \to BO(1)$ and injective $\pi_1(Y)\to\pi_1(X)$ decompose as

$$LS_{m-1}(X,Y) \;=\; \widehat{H}^m(\mathbb{Z}_2;I)\oplus \mathrm{UNil}_{m+1}$$

with $I=\ker(\widetilde{K}_0(\mathbb{Z}[\pi_1(Y)])\to\widetilde{K}_0(\mathbb{Z}[\pi_1(X\backslash Y)]))$.
- The UNil-groups are the obstruction groups to a Mayer-Vietoris exact sequence in $L$-theory, with

$$L_{m+1}(\mathbb{Z}[\pi_1(X)]) \;=\; L^I_{m+1}(\mathbb{Z}[\pi_1(Y)]\to\mathbb{Z}[\pi_1(X\backslash Y)])\oplus \mathrm{UNil}_{m+1}$$

in the first instance geometrically.
- The splitting obstruction of a homotopy equivalence $f: M\to X$ of $m$-dimensional manifolds at $Y\subset X$ is

$$s(f) \;=\; (\tau(f),\mathrm{unil}(f))\in LS_{m-1}(X,Y) \;=\; \widehat{H}^m(\mathbb{Z}_2;I)\oplus\mathrm{UNil}_{m+1}$$

with $\mathrm{unil}(f)$ the surgery obstruction of the 'unitary nilpotent' bordism of $f$ to a split homotopy equivalence.

## The UNil-groups since 1974

- Farrell (1979) $\mathrm{UNil}_*$ has exponent 4.
- R. (1980, 1995) Chain complex definition of $\mathrm{UNil}_*$.
- (Connolly-R., Connolly-Davis, Banagl-R., 2004-2006) Computation of $\mathrm{UNil}_*$ for the $L$-theory of the infinite dihedral group $\pi_1(\mathbb{R}\,\mathbb{P}^n \# \mathbb{R}\,\mathbb{P}^n) = \mathbb{Z}_2 * \mathbb{Z}_2$

$$\mathrm{Unil}_n = \begin{cases} 0 & \text{if } n \equiv 0, 1 (\bmod 4) \\ \bigoplus_\infty \mathbb{Z}_2 & \text{if } n \equiv 2 (\bmod 4) \\ \bigoplus_\infty \mathbb{Z}_4 \oplus \bigoplus_\infty \mathbb{Z}_2 & \text{if } n \equiv 3 (\bmod 4) \end{cases}$$

  with each $\infty$ countable.
- Brookman (Edinburgh Ph.D. thesis, 2004) Algebraic definition of $\mathrm{UNil}_{2*+1}$, using short odd-dimensional quadratic Poincaré complexes ('unil formations')
- Davis-Khan-R. (2008?) Algebraic $L$-theory over the infinite dihedral group.

## Algebraic and geometric transversality

- Cappell's geometric proofs use geometric transversality: **algebraic proofs require algebraic transversality!**
- The first step in getting to the splitting obstruction $\mathrm{unil}(f) \in \mathrm{UNil}_{m+1}$ of a homotopy equivalence $f : M \to X$ of $m$-dimensional manifolds at a codimension 1 submanifold $Y \subset X$ is to prove that $f$ can be made transverse in such a way that the restrictions

$$g = f| : N = f^{-1}(Y) \to Y \ , \ h = f| : M \backslash N \to X \backslash Y$$

  are connected below the middle dimension(s).
- The split surjection

$$L_{m+1}(\mathbb{Z}[\pi_1(X)]) \to \mathrm{UNil}_{m+1} \ ; \ x \mapsto \partial x$$

  is defined geometrically by the Wall realization $x = \sigma(F)$ with $F : V \to W$ an $(m+1)$-dimensional normal map with $f = \partial F : M = \partial V \to X = \partial W$ a homotopy equivalence, and setting $\partial x = \mathrm{unil}(f)$.

## Induction and restriction

- Given a ring morphism $i : A \to B$ define the **induction** functor

  $$i_! \ : \ \{A\text{-modules}\} \to \{B\text{-modules}\} \ ; \ M \mapsto i_! M \ = \ B \otimes_A M$$

  and the adjoint **restriction** functor

  $$i^! \ : \ \{B\text{-modules}\} \to \{A\text{-modules}\} \ ; \ N \mapsto i^! N \ = \ N$$

- **Frobenius reciprocity**

  $$\mathrm{Hom}_A(M, i^! N) \ = \ \mathrm{Hom}_B(i_! M, N) \ ,$$
  $$M \otimes_A i^! N \ = \ i_! M \otimes_B N \ .$$

## Amalgamated free products

- Let $A = A_1 *_B A_2$ be an amalgamated free product of rings, a pushout square

$$\begin{tikzcd} B \arrow[r, "i_1"] \arrow[d, "i_2"] & A_1 \arrow[d, "j_1"'] \\ A_2 \arrow[r, "j_2"] & A \end{tikzcd}$$

  and let $k = j_1 i_1 = j_2 i_2 : B \to A$.
- Will only consider $A = A_1 *_B A_2$ with $A_1 = B \oplus A_1'$, $A_2 = B \oplus A_2'$ for $(B, B)$-bimodules $A_1', A_2'$, so that

$$A \ = \ B \oplus A_1' \oplus A_2' \oplus A_1' \otimes_B A_2' \oplus A_2' \otimes_B A_1' \oplus \dots \ .$$

  Always assume $A_1', A_2'$ are free as right $B$-modules.
- **Example** If $\pi = \pi_1 *_\rho \pi_2$ then $\mathbb{Z}[\pi] = \mathbb{Z}[\pi_1] *_{\mathbb{Z}[\rho]} \mathbb{Z}[\pi_2]$.
- Similarly for *HNN* extensions.

## Mayer Vietoris presentations

- An **MV presentation** of an $A$-module chain complex $C$ is an exact sequence

$$0 \to k_! E \to (j_1)_! D_1 \oplus (j_2)_! D_2 \to C \to 0$$

with $D_r$ a $\mathbb{Z}[A_r]$-module chain complex $(r = 1, 2)$ and $E$ a $\mathbb{Z}[B]$-module chain complex.

- **Definition** An $A$-module chain complex $C$ is **finite** if it is bounded f.g. free. An MV presentation is **finite** if every chain complex in it is finite.
- **Proposition** (Waldhausen 1974, R.) For every finite $A$-module chain complex $C$ there exist finite $A_r$-module subcomplexes $D_r \subset i_r^! C$ $(r = 1, 2)$ such that $E = D_1 \cap D_2 \subset k^! C$ is a finite $B$-module subcomplex, with a finite MV presentation

$$0 \to k_! E \to (j_1)_! D_1 \oplus (j_2)_! D_2 \to C \to 0$$

## Algebraic transversality for chain complexes

- **Idea of proof** The ring $A = A_1 *_B A_2$ has a universal infinite MV presentation

$$0 \to k_! k^! A \to (j_1)_! j_1^! A \oplus (j_2)_! j_2^! A \to A \to 0$$

so that any $A$-module chain complex $C$ has a universal infinite MV presentation

$$0 \to k_! k^! C \to (j_1)_! j_1^! C \oplus (j_2)_! j_2^! C \to C \to 0 \ .$$

- If $C$ is finite can use the expression

$$\text{Bass-Serre tree} = \text{union of finite subtrees}$$

to prove there exists a cofinal family of finite subcomplexes $D_1 \subset j_1^! C$, $D_2 \subset j_2^! C$ with $E = D_1 \cap D_2 \subset k^! C$ finite.

## Quadratic Poincaré complexes

- An $m$-**dimensional quadratic complex** $(C, \psi)$ over a ring with an involution $A$ is an $m$-dimensional $A$-module chain complex $C$ together with an element

$$\psi \in Q_m(C) \ = \ H_m(W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes_A C))$$

  where $T(x \otimes y) = \pm y \otimes x$ and $W$ is the standard free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$.
- A quadratic complex $(C, \psi)$ is **finite** if $C$ is finite.
- A quadratic complex $(C, \psi)$ is **Poincaré** if the $A$-module chain map

$$(1 + T)\psi_0 \ : \ C^{m-*} \ = \ \mathrm{Hom}_A(C, A)_{*-m} \to C$$

  is a chain equivalence.
- **Definition/Theorem** $L_m(A) =$ the cobordism group of finite quadratic Poincaré complexes $(C, \psi)$ over $A$.

## Mayer-Vietoris presentations for quadratic complexes

- **Definition** (i) An **MV presentation** of an $m$-dimensional quadratic complex $(C,\psi)$ over $A = A_1 *_B A_2$ is an expression

$$(C,\psi) \ = \ (j_1)_!(D_1,\delta\theta_1) \cup_{k_!(E,\theta)} (j_2)_!(D_2,\delta\theta_2)$$

  with $(E,\theta)$ is an $(m-1)$-dimensional quadratic complex over $B$, and $((i_r)_!E \to D_r, (\delta\theta_r,\theta))$ $(r=1,2)$ is an $m$-dimensional quadratic pair over $A_r$.
- (ii) The MV presentation is **finite** if each chain complex is bounded f.g. free.
- (iii) The MV presentation is **Poincaré** if each quadratic complex/pair is Poincaré.

## Algebraic transversality for quadratic Poincaré complexes

- **Proposition** (i) Every finite quadratic complex $(C, \psi)$ over $A = A_1 *_B A_2$ has a finite MV presentation.
  (ii) A finite quadratic Poincaré complex $(C, \psi)$ over $A$ admits a finite Poincaré MV presentation if and only if the **algebraic splitting obstruction** $\mathrm{unil}(C, \psi) \in \mathrm{UNil}_m(B; A'_1, A'_2)$ is 0.
- **Proof** Start with the universal infinite MV presentation

  $$0 \to k_! k^! C \to (j_1)_! j_1^! C \oplus (j_2)_! j_2^! C \to C \to 0 \ .$$

  Apply $C \otimes_A -$ and Frobenius reciprocity

  $$0 \to k^! C \otimes_B k^! C \to j_1^! C \otimes_{A_1} j_1^! C \oplus j_2^! C \otimes_{A_2} j_2^! C \to C \otimes_A C \to 0 \ .$$

  Can choose finite subcomplexes $D_1 \subset j_1^! C$, $D_2 \subset j_2^! C$ with $E = D_1 \cap D_2 \subset k^! C$ finite, **with quadratic structures**.

## Algebraic $L$-theory of generalized free products

- The algebraic splitting obstruction theory can be used to obtain Mayer-Vietoris decompositions of the algebraic $L$-theory of injective amalgamated free products and $HNN$ extensions.
- **Theorem** (Cappell 1974, R.) If $A = A_1 *_B A_2$ then

$$L_*(A) \;=\; L^I_*(B \to A_1 \times A_2) \oplus \mathrm{UNil}_*(B; A'_1, A'_2)$$

  with $I = \ker(\widetilde{K}_0(B) \to \widetilde{K}_0(A_1) \oplus \widetilde{K}_0(A_2))$.
- **Proof** Replace manifold transversality by MV presentations of quadratic Poincaré complexes. The first summand is the $L$-theory of quadratic Poincaré MV presentations. The second summand is the $L$-theory of quadratic Poincaré MV presentations of 0.

## Algebraic $L$-theory of a tensor algebra

- **Definition** Let $A$ be a ring and $B$ an $(A,A)$-bimodule. The **tensor algebra** is the ring

$$T(B) \;=\; A \oplus B \oplus (B \otimes_A B) \oplus (B \otimes_A B \otimes_A B) \oplus \dots$$

- **Theorem** (2008) If $A, B$ have involutions then $T(B)$ has an involution, and $L_*(T(B))$ is the quadratic $L$-theory of $B$-nilpotent quadratic Poincaré complexes $(C, \nu, \psi)$ over $A$, with $C$ a f.g. free $A$-module chain complex, $\nu : C \to B \otimes_A C$ chain homotopy nilpotent, with

$$L_*(T(B)) \;=\; L_*(A) \oplus \mathrm{UNil}_*(A, A, B)\ .$$

- **Example** If $B = A$ then $T(B) = A[x]$ with $\bar{x} = x$: in this case $L_*(A[x]) = L_*(A) \oplus \mathrm{UNil}_*(A, A, A)$ was first obtained in 1974.

## References


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