THE HOPF INVARIANT
IN TOPOLOGY AND ALGEBRA

Andrew Ranicki (Edinburgh and Münster)

http://www.maths.ed.ac.uk/~aar

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Three quotations

- It is a fact of sociology that topologists are interested in quadratic forms.

  Serge Lang

- A quadratic form is the basic discretization of a compact manifold.

  Elmar Winkelnkemper

- A talk on the Hopf invariant shouldn't take more than 5 minutes.

  Elmer Rees

Heinz Hopf (1894–1971)

30 Murray Place, Princeton, 1928

Princeton, N.J., 30 Murray Place, den 17. August 1928.

Lieber Herr Freudenthal!

Für den Fall, dass Sie sich noch für die Frage nach den Klassen der Abbildungen der 3-dimensionalen Kugel S^3 auf die 2-dimensionale Kugel S^2 interessieren, möchte ich Ihnen mitteilen, dass ich diese Frage jetzt beantworten kann: es existieren unendlich viele Klassen. Und zwar gibt es eine Klasseninvariante folgender Art: x, y seien Punkte der S^2; dann besteht bei hinreichend anständiger Approximation der gegebenen Abbildung die Originalmenge von x aus endlich vielen einfach geschlossenen, orientierten Polygonen $P_1, P_2, \ldots, P_a$ und ebenso die Originalmenge von y aus Polygonen $Q_1, Q_2, \ldots, Q_b$. Bezeichnet v_{ij} die Verschlingungszahl von P_i mit Q_j, so ist $\sum_{i,j} v_{ij} = \gamma$ unabhängig von x, y und von der Approximation und ändert sich nicht bei stetiger Änderung der Abbildung. Zu jedem γ gibt es Abbildungen. Ob es zu einem jeden γ nur eine Klasse gibt, weiss ich nicht. Wird nicht die ganze S^2 von der Bildmenge bedeckt, so ist $\gamma = 0$. Eine Folgerung davon ist dass man die Linienelemente auf einer S^2 nicht stetig in einen Punkt zusammenfegen kann.

Es bleiben noch eine Anzahl von Fragen offen, die mir interessant zu sein scheinen, besonders solche, die sich auf Vektorfelder auf der S^3 beziehen und mit analytischen Fragen zusammenhängen (Existenz geschlossener Integralkurven). Wenn Sie sich dafür interessieren, so schreiben Sie mir doch einmal. Meine Adresse ist bis 20. Mai die oben angegebene, im Juni und Juli: Göttingen, Mathematisches Institut der Universität, Weender Landstrasse.

Mit den besten Grüssen, auch an die übrigen Bekannten im Seminar,

Heinz Hopf.

From $\pi_3(S^2)$, *H. Hopf, W. K. Clifford and F. Klein*
by H. Samelson, History of Topology (ed. I. M. James), 1999.

30 Murray Place, Princeton, 1980

OBERWOLFACH PHOTO COLLECTION

Total number of photos: 10018

1980

On the Photo:

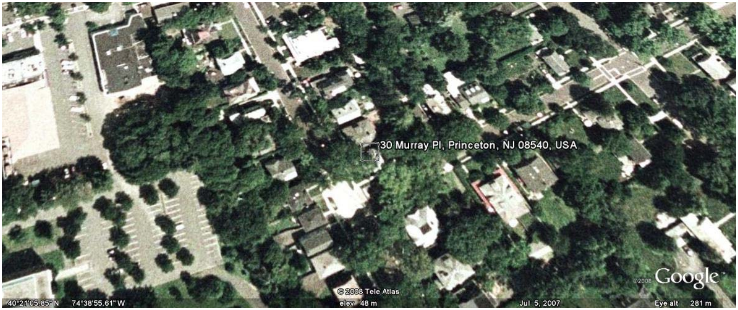
Andrew A. Ranicki

Ida Thompson (left)

Carla Ranicki (middle)

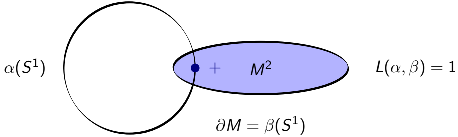
Location: 30 Murray Place, Princeton

30 Murray Place, Princeton, 2008

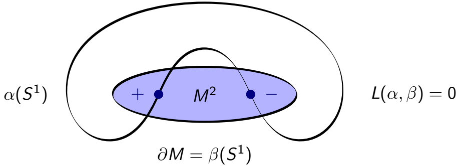


Linking

- The **linking number** of disjoint embeddings $\alpha, \beta : S^1 \hookrightarrow S^3$ is
$$L(\alpha,\beta) \ = \ \alpha(S^1) \cap M^2 \in \mathbb{Z}$$
with $M^2 \subset S^3$ a surface with boundary $\partial M = \beta(S^1)$.
- **Example**



- **Example**



The original Hopf invariant (1928)

- The **Hopf invariant** of a map $F : S^3 \to S^2$ is the linking number

$$H(F) \ = \ L(F^{-1}(x), F^{-1}(y)) \in \mathbb{Z}$$

of the disjoint inverse image circles (or unions of circles)

$$F^{-1}(x)\ ,\ F^{-1}(y)\ :\ S^1 \hookrightarrow S^3$$

of generic $x \neq y \in S^2$.

- The projection of the **Hopf fibration**

$$S^1 \longrightarrow S^3 \xrightarrow{F} S^2$$

is a map $F : S^3 \to S^2$ with Hopf invariant 1.

- Film: http://www.dimensions-math.org

Rings with involution

- I want to describe a generalization of the Hopf invariant to more general maps than just $S^3 \to S^2$, which is particularly useful in the classification of manifolds with non-trivial fundamental group π.
- The generalized Hopf invariant involves the modern algebraic theory of symmetric and quadratic forms on chain complexes over a ring with involution.
- An **involution** on a ring A is a function $A \to A; a \mapsto \bar{a}$ with

$$\overline{a+b} = \bar{a} + \bar{b}\ ,\ \overline{ab} = \bar{b}.\bar{a}\ ,\ \bar{\bar{a}} = a\ ,\ \bar{1} = 1 \in A\ .$$

- Use involution to identify

$$\text{left } A\text{-modules} = \text{right } A\text{-modules}$$

- **Example** A = commutative ring, involution = identity.
- **Example** $A = \mathbb{C}$, involution = complex conjugation.
- **Example** $A = \mathbb{Z}[\pi]$, involution by $\bar{g} = g^{-1}$ for $g \in \pi$.

Symmetric forms in algebra

- Given an A-module M let $S(M)$ be the abelian group of all sesquilinear pairings
$$\lambda \ : \ M \times M \to A \ ; \ (x,y) \mapsto \lambda(x,y)$$
such that $\lambda(ax,by) = b\lambda(x,y)\overline{a} \in A$.
- The pairing is **nonsingular** if the adjoint A-module morphism
$$\mathrm{adj}(\lambda) \ : \ M \to \mathrm{Hom}_A(M,A) \ ; \ x \mapsto (y \mapsto \lambda(x,y))$$
is an isomorphism.
- For $\epsilon = 1$ or -1 regard $S(M)$ as a $\mathbb{Z}[\mathbb{Z}_2]$-module by the ϵ-**transposition** involution
$$T_\epsilon \ : \ S(M) \to S(M) \ ; \ \lambda \mapsto (T_\epsilon\lambda : (x,y) \mapsto \overline{\epsilon\lambda(y,x)}) \ .$$
- An ϵ-**symmetric form** (M,λ) over A is an A-module M with an element
$$\lambda \in Q^\epsilon(M) \ = \ H^0(\mathbb{Z}_2; S(M), T_\epsilon) \ = \ \ker(1-T_\epsilon : S(M) \to S(M)) \ .$$

Symmetric forms in topology

- For any space X let $\mathbb{Z}_2$ act on $X \times X$ by the **transposition**

$$T \ : \ X \times X \to X \times X \ ; \ (x,y) \mapsto (y,x) \ .$$

- The **diagonal map** $\Delta_X : X \to X \times X ; x \mapsto (x,x)$ is $\mathbb{Z}_2$-equivariant, with the identity $\mathbb{Z}_2$-action on X.
- The **cup product** in cohomology

$$\cup \ : \ H^p(X) \otimes H^q(X) \to H^{p+q}(X \times X) \xrightarrow{\Delta_X^*} H^{p+q}(X)$$

  is $\mathbb{Z}_2$-equivariant, with $x \cup y = (-)^{pq} y \cup x$.
- An oriented $2i$-dimensional manifold M^{2i} has a $(-)^i$-symmetric **intersection form** $(H^i(M), \lambda)$ over $A = \mathbb{Z}$, with $\lambda \in Q^{(-)^i}(H^i(M))$ given by

$$\lambda(x,y) \ = \ \langle x \cup y, [M] \rangle \in \mathbb{Z} \ .$$

- **Example** The **signature** of M^{4j} is a cobordism invariant

$$\text{signature}(M) \ = \ \text{signature}(H^{2j}(M), \lambda) \in \mathbb{Z}$$

Symmetric forms on chain complexes

- Let $\Lambda = \mathbb{Z}[\mathbb{Z}_2]$, W = standard free Λ-module resolution of $\mathbb{Z}$

$$W \ : \ \dots \longrightarrow W_3 = \Lambda \xrightarrow{1-T} W_2 = \Lambda \xrightarrow{1+T} W_1 = \Lambda \xrightarrow{1-T} W_0 = \Lambda$$

- For A-module chain complex C define an involution

$$T \ : \ C_p \otimes_A C_q \to C_q \otimes_A C_p \ ; \ x \otimes y \mapsto (-)^{pq} y \otimes x \ ,$$

  so that $C \otimes_A C$ is a Λ-module chain complex.
- **Definition** (Mishchenko, 1972) The **symmetric Q-groups** of C

$$Q^n(C) \ = \ H^n(\mathbb{Z}_2; C \otimes_A C) \ = \ H_n(\mathrm{Hom}_\Lambda(W, C \otimes_A C)) \ .$$

  An element $\phi \in Q^n(C)$ is a chain map $\phi_0 : C^{n-*} \to C$ with chain homotopies $\phi_s : \phi_{s-1} \simeq T\phi_{s-1}$ $(s \geqslant 1)$.
- For $n = 2i$ forgetful map

$$Q^{2i}(C) \to Q^{(-)^i}(H^i(C)) \ ; \ \phi \mapsto \phi_0 \ .$$

  A $2i$-dimensional symmetric structure $\phi \in Q^{2i}(C)$ determines a $(-)^i$-symmetric form $(H^i(C), \phi_0)$ over A.

The symmetric construction

- For any space X the Alexander-Whitney-Steenrod diagonal chain approximation

$$\phi_X \ : \ C(X) \to \mathrm{Hom}_\Lambda(W, C(X) \otimes_{\mathbb{Z}} C(X))$$

induces the **symmetric construction**

$$\phi_X \ : \ H_n(X) \to Q^n(C(X)) \ \ (A = \mathbb{Z})$$

such that

$$\Delta_X \ : \ H_n(X) \xrightarrow{\phi_X} Q^n(C(X)) \longrightarrow H_n(X \times X) \ .$$

- **Theorem** (Mishchenko, 1972) If X is an n-dimensional manifold (or even just a Poincaré duality space) then $\phi_X([X]) \in Q^n(C(X))$ has

$$\phi_X([X])_0 \ = \ [X] \cap - \ : \ C(X)^{n-*} \xrightarrow{\simeq} C(X) \ .$$

- There is also a $\pi_1(X)$-equivariant version, involving the universal cover $\widetilde{X}$, with $A = \mathbb{Z}[\pi_1(X)]$.

Quadratic forms

- (Tits 1968, Wall 1970) An ϵ-**quadratic form** (M,ψ) over A is an A-module M with an element

$$\psi \in Q_\epsilon(M) \;=\; H_0(\mathbb{Z}_2; S(M), T_\epsilon) \;=\; \text{coker}(1-T_\epsilon : S(M) \to S(M))\,.$$

- For a f.g. projective A-module M an ϵ-quadratic form (M,ψ) is an ϵ-symmetric form (M,λ) with an ϵ-quadratic function

$$\mu \;:\; M \to Q_\epsilon(A) \;=\; A/\{a-\epsilon\bar{a}\,|\,a\in A\}\;;\; x \mapsto \psi(x,x)$$

such that

$$\begin{aligned}\lambda(x,x) &= \mu(x)+\epsilon\overline{\mu(x)} \in Q^\epsilon(A)\;,\\ \lambda(x,y) &= \mu(x+y)-\mu(x)-\mu(y)\in Q_\epsilon(A)\end{aligned}$$

where $Q^\epsilon(A)=\{b\in A\,|\,\epsilon\overline{b}=b\}$.

Homotopy groups

- Given pointed spaces X, Y let $[X, Y]$ be the set of homotopy classes of maps $F : X \to Y$.
- The **homotopy groups** of a pointed space X

$$\pi_n(X) \ = \ [S^n, X] \ .$$

  Abelian for $n \geqslant 2$.
- A space X is k-**connected** if

$$\pi_n(X) \ = \ 0 \text{ for } n \leqslant k \ .$$

- **Example** The k-sphere S^k is $(k-1)$-connected with

$$\pi_n(S^k) \ = \ \begin{cases} 0 & \text{if } n \leqslant k-1 \\ \mathbb{Z} & \text{if } n = k \ . \end{cases}$$

Framing

- A **framing** of an m-dimensional manifold M^m is an embedding $M^m \subset S^{m+k}$ for some $k \geqslant 1$ together with a trivialization $b : \nu_M \cong \epsilon^k$ of the normal bundle ν_M, or equivalently a stable trivialization $b : \tau_M \oplus \epsilon^k \cong \tau_{S^{m+k}}|_M$ of the tangent bundle τ_M.
- **Example** The standard embedding

$$S^m \ = \ S^m \times \{0\} \subset S^{m+k} \ = \ S^m \times D^k \cup D^{m+1} \times S^{k-1}$$

has trivial normal bundle ϵ^k, with $b : \tau_{S^m} \oplus \epsilon^k \cong \epsilon^{m+k}$.
- **Theorem** (i) (Pontrjagin 1950) The function $(F : S^{m+k} \to S^k) \mapsto M^m = F^{-1}(x)$ defines an isomorphism

$$\pi_{m+k}(S^k) \xrightarrow{\cong} \{\text{cobordism of framed } M^m \subset S^{m+k}\} \ .$$

(ii) (Hopf 1931) $H : \pi_{k+1}(S^k) \xrightarrow{\cong} \begin{cases} \mathbb{Z} & \text{if } k = 2 \\ \mathbb{Z}_2 & \text{if } k \geqslant 3 \end{cases}$
- Oriented manifolds $\to$ symmetric forms.
- Framed manifolds $\to$ quadratic forms.

Quadratic forms in topology

- (Stiefel 1935, Steenrod 1951) A stably trivialized i-plane bundle over S^i (such as τ_{S^i}) has a Hopf-type invariant in

$$\pi_{i+1}(BO, BO(i)) \;=\; Q_{(-)^i}(\mathbb{Z}) \;=\; \begin{cases} \mathbb{Z} & \text{if } i \equiv 0(\text{mod} 2) \\ \mathbb{Z}_2 & \text{if } i \equiv 1(\text{mod} 2) \end{cases}$$

- (Smale 1959) An immersion $f : S^i \looparrowright S^{2i}$ is classified up to homotopy through immersions by $\mu(f) = (\nu_f, b) \in Q_{(-)^i}(\mathbb{Z})$.
- (P. 1950 for $j = 0$, Kervaire-Milnor 1962 for $j \geqslant 1$) A framing b of M^{4j+2} determines a quadratic form $(H^{2j+1}(M;\mathbb{Z}_2), \mu_b)$ over $A = \mathbb{Z}_2$. Framed cobordism invariant

$$\text{Arf}(M, b) \;=\; \text{Arf}(H^{2j+1}(M;\mathbb{Z}_2), \mu_b) \in \mathbb{Z}_2 \;.$$

- (P.) For $m \geqslant 2$ the Arf invariant defines isomorphism

$$\pi_{m+2}(S^m) \;=\; \{\text{cobordism of framed } M^2 \subset S^{m+2}\} \;\cong\; \mathbb{Z}_2 \;.$$

- (Wall 1970) Generalization of μ_b to a $(-)^i$-quadratic form over $\mathbb{Z}[\pi_1(X)]$ for normal map $(f, b) : M^{2i} \to X$.

Quadratic forms on chain complexes

- **Definition** (R., 1980) The **quadratic Q-groups** of an A-module chain complex C are the $\mathbb{Z}_2$-hyperhomology groups

$$Q_n(C) \;=\; H_n(\mathbb{Z}_2; C \otimes_A C) \;=\; H_n(W \otimes_\Lambda (C \otimes_A C)) \ .$$

with forgetful maps

$$H_n(C \otimes C) \longrightarrow Q_n(C) \xrightarrow{1+T} Q^n(C) \longrightarrow H_n(C \otimes_A C) \ .$$

- **Example** For $n = 2i$ forgetful map

$$Q_{2i}(C) \to Q_{(-)^i}(H^i(C)) \ ; \ \psi \mapsto \psi_0 \ .$$

A $2i$-dimensional quadratic structure $\psi \in Q_{2i}(C)$ determines a $(-)^i$-quadratic form $(H^i(C), \psi_0)$ over A.

Suspension

- The **smash product** of pointed spaces X, Y is

$$X \wedge Y \ = \ X \times Y/(X \times \{\mathrm{pt.}_Y\} \cup \{\mathrm{pt.}_X\} \times Y) \ .$$

- (Freudenthal, 1938) The **suspension** of a pointed space X is

$$\Sigma X \ = \ S^1 \wedge X$$

with suspension map

$$E \ = \ \text{Einhängung} \ : \ [X, Y] \to [\Sigma X, \Sigma Y] \ .$$

- The relationship between the Hopf invariant and

$$E \ : \ \pi_n(X) \to \pi_{n+1}(\Sigma X)$$

provides insights into homotopy theoretic quadratic and symmetric forms with $A = \mathbb{Z}[\pi_1(X)]$.

The quadratic construction in chain homotopy theory

- (R., 1980) The **quadratic construction** of a stable map $F : \Sigma^k X \to \Sigma^k Y$ is a natural transformation

$$\psi_F \;:\; H_n(X) \to Q_n(C(Y)) \;\; (A = \mathbb{Z})$$

with $(1+T)\psi_F = (F \otimes F)\phi_X - \phi_Y F : H_n(X) \to Q^n(C(Y))$.

- For $k = 1$ $\psi_F : H_n(X) \xrightarrow{\widehat{\psi}_F} H_n(Y \times Y) \longrightarrow Q_n(C(Y))$.
- **Example** For $F : S^3 = \Sigma S^2 \to S^2 = \Sigma S^1$

$$\widehat{\psi}_F \;=\; \text{Hopf invariant}(F) \;:\; H_2(S^2) = \mathbb{Z} \to H_2(S^1 \times S^1) \;=\; \mathbb{Z}\,.$$

- The quadratic construction counts the double points of immersions of manifolds. A $\pi_1(X)$-equivariant version gave a chain complex treatment of Wall's nonsimply connected surgery obstruction theory.
- **Key idea** (with Crabb) ψ_F is induced by the 'geometric Hopf invariant' construction in homotopy theory.

Mapping cones and cofibrations

- The **mapping cone** of a map $F : X \to Y$ is the identification space

$$\mathcal{C}(F) \;=\; (Y \cup X \times I)/\{(x,0) \sim F(x),\; (x,1) \sim (x',1)\}$$

- (Barratt-Puppe 1961) The **cofibration sequence**

$$X \xrightarrow{F} Y \longrightarrow \mathcal{C}(F) \longrightarrow \Sigma X$$

induces a long exact sequence of homotopy sets

$$\cdots \longrightarrow [\Sigma X, A] \longrightarrow [\mathcal{C}(F), A] \longrightarrow [Y, A] \xrightarrow{F^*} [X, A] \longrightarrow \cdots$$

for any pointed space A.

The Hopf invariant is a desuspension obstruction

- (Steenrod 1948) For any map $F : S^3 \to S^2$
$$H^2(\mathcal{C}(F)) \;=\; H^4(\mathcal{C}(F)) \;=\; \mathbb{Z}\ .$$
The Hopf invariant is given by the cup product
$$H^2(\mathcal{C}(F)) \otimes H^2(\mathcal{C}(F)) \to H^4(\mathcal{C}(F))\ ;\ 1 \otimes 1 \mapsto H(F)\ .$$
In particular, if $H(F) = 1$ then $\mathcal{C}(F) = \mathbb{CP}^2$.
- Cup products vanish in suspensions:
$$\text{if } F \simeq \Sigma F_0 \text{ for } F_0 : S^2 \to S^1 \text{ then } H(F) = 0.$$
- (G.W.Whitehead 1950) The *EHP* exact sequence
$$\cdots \longrightarrow \pi_n(X) \xrightarrow{E} \pi_{n+1}(\Sigma X) \xrightarrow{H} \pi_n(X \wedge X) \xrightarrow{P} \pi_{n-1}(X) \longrightarrow \cdots$$
is defined for any $(m-1)$-connected space X, $n \leqslant 3m-2$.
- For $n = 2m$, $X = S^m$ have exact sequence
$$\cdots \longrightarrow \pi_{2m}(S^m) \xrightarrow{E} \pi_{2m+1}(S^{m+1}) \xrightarrow{H} \mathbb{Z} \xrightarrow{P} \pi_{2m-1}(S^m) \longrightarrow \cdots$$

Stable homotopy theory

- Let $X^\infty = X \cup \{\infty\}$ be the one-point compactification of X.
- For $V = \mathbb{R}^k$ have $V^\infty = S^k$, $V^\infty \wedge X = \Sigma^k X$.
- A **stable map** $F : X \rightsquigarrow Y$ is a map

$$F \ : \ V^\infty \wedge X \to V^\infty \wedge Y$$

  for some finite-dimensional inner product space V.
- **Definition** The **stable homotopy group** is

$$\{X;Y\} \ = \ \varinjlim_V [V^\infty \wedge X, V^\infty \wedge Y]$$

  with V finite-dimensional inner product spaces.
- A map $F : X \to Y$ induces a long exact sequence of stable homotopy groups

$$\cdots \longrightarrow \{A;X\} \xrightarrow{F_*} \{A;Y\} \longrightarrow \{A;\mathcal{C}(F)\} \longrightarrow \{A;\Sigma X\} \longrightarrow \cdots$$

Stable $\mathbb{Z}_2$-equivariant homotopy theory

- Given pointed $\mathbb{Z}_2$-spaces X, Y let $[X, Y]_{\mathbb{Z}_2}$ be the set of homotopy classes of $\mathbb{Z}_2$-equivariant maps $X \to Y$.
- Given an inner product space V let LV be the inner product $\mathbb{Z}_2$-space with $LV = V$ nonequivariantly and

$$T \; : \; LV \to LV \; ; \; v \mapsto -v \; .$$

- **Definition** The **stable $\mathbb{Z}_2$-equivariant homotopy group** is

$$\{X; Y\}_{\mathbb{Z}_2} \;=\; \varinjlim_{U} \varinjlim_{V} [U^\infty \wedge LV^\infty \wedge X, U^\infty \wedge LV^\infty \wedge Y]_{\mathbb{Z}_2}$$

with U, V finite-dimensional inner product spaces.

A definition and a theorem

- Joint work with Michael Crabb, since 1998.
- **Definition** The **geometric Hopf invariant** of a k-stable map $F : \Sigma^k X \to \Sigma^k Y$ is the stable $\mathbb{Z}_2$-equivariant map

$$h(F) \ = \ (F \wedge F)\Delta_X - \Delta_Y F \ : \ X \rightsquigarrow Y \wedge Y$$

  with $\mathbb{Z}_2$ acting by identity on X, transposition on $Y \wedge Y$.
- **Theorem** (i) The stable $\mathbb{Z}_2$-equivariant homotopy class of $h(F)$ is the primary obstruction to F being homotopy to the k-fold suspension of a map $F_0 : X \to Y$.
- (ii) The geometric Hopf invariant $h(F)$ induces the quadratic construction ψ_F on the chain level.
- (iii) The geometric Hopf invariant $h(F)$ counts the double points of an immersion of manifolds $f : N \looparrowright M$, with $F : M^+ \ \rightsquigarrow \ T(\nu_f)$ the Pontrjagin-Thom stable Umkehr map of an embedding $N \hookrightarrow M \times \mathbb{R}^k$ approximating f.
- There is also a $\pi_1(X)$-equivariant version.

k-fold desuspension = compression into $X \subset \Omega^k\Sigma^k X$

- The **loop space** ΩX of a connected pointed space X is the space of loops $S^1 \to X$, with

$$\pi_n(\Omega X) \ = \ \pi_{n+1}(X) \ .$$

- For any connected pointed space X and $k \geqslant 1$ the k-fold suspension map is induced by the inclusion $X \subset \Omega^k\Sigma^k X$

$$E^k : \pi_n(X) \xrightarrow{E} \pi_{n+1}(\Sigma X) \xrightarrow{E} \pi_{n+2}(\Sigma^2 X) \xrightarrow{E} \cdots \xrightarrow{E} \pi_{n+k}(\Sigma^k X) = \pi_n(\Omega^k\Sigma^k X) \ .$$

- A map $F : \Sigma^k Y \to \Sigma^k X$ is homotopic to $\Sigma^k F_0$ for $F_0 : Y \to X$ if and only if the adjoint map

$$\mathrm{adj}(F) \ : \ Y \to \Omega^k\Sigma^k X \ ; \ y \mapsto (s \mapsto F(s,y))$$

can be factored up to homotopy through $X \subset \Omega^k\Sigma^k X$.

The quadratic construction in homotopy theory

- (Toda, ..., 1950's) The V-**quadratic construction** on X is

$$Q_V(X) \ = \ S(LV)^+ \wedge_{\mathbb{Z}_2} (X \wedge X)$$

  with V an inner product space and $T(x,y) = (y,x)$.
- The projection

$$\overline{Q}_V(X) \ = \ S(LV)^+ \wedge (X \wedge X) \to Q_V(X)$$

  is a double cover away from the base point.
- For $1 \leqslant k \leqslant \infty$ write

$$Q_k(X) \ = \ Q_{\mathbb{R}^k}(X) \ , \ \overline{Q}_k(X) \ = \ \overline{Q}_{\mathbb{R}^k}(X) \ .$$

- **Example** $Q_0(X) = \{\text{pt.}\}$
- **Example** $Q_1(X) = X \wedge X$
- **Example** $Q_k(S^0) = (S^{k-1})^+/\mathbb{Z}_2 = (\mathbb{RP}^{k-1})^+$.

Combinatorial models for $\Omega^k\Sigma^k X$

- **Theorem** (James 1955) The case $k = 1$. For any connected space X the addition map

$$X \times X \to \Omega\Sigma X \; ; \; (x,y) \mapsto (t \mapsto \begin{cases} (2t, x) & \text{if } 0 \leqslant t \leqslant 1/2 \\ (2t-1, y) & \text{if } 1/2 \leqslant t \leqslant 1 \end{cases})$$

extends to a stable homotopy decomposition

$$\Omega\Sigma X \; \simeq_s \bigvee_{j=1}^{\infty} (\bigwedge_j X) \; .$$

- The component for $j = 2$ is $Q_1(X) \; = \; X \wedge X$.
- (Snaith 1974, May 1975) Generalizations to stable homotopy decompositions of $\Omega^k\Sigma^k X$ for $1 \leqslant k \leqslant \infty$, using configuration spaces, with $Q_k(X)$ in the quadratic part.

Splitting off the quadratic information

- For $k = \infty$ and connected X have stable homotopy decomposition
$$\Omega^\infty \Sigma^\infty X \ \simeq_s \bigvee_{j=1}^{\infty} (E\Sigma_j)^+ \wedge_{\Sigma_j} (\bigwedge_j X)\ .$$
- The component for $j = 2$ is
$$Q_\infty(X) \ = \ \varinjlim_k Q_k(X) \ = \ (S^\infty)^+ \wedge_{\mathbb{Z}_2} (X \wedge X)$$
with $C(S^\infty) = W$ and $H_*(Q_\infty(X)) = Q_*(C(X))$.
- Also for disconnected X (Barratt, Quillen, 1970's).
- For $1 \leqslant k \leqslant \infty$ and any k-stable map $F : \Sigma^k Y \to \Sigma^k X$ the geometric Hopf invariant provides a direct expression for the composite stable map
$$Y \xrightarrow{\text{adj}(F)} \Omega^k \Sigma^k X \ \rightsquigarrow Q_k(X)\ .$$

The stable homotopy groups

- The **stable homotopy groups** of a pointed space X are

$$\widetilde{\omega}_n(X) = \{S^n; X\}$$

- For an unpointed space X let

$$X^+ = X \sqcup \{\text{pt.}\} \ , \ \omega_n(X) = \widetilde{\omega}_n(X^+) \ .$$

- The **stable $\mathbb{Z}_2$-equivariant homotopy groups** of a pointed $\mathbb{Z}_2$-space X are

$$\widetilde{\omega}_n^{\mathbb{Z}_2}(X) = \{S^n; X\}_{\mathbb{Z}_2} \ .$$

- For an unpointed $\mathbb{Z}_2$-space X let

$$\omega_n^{\mathbb{Z}_2}(X) = \widetilde{\omega}_n^{\mathbb{Z}_2}(X^+) \ .$$

- **Example** If the $\mathbb{Z}_2$-action on X is free away from the base point then

$$\widetilde{\omega}_*^{\mathbb{Z}_2}(X) = \widetilde{\omega}_*(X/\mathbb{Z}_2) \ .$$

The cofibration sequence

- Given an inner product space V let

$$S(V) = \{v \in V \mid \|v\| = 1\}$$

be the unit sphere.
- The homeomorphism

$$(0,1) \times S(V) \to V \backslash \{0\} \; ; \; (t,u) \mapsto \frac{tu}{1-t}$$

has one-point compactification the homeomorphism

$$\Sigma S(V)^+ \to V^\infty / 0^\infty \; ; \; (t,u) \mapsto [t,u] = \frac{tu}{1-t} \; (t \in [0,1]) \; .$$

- **Proposition** The Pontrjagin-Thom map of $S(V) \subset V$

$$PT \; : \; V^\infty \to V^\infty / 0^\infty = \Sigma S(V)^+$$

extends to a cofibration of pointed spaces

$$S(V)^+ \longrightarrow S^0 = 0^\infty \xrightarrow{0} V^\infty \xrightarrow{PT} \Sigma S(V)^+ \longrightarrow S^1 \longrightarrow \cdots$$

The relative difference construction I.

- **Definition** The **relative difference** of maps $p,q:V^\infty\wedge X\to Y$ such that

$$p(0,x)\ =\ q(0,x)\in Y\ (x\in X)$$

is the map

$$\begin{array}{l}\delta(p,q)\ :\ \Sigma S(V)^+\wedge X\to Y\ ;\\ \qquad (t,u,x)\mapsto\begin{cases}p([1-2t,u],x) & \text{if } 0\leqslant t\leqslant 1/2\\ q([2t-1,u],x) & \text{if } 1/2\leqslant t\leqslant 1\ .\end{cases}\end{array}$$

- The cofibration induces a Barratt-Puppe exact sequence of homotopy sets

$$[\Sigma S(V)^+\wedge X,Y]\xrightarrow{PT^*}[V^\infty\wedge X,Y]\xrightarrow{0^*}[X,Y]$$

with $PT^*\delta(p,q)=q-p\in \mathrm{im}(PT^*)=\ker(0^*)$.
- The homotopy class $\delta(p,q)\in[\Sigma S(V)^+\wedge X,Y]$ is the obstruction to a rel $0^\infty\wedge X$ homotopy $p\simeq q:V^\infty\wedge X\to Y$.

The relative difference construction II.

- The composite

$$V^\infty \xrightarrow{PT} \Sigma S(V)^+ \xrightarrow{\Delta} S(V)^+ \wedge \Sigma S(V)^+$$

is an S-duality map.

- Stable maps $p, q : V^\infty \wedge X \to V^\infty \wedge Y$ with

$$p(0,x) \ = \ q(0,x) \in V^\infty \wedge Y \ (x \in X)$$

have a relative difference

$$\delta(p,q) \in \{\Sigma S(V)^+ \wedge X; V^\infty \wedge Y\} \ = \ \{X; S(V)^+ \wedge Y\}$$

with image

$$\begin{aligned}[\delta(p,q)] \ &= \ q - p \in \text{im}(\{X; S(V)^+ \wedge Y\} \to \{X; Y\}) \\ &= \ \ker(0_*^\infty : \{X; Y\} \to \{X; V^\infty \wedge Y\}) \ .\end{aligned}$$

The splitting of the stable $\mathbb{Z}_2$-equivariant homotopy groups

- **Definition** The **homotopy $\mathbb{Z}_2$-orbit space** of a pointed $\mathbb{Z}_2$-space X is $S(\infty)^+ \wedge_{\mathbb{Z}_2} X$, with $S(\infty) = \varinjlim_k S(L\mathbb{R}^k)$ a contractible space with a free $\mathbb{Z}_2$-action.
- **Theorem** (Segal, tom Dieck, ..., 1970's)
  For any pointed CW-$\mathbb{Z}_2$-complex X
  $$\widetilde{\omega}_*^{\mathbb{Z}_2}(X) \;=\; \widetilde{\omega}_*(X^{\mathbb{Z}_2}) \oplus \widetilde{\omega}_*(S(\infty)^+ \wedge_{\mathbb{Z}_2} X)$$
  with $X^{\mathbb{Z}_2} = \{x \in X \,|\, Tx = x\}$ the fixed point set.
- **Idea of proof** The $\mathbb{Z}_2$-equivariant cofibration sequence
  $$S(\infty)^+ \to S^0 \to L\mathbb{R}(\infty)^\infty = \varinjlim_k (L\mathbb{R}^k)^\infty$$
  induces a short exact sequence
  $$\widetilde{\omega}_*(S(\infty)^+ \wedge_{\mathbb{Z}_2} X) \rightarrowtail \widetilde{\omega}_*^{\mathbb{Z}_2}(X) \xrightarrow{\rho} \widetilde{\omega}_*^{\mathbb{Z}_2}(L\mathbb{R}(\infty)^\infty \wedge X) = \widetilde{\omega}_*(X^{\mathbb{Z}_2})$$
  Fixed point map $\rho : F \mapsto F^{\mathbb{Z}_2}$ split by inclusion $i_X : X^{\mathbb{Z}_2} \subset X$.
- Special case $G = \mathbb{Z}_2$ of splitting for any compact Lie group G.

The geometric Hopf invariant I.

- **Theorem** (C+R) **The geometric Hopf invariant** of a stable $\mathbb{Z}_2$-equivariant map $F : U^\infty \wedge LV^\infty \wedge X \to U^\infty \wedge LV^\infty \wedge Y$ is the stable map

$$\begin{aligned} h^{\mathbb{Z}_2}(F) &= \delta((1 \wedge i_Y)(1 \wedge F^{\mathbb{Z}_2}), F(1 \wedge i_X)) \\ &\quad : X^{\mathbb{Z}_2} \to S(LV)^+ \wedge_{\mathbb{Z}_2} Y \subset S(\infty)^+ \wedge_{\mathbb{Z}_2} Y \ (V \subset \mathbb{R}(\infty)) \end{aligned}$$

  defining $h^{\mathbb{Z}_2} : \{X; Y\}_{\mathbb{Z}_2} \to \{X; S(\infty)^+ \wedge_{\mathbb{Z}_2} Y\}$.
- $h^{\mathbb{Z}_2}(F)$ measures the noncommutativity of the diagram

$$\begin{array}{ccc} U^\infty \wedge LV^\infty \wedge X^{\mathbb{Z}_2} & \xrightarrow{1_{LV^\infty} \wedge F^{\mathbb{Z}_2}} & U^\infty \wedge LV^\infty \wedge Y^{\mathbb{Z}_2} \\ \Big\downarrow 1 \wedge i_X & & \Big\downarrow 1 \wedge i_Y \\ U^\infty \wedge LV^\infty \wedge X & \xrightarrow{F} & U^\infty \wedge LV^\infty \wedge Y \end{array}$$

  given that it commutes on

$$(U^\infty \wedge LV^\infty \wedge X)^{\mathbb{Z}_2} = U^\infty \wedge 0^\infty \wedge X^{\mathbb{Z}_2} .$$

The geometric Hopf invariant II.

- The geometric Hopf invariant of a stable $\mathbb{Z}_2$-equivariant map $F : U^\infty \wedge LV^\infty \wedge X \to U^\infty \wedge LV^\infty \wedge Y$ induces the off-diagonal terms

$$h^{\mathbb{Z}_2}(F) \;:\; \widetilde{\omega}_*(X^{\mathbb{Z}_2}) \to \widetilde{\omega}_*(S(\infty)^+ \wedge_{\mathbb{Z}_2} Y)$$

  in the morphisms induced by F in the stable $\mathbb{Z}_2$-equivariant homotopy groups

$$\begin{aligned} F_* \;=\; \begin{pmatrix} F_*^{\mathbb{Z}_2} & 0 \\ h^{\mathbb{Z}_2}(F) & 1 \wedge F_* \end{pmatrix} \;&: \\ \widetilde{\omega}_*^{\mathbb{Z}_2}(X) \;=\; \widetilde{\omega}_*(X^{\mathbb{Z}_2}) \oplus \widetilde{\omega}_*(S(\infty)^+ \wedge_{\mathbb{Z}_2} X) & \\ \to \widetilde{\omega}_*^{\mathbb{Z}_2}(Y) \;=\; \widetilde{\omega}_*(Y^{\mathbb{Z}_2}) \oplus \widetilde{\omega}_*(S(\infty)^+ \wedge_{\mathbb{Z}_2} Y)\;. & \end{aligned}$$

- If $F \simeq 1_{LV^\infty} \wedge F_0$ for some $F_0 : U^\infty \wedge X \to U^\infty \wedge Y$ then

$$h^{\mathbb{Z}_2}(F) = 0 \in \{X^{\mathbb{Z}_2}; S(\infty)^+ \wedge Y\}_{\mathbb{Z}_2} \;=\; \{X^{\mathbb{Z}_2}; S(\infty)^+ \wedge_{\mathbb{Z}_2} Y\}\;.$$

$\mathbb{Z}_2$-equivariant stable homotopy theory = fixed-point + fixed-point-free

- **Proposition** (C+R) For any pointed spaces X, Y there is an exact sequence of abelian groups

$$0 \to \{X; Q_\infty(Y)\} \xrightarrow{1+T} \{X; Y \wedge Y\}_{\mathbb{Z}_2} \xrightarrow{\rho} \{X; Y\} \to 0$$

with

$$\{X; Q_\infty(Y)\} = \varinjlim_V [\Sigma S(LV)^+ \wedge X, LV^\infty \wedge Y \wedge Y]_{\mathbb{Z}_2} \ (S\text{-duality}) .$$

- ρ is given by the $\mathbb{Z}_2$-fixed points, split by

$$\Delta_Y \ : \ \{X; Y\} \to \{X; Y \wedge Y\}_{\mathbb{Z}_2} \ ; \ F \mapsto \Delta_Y F$$

with $\Delta_Y = i_{Y \wedge Y} : (Y \wedge Y)^{\mathbb{Z}_2} = Y \subset Y \wedge Y$.

- The injection $1 + T$ is induced by projection $(S^\infty)^+ \to 0^\infty$

$$1 + T \ : \ \{X; Q_\infty(Y)\} = \{X; \overline{Q}_\infty(Y)\}_{\mathbb{Z}_2} \to \{X; Y \wedge Y\}_{\mathbb{Z}_2} ,$$

split by

$$h \ : \ \{X; Y \wedge Y\}_{\mathbb{Z}_2} \to \{X; Q_\infty(Y)\} \ ; \ G \mapsto \delta(\Delta_Y \rho(G), G) .$$

The stable $\mathbb{Z}_2$-equivariant homotopy theory of a square

- The **square** of a pointed space X is a $\mathbb{Z}_2$-space $X \wedge X$ with fixed point set

$$(X \wedge X)^{\mathbb{Z}_2} \;=\; \{(x,x)\,|\,x \in X\} \;=\; X \subset X \wedge X \;.$$

- $\widetilde{\omega}^{\mathbb{Z}_2}_*(X \wedge X) \;=\; \widetilde{\omega}_*(X) \oplus \widetilde{\omega}_*(Q_\infty(X))$
- The square of a stable map

$$F : V^\infty \wedge X \to V^\infty \wedge Y$$

is a $\mathbb{Z}_2$-equivariant map

$$F \wedge F \;:\; (V^\infty \wedge X) \wedge (V^\infty \wedge X) \to (V^\infty \wedge Y) \wedge (V^\infty \wedge Y)$$

with $(F \wedge F)^{\mathbb{Z}_2} = F$.

The geometric Hopf invariant III.

- The **geometric Hopf invariant** of a stable map $F : X \rightsquigarrow Y$ is

$$\begin{aligned} h(F) = h^{\mathbb{Z}_2}(F \wedge F) &= (F \wedge F)\Delta_X - \Delta_Y F \\ &= \delta(\Delta_Y F, (F \wedge F)\Delta_X) \end{aligned}$$

$$\begin{aligned} &\in \ker(\rho : \{X; Y \wedge Y\}_{\mathbb{Z}_2} \to \{X; Y\}) \\ &\quad = \operatorname{im}(1 + T : \{X; Q_\infty(Y)\} \hookrightarrow \{X; Y \wedge Y\}_{\mathbb{Z}_2})\ . \end{aligned}$$

- **Proposition** (i) The function

$$h\ :\ \{X; Y\} \to \{X; Q_\infty(Y)\}\ ;\ F \mapsto h(F)$$

is nonadditive, being quadratic in nature:

$$h(F + G)\ =\ h(F) + h(G) + (F \wedge G)\Delta_X$$

(ii) If $F \in \operatorname{im}([X, Y] \to \{X; Y\})$ then $h(F) = 0$.

Two examples

- **Example** If $X = Y = S^0$, $k = 1$, $d \in \mathbb{Z}$ the stable map $F = d : \Sigma X = S^1 \to \Sigma Y = S^1$ has geometric Hopf invariant

$$h(F) \; = \; d(d-1)/2 \in \{0^\infty; Q_\infty(0^\infty)\} \; = \; \mathbb{Z} \; .$$

  This is the number of double points of the immersion $\{1, 2, \dots, d\} \looparrowright \{0\}$ of 0-dimensional manifolds.
- **Example** If $X = S^2$, $Y = S^1$, $k = 1$ the geometric Hopf invariant $F : \Sigma X = S^3 \to \Sigma Y = S^2$ then

$$h(F) = \text{mod 2 Hopf invariant}(F) \in \{S^2; Q_\infty(S^1)\} = \mathbb{Z}_2$$

  Working a little harder can lift $h(F)$ to an integer invariant

$$h_{\mathbb{R}}(F) = \text{original Hopf invariant}(F) \in \{S^2; Q_1(S^1)\} = \mathbb{Z}$$

Double points

- For any map $f : M \to N$ the group $\mathbb{Z}_2$ acts on
$$(f \times f)^{-1}(\Delta_N) \;=\; \{(x, y) \in M \times M \,|\, f(x) = f(y) \in N\}$$
by $T(x, y) = (y, x)$.
- The **ordered double point set** of f is the $\mathbb{Z}_2$-free set
$$\overline{D}_2(f) = \{(x, y) \,|\, x \neq y \in M,\, f(x) = f(y) \in N\}$$
and
$$(f \times f)^{-1}(\Delta_N) \;=\; \Delta_M \cup \overline{D}_2(f) \;=\; \mathbb{Z}_2\text{-fixed points} \cup \mathbb{Z}_2\text{-free} \;.$$
- The **unordered double point set** is
$$D_2(f) \;=\; \overline{D}_2(f)/\mathbb{Z}_2 \;.$$
- f is an embedding if and only if $D_2(f) = \emptyset$.
- **The geometric Hopf invariant is the primary homotopy theoretic method of capturing $D_2(f)$.**

Immersions of spaces

- **Definition** A **immersion of spaces** $f : M \looparrowright N$ is a map $f : M \to N$ with an open embedding of the type

$$g \;=\; (e,f) : V \times M \hookrightarrow V \times N \;;\; (v,x) \mapsto (e(v,x), f(x))$$

with V finite dimensional, for some map $e : V \times M \to V$, so

$$\begin{array}{ccc} V \times M & \overset{g}{\hookrightarrow} & V \times N \\ \downarrow & & \downarrow \\ M & \xrightarrow{\;f\;} & N \end{array}$$

- The **Umkehr** map of g is the stable map

$$F : (V \times N)^\infty = V^\infty \wedge N^\infty \to (V \times M)^\infty = V^\infty \wedge M^\infty \;;$$

$$(w,y) \mapsto \begin{cases} (v,x) & \text{if } (w,y) = g(v,x) \\ \infty & \text{if } (w,y) \notin \mathrm{im}(g) \;. \end{cases}$$

- **Example** A codimension 0 immersion of manifolds.

Capturing double points with $\mathbb{Z}_2$-homotopy theory

- Let $f : M \to N$ be an immersion of spaces, with embedding $g = (e, f) : V \times M \hookrightarrow V \times N$. The $\mathbb{Z}_2$-equivariant product embedding

$$g \times g \ : \ V \times V \times M \times M \hookrightarrow V \times V \times N \times N$$

restricts to a $\mathbb{Z}_2$-equivariant embedding

$$g \times g| \ : \ V \times V \times D_2(f) \hookrightarrow V \times V \times N$$

with $\mathbb{Z}_2$-equivariant Umkehr map

$$G \ : \ V^\infty \wedge V^\infty \wedge N^\infty \to V^\infty \wedge V^\infty \wedge D_2(f)^+ \ .$$

- Define also the $\mathbb{Z}_2$-equivariant map

$$H \ : \ D_2(f)^+ \to \overline{Q}_V(M^\infty) \ = \ S(LV)^+ \wedge (M^\infty \wedge M^\infty) \ ;$$

$$(x, y) \mapsto \Big(\frac{e(0,x) - e(0,y)}{\|e(0,x) - e(0,y)\|}, x, y\Big) \ .$$

Double points of immersions of manifolds

- The double point set $D_2(f)$ of a generic immersion $f : M^m \looparrowright N^n$ with normal bundle $\nu_f : M \to BO(n-m)$ is a $(2m-n)$-dimensional manifold. For $k \geqslant 2m-n+1$ there exists a map $e : M \to V = \mathbb{R}^k$ such that

$$g \ = \ (e, f) \ : \ M \hookrightarrow V \times N \ ; \ x \mapsto (e(x), f(x))$$

  is an embedding with normal bundle

$$\nu_g \ = \ \nu_f \oplus \epsilon^k \ : \ M \to BO(n-m+k) \ .$$

- By the tubular neighbourhood theorem can approximate the product immersion $1 \times f : V \times M \looparrowright V \times N$ by an embedding

$$\overline{g} \ = \ (\overline{e}, \overline{f}) \ : \ V \times E(\nu_f) \hookrightarrow V \times N$$

  extending g, with $\overline{f} : E(\nu_f) \looparrowright N$ a codimension 0 immersion. $E(\nu_f)^\infty = T(\nu_f) =$ Thom space. Stable **Umkehr** map $F : N^+ \ \rightsquigarrow \ T(\nu_f)$ represented by

$$F \ : \ V^\infty \wedge N^+ \ = \ \Sigma^k N^+ \to \Sigma^k T(\nu_f) \ .$$

The Double Point Theorem

- **Theorem** (C+R) If $f : M^m \looparrowright N^n$ is an immersion of manifolds with Umkehr map $F : N^+ \rightsquigarrow T(\nu_f)$ then

$$\begin{aligned} h(F) &= HG \\ &\in \ker\big(\rho : \{N^+; T(\nu_f) \wedge T(\nu_f)\}_{\mathbb{Z}_2} \to \{N^+; T(\nu_f)\}\big) \\ &= \operatorname{im}\big(\{N^+; Q_\infty(T(\nu_f))\} \hookrightarrow \{N^+; T(\nu_f) \wedge T(\nu_f)\}_{\mathbb{Z}_2}\big) \end{aligned}$$

  is a factorization of $h(F)$ through $D_2(f)^+$, with

$$N^+ \xrightarrow{G} T(\nu_f \times \nu_f|_{D_2(f)}) \xrightarrow{H} T(\nu_f \times \nu_f) = T(\nu_f) \wedge T(\nu_f) \ .$$

- If $f : M \looparrowright N$ is regular homotopic to an embedding $f_0 : M \hookrightarrow N$ with Umkehr map $F_0 : N^+ \to T(\nu_f)$ then F is stably homotopic to F_0, and $h(F)$ is stably null-homotopic.

The difference of diagonals

- For any space X the diagonal map
$$\Delta_X \ : \ X \to X \wedge X \ ; \ x \mapsto (x, x)$$
is $\mathbb{Z}_2$-equivariant.
- For any f.d. inner product space V define $\mathbb{Z}_2$-equivariant homeomorphism
$$\kappa_V \ : \ LV^\infty \wedge V^\infty \to V^\infty \wedge V^\infty \ ; \ (x, y) \mapsto (x + y, -x + y) \ .$$
- Given a map $F : V^\infty \wedge X \to V^\infty \wedge Y$ define the **noncommutative** square of $\mathbb{Z}_2$-equivariant maps
$$\begin{array}{ccc} LV^\infty \wedge V^\infty \wedge X & \xrightarrow{1 \wedge \Delta_X} & LV^\infty \wedge V^\infty \wedge X \wedge X \\ \Big\downarrow {\scriptstyle 1 \wedge F} & (\kappa_V^{-1} \wedge 1)(F \wedge F)(\kappa_V \wedge 1) & \Big\downarrow \\ LV^\infty \wedge V^\infty \wedge Y & \xrightarrow{1 \wedge \Delta_Y} & LV^\infty \wedge V^\infty \wedge Y \wedge Y \end{array}$$

The unstable geometric Hopf invariant $h_V(F)$

- **Definition** The **unstable geometric Hopf invariant** of a map $F : V^\infty \wedge X \to V^\infty \wedge Y$ is the $\mathbb{Z}_2$-equivariant relative difference map

  $$h_V(F) \;=\; \delta(p,q) \;:\; \Sigma S(LV)^+ \wedge V^\infty \wedge X \to LV^\infty \wedge V^\infty \wedge Y \wedge Y$$

  of the $\mathbb{Z}_2$-equivariant maps

  $$p \;=\; (1 \wedge \Delta_Y)F \;,\; q \;=\; (\kappa_V^{-1} \wedge 1)(F \wedge F)(\kappa_V \wedge 1) \;: \\ LV^\infty \wedge V^\infty \wedge X \to LV^\infty \wedge V^\infty \wedge Y \wedge Y$$

  with $\Sigma S(LV)^+ = LV^\infty / 0^\infty = (LV \backslash \{0\})^\infty$.
- The stable $\mathbb{Z}_2$-equivariant homotopy class of $h_V(F)$ depends only on the homotopy class of F, defining a function

  $$h_V \;:\; [V^\infty \wedge X, V^\infty \wedge Y] \to \\ \{\Sigma S(LV)^+ \wedge V^\infty \wedge X, LV^\infty \wedge V^\infty \wedge Y \wedge Y\}_{\mathbb{Z}_2} = \{X; Q_V(Y)\}$$
- Boardman and Steer (1967) $h_{\mathbb{R}}(F) : X \rightsquigarrow Y \wedge Y$.

Some properties of $h_V(F)$

- Composition and addition formulae

$$h_V(GF) = (G \wedge G)h_V(F) + h_V(G)F \ ,$$
$$h_V(F+F') = h_V(F) + h_V(F') + (F \wedge F')\Delta \ .$$

- If $F \simeq 1_V \wedge F_0$ for some $F_0 : X \to Y$ then

$$h_V(F) = 0 \in \{X; Q_V(Y)\} \ .$$

- The Double Point Theorem has unstable version, with $h_V(F)$.
- The original Hopf invariant of a map

$$F \ : \ S^{2m+1} \ = \ \Sigma(S^{2m}) \to S^{m+1} \ = \ \Sigma(S^m)$$

is

$$H(F) \ = \ h_{\mathbb{R}}(F) \in \{S^{2m}; Q_{\mathbb{R}}(S^m)\} \ = \ \{S^{2m}; S^{2m}\} = \mathbb{Z} \ .$$

Immersions of S^n in S^{2n}

- For every $n \geqslant 1$ Whitney (1944) constructed an immersion $f : S^n \looparrowright S^{2n}$ with normal bundle $\nu_f = \tau_{S^n}$ and a single double point. The composite immersion

$$S^n \overset{f}{\looparrowright} S^{2n} \hookrightarrow S^{2n+1}$$

  is homotopic through immersions to an embedding $g : S^n \hookrightarrow S^{2n+1}$ with a framing

$$b \ : \ \nu_g \ = \ \tau_{S^n} \oplus \epsilon \ \cong \ \epsilon^{n+1} \ .$$

- The Umkehr $F : S^{2n+1} \to \Sigma T(\tau_{S^n})$ has geometric Hopf invariant

$$h_{\mathbb{R}}(F) \ = \ \mu(f) \ = \ 1 \in \mathbb{Z} \ .$$

Hopf invariant 1?

- The computation $h_{\mathbb{R}}(F) = 1$ for $f : S^n \looparrowright S^{2n}$ for all $n \geqslant 1$ does **not** contradict the result of Adams (1960) that there exists a map $G : S^{2n+1} \to S^{n+1}$ with Hopf invariant $H(G) = 1$ if and only if $n = 1, 3, 7$.
- The Thom space of τ_{S^n} is

$$T(\tau_{S^n}) \ = \ S^n \cup_{[\iota_n, \iota_n]} D^{2n}$$

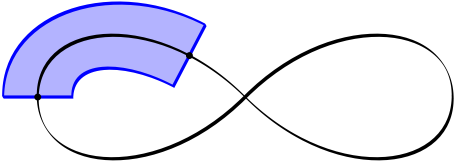
with $[\iota_n, \iota_n] \in \ker(E : \pi_{2n-1}(S^n) \to \pi_{2n}(S^{n+1})) = \mathrm{im}(P)$ such that

$$H([\iota_n, \iota_n]) \ = \ \chi(S^n) \ = \ 1 + (-)^n \in \mathbb{Z} \ .$$

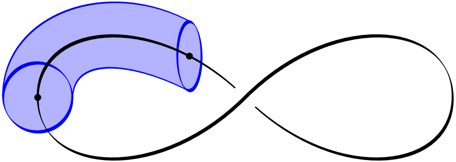
- By Bott and Milnor (1958) $\tau_{S^n} \cong \epsilon^n$ if and only if $n = 1, 3, 7$. For $n \neq 1, 3, 7$ $T(\tau_{S^n}) \not\simeq S^n \vee S^{2n}$, so cannot use $F : \Sigma T(\tau_{S^n}) \to S^{n+1}$ with $h_{\mathbb{R}}(F) = 1$ to construct a map $G : S^{2n+1} \to S^{n+1}$ with $H(G) = 1$.

The figure eight immersion I.

- The figure eight immersion $f : S^1 \looparrowright S^2$ with a framing $a : \nu_f = \tau_{S^1} \cong \epsilon$ and a single double point $\mu(f) = 1 \in \mathbb{Z}$



- The composite immersion $S^1 \looparrowright^f S^2 \hookrightarrow S^3$ is homotopic through immersions to an embedding $g : S^1 \hookrightarrow S^3$ with a framing $b : \nu_g = \nu_f \oplus \epsilon \cong \epsilon^2$.



The figure eight immersion II.

- The Umkehr map of the figure eight immersion

$$F \ : \ S^3 \to T(\nu_g) \ = \ \Sigma T(\nu_f)$$

is such that

$$h_{\mathbb{R}}(F) \ = \ \mu(f) \ = \ 1 \in \mathbb{Z} \ .$$

- The framings a, b are such that $b(a \oplus 1) : S^1 \to SO(2)$ is the rotation homeomorphism. The composite

$$p_2 F \ : \ S^3 \to T(\nu_g) = T(\nu_f \oplus \epsilon) \xrightarrow{T(a \oplus 1)} T(\epsilon^2) \ = \ S^3 \vee S^2 \to S^2$$

has Hopf invariant

$$H(p_2 F) \ = \ \mu(f) \ = \ 1 \in \mathbb{Z} \ .$$