ASPECTS OF QUADRATIC FORMS IN THE WORK OF HIRZEBRUCH AND ATIYAH - the director's cut

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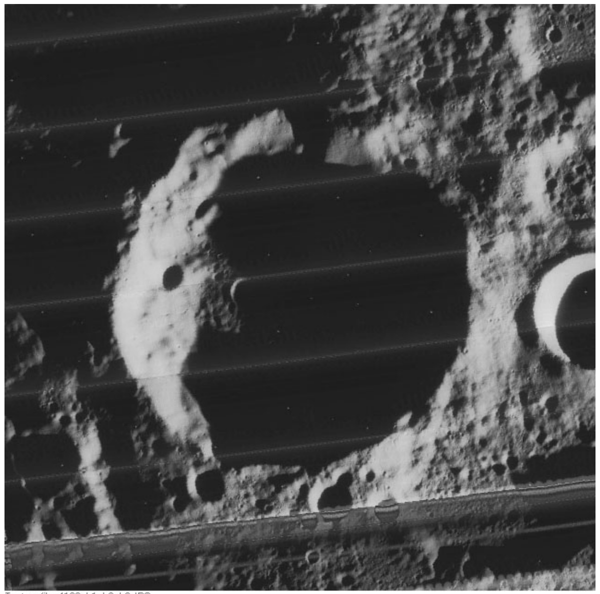
James Joseph Sylvester (1814–1897)

Honorary Fellow of the RSE, 1874

Sylvester's 1852 paper

A DEMONSTRATION OF THE THEOREM THAT EVERY HOMOGENEOUS QUADRATIC POLYNOMIAL IS REDUCIBLE BY REAL ORTHOGONAL SUBSTITUTIONS TO THE FORM OF A SUM OF POSITIVE AND NEGATIVE SQUARES.

- Fundamental insight: the invariance of the numbers of positive and negative eigenvalues of a quadratic polynomial under linear substitutions.
- Impact statement: the **Sylvester crater** on the Moon



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Symmetric matrices

- An $m \times n$ matrix $S = (s_{ij} \in \mathbb{R})$ corresponds to a bilinear pairing

$$S : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R} ; ((x_1, x_2, \dots, x_m), (y_1, y_2, \dots, y_n)) \mapsto \sum_{i=1}^{m} \sum_{j=1}^{n} s_{ij} x_i y_j .$$

- The **transpose** of an $m \times n$ matrix S is the $n \times m$ matrix $S^* = (s^*_{ji})$

$$s^*_{ji} = s_{ij} , \ S^*(x, y) = S(y, x) .$$

- An $n \times n$ matrix S is **symmetric** if $S^* = S$, i.e. $S(x, y) = S(y, x)$.
- Quadratic polynomials $Q(x)$ correspond to symmetric matrices S

$$Q(x) = S(x, x) , \text{ with } S(x, y) = (Q(x + y) - Q(x) - Q(y))/2 .$$

- **Spectral theorem** (Cauchy, 1829) The eigenvalues of a symmetric $n \times n$ matrix S are real: the characteristic polynomial of S

$$\mathrm{ch}_z(S) = \det(zI_n - S) \in \mathbb{R}[z]$$

has real roots $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R} \subset \mathbb{C}$.

Linear and orthogonal congruence

- Two $n \times n$ matrices S, T are **linearly congruent** if $T = A^*SA$ for an invertible $n \times n$ matrix A, i.e.

$$T(x,y) \;=\; S(Ax,Ay) \in \mathbb{R}\ (x,y \in \mathbb{R}^n)\ .$$

- An $n \times n$ matrix A is **orthogonal** if it is invertible and $A^{-1} = A^*$.
- Two $n \times n$ matrices S, T are **orthogonally congruent** if $T = A^*SA$ for an orthogonal $n \times n$ matrix A. Then $T = A^{-1}SA$ is conjugate to S.
- **Diagonalization** A symmetric $n \times n$ matrix S is orthogonally congruent to a diagonal matrix

$$A^*SA \;=\; D(\chi_1,\chi_2,\dots,\chi_n) \;=\; \begin{pmatrix} \chi_1 & 0 & \dots & 0 \\ 0 & \chi_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \chi_n \end{pmatrix}$$

with $A = (b_1\ b_2\ \dots\ b_n)$ the orthogonal $n \times n$ matrix with columns an orthonormal basis of $\mathbb{R}^n$ of eigenvectors $b_k \in \mathbb{R}^n$, $Ab_k = \chi_k b_k \in \mathbb{R}^n$.
- **Proposition** Symmetric $n \times n$ matrices S, T are orthogonally congruent if and only if they have the same eigenvalues.

The indices of inertia and the signature

- The **positive** and **negative index of inertia** of a symmetric $n \times n$ matrix S are

$$\begin{aligned} \tau_+(S) &= (\text{no. of eigenvalues } \lambda_k > 0) \ , \\ \tau_-(S) &= (\text{no. of eigenvalues } \lambda_k < 0) \in \{0, 1, 2, \dots, n\} \ . \end{aligned}$$

- $\tau_+(S) = \dim(V_+)$ is the dimension of any maximal subspace $V_+ \subseteq \mathbb{R}^n$ with $S(x, y) > 0$ for all $x, y \in V_+ \backslash \{0\}$. Similarly for $\tau_-(S)$.
- The **signature** ($=$ **index of inertia**) of S is the difference

$$\tau(S) = \tau_+(S) - \tau_-(S) = \sum_{k=1}^{n} \text{sign}(\lambda_k) \in \{-n, \dots, -1, 0, 1, \dots, n\} \ .$$

- The rank of S is the sum

$$\dim_{\mathbb{R}}(S(\mathbb{R}^n)) = \tau_+(S) + \tau_-(S) = \sum_{k=1}^{n} |\text{sign}(\lambda_k)| \ \in \{0, 1, 2, \dots, n\} \ .$$

S is invertible if and only if $\tau_+(S) + \tau_-(S) = n$.

Sylvester's Law of Inertia (1852)

- **Law of Inertia** Symmetric $n \times n$ matrices S, T are linearly congruent if and only if they have eigenvalues of the same signs, i.e. same indices

$$\tau_+(S) \; = \; \tau_+(T) \text{ and } \tau_-(S) \; = \; \tau_-(T) \in \{0, 1, \ldots, n\} \; .$$

- **Proof** (i) If $x \in \mathbb{R}^n$ is such that $S(x, x) \neq 0$ then S is linearly congruent to $\begin{pmatrix} S(x,x) & 0 \\ 0 & S' \end{pmatrix}$ with S' the $(n-1) \times (n-1)$-matrix of S restricted to the $(n-1)$-dimensional subspace $x^\perp = \{y \in \mathbb{R}^n \,|\, S(x, y) = 0\} \subset \mathbb{R}^n$.
- (ii) A symmetric $n \times n$ matrix S with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ is linearly congruent to the diagonal matrix

$$D(\text{sign}(\lambda_1), \text{sign}(\lambda_2), \ldots, \text{sign}(\lambda_n)) \; = \; \begin{pmatrix} I_p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I_q \end{pmatrix}$$

with $\tau_+(S) = p$, $\tau_-(S) = q$, $\tau(S) = p - q$, $\text{rank}(S) = p + q$.
- Invertible symmetric $n \times n$ matrices S, T are linearly congruent if and only if they have the same signature $\tau(S) = \tau(T)$.

Regular symmetric matrices

- The **principal** $k \times k$ **minor** of an $n \times n$ matrix $S = (s_{ij})_{1 \leqslant i,j \leqslant n}$ is

$$\mu_k(S) \ = \ \det(S_k) \in \mathbb{R}$$

with $S_k = (s_{ij})_{1 \leqslant i,j \leqslant k}$ the principal $k \times k$ submatrix.

$$S \ = \ \begin{pmatrix} S_k & \dots \\ \vdots & \ddots \end{pmatrix} .$$

- An $n \times n$ matrix S is **regular** if

$$\mu_k(S) \neq 0 \in \mathbb{R} \ (1 \leqslant k \leqslant n) \ ,$$

that is if each S_k is invertible.

- In particular, $S_n = S$ is invertible, and the eigenvalues are $\lambda_k \neq 0$.

The Sylvester-Gundelfinger-Frobenius theorem

- **Theorem** (Sylvester 1852, Gundelfinger 1881, Frobenius 1895)
  The eigenvalues $\lambda_k(S)$ of a regular symmetric $n \times n$ matrix S have the signs of the successive minor quotients

  $$\text{sign}(\lambda_k(S)) \;=\; \text{sign}(\mu_k(S)/\mu_{k-1}(S)) \in \{-1, 1\}$$

  for $k = 1, 2, \dots, n$, with $\mu_0(S) = 1$. The signature is

  $$\tau(S) \;=\; \sum_{k=1}^{n} \text{sign}(\mu_k(S)/\mu_{k-1}(S)) \in \{-n, -n+1, \dots, n\} \; .$$

- Proved by the "algebraic plumbing" of matrices – the algebraic analogue of the geometric plumbing of manifolds.
- **Corollary** If S is an invertible symmetric $n \times n$ matrix which is not regular then for sufficiently small $\epsilon \neq 0$ the symmetric $n \times n$ matrix $S_\epsilon = S + \epsilon I_n$ is regular, with eigenvalues $\lambda_k(S_\epsilon) = \lambda_k(S) + \epsilon \neq 0$, and

  $$\text{sign}(\lambda_k(S_\epsilon)) \;=\; \text{sign}(\lambda_k(S)) \in \{-1, 1\} \; ,$$
  $$\tau(S) \;=\; \tau(S_\epsilon) \;=\; \sum_{k=1}^{n} \text{sign}(\mu_k(S_\epsilon)/\mu_{k-1}(S_\epsilon)) \in \mathbb{Z} \; .$$

Algebraic plumbing

- **Definition** The **plumbing** of a regular symmetric $n \times n$ matrix S with respect to $v \in \mathbb{R}^n$, $w \neq vS^{-1}v^* \in \mathbb{R}$ is the regular symmetric $(n+1) \times (n+1)$ matrix $S' = \begin{pmatrix} S & v^* \\ v & w \end{pmatrix}$.
- **Proof of the Sylvester-Gundelfinger-Frobenius Theorem**
  It suffices to calculate the jump in signature under plumbing.
  The matrix identity

$$S' = \begin{pmatrix} 1 & 0 \\ vS^{-1} & 1 \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & w - vS^{-1}v^* \end{pmatrix} \begin{pmatrix} 1 & S^{-1}v^* \\ 0 & 1 \end{pmatrix}$$

shows that S' is linearly congruent to $\begin{pmatrix} S & 0 \\ 0 & w - vS^{-1}v^* \end{pmatrix}$.

By the Law of Inertia

$$\tau(S') = \tau(S) + \text{sign}(w - vS^{-1}v^*) \in \mathbb{Z} ,$$

so that $\tau(S') - \tau(S) = \text{sign}(\mu_n(S')/\mu_{n-1}(S'))$.

Tridiagonal matrices

- The **tridiagonal symmetric** $n \times n$ **matrix** of $\chi = (\chi_1, \chi_2, \ldots, \chi_n) \in \mathbb{R}^n$

$$\mathrm{Tri}(\chi) \;=\; \begin{pmatrix} \chi_1 & 1 & 0 & \ldots & 0 & 0 \\ 1 & \chi_2 & 1 & \ldots & 0 & 0 \\ 0 & 1 & \chi_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \chi_{n-1} & 1 \\ 0 & 0 & 0 & \ldots & 1 & \chi_n \end{pmatrix}$$

- Jacobi *Ein leichtes Verfahren, die in der Theorie der Säcularstörungen vorkommenden Gleichungen numerisch aufzulösen* (1846). Tridiagonal matrices first used in the numerical solution of simultaneous linear equations.
- Tridiagonal matrices and continued fractions feature in recurrences, Sturm theory, numerical analysis, orthogonal polynomials, integrable systems ... and in the Hirzebruch-Jung resolution of singularities.

Sylvester's 1853 paper

ON A REMARKABLE MODIFICATION OF STURM'S THEOREM.

- Sturm's theorem gave a formula for the number of roots in an interval of a generic real polynomial $f(x) \in \mathbb{R}[x]$.
- The formula was in terms of the numbers of changes of signs at the ends of the interval in the polynomials which occur as the successive remainders in the Euclidean algorithm in the polynomial ring $\mathbb{R}[x]$ applied to $f(x)/f'(x)$.
- Sylvester recast the formula as a difference of signatures, using an expression for the signature of a tridiagonal matrix in terms of continued fractions.
- Barge and Lannes, *Suites de Sturm, indice de Maslov et périodicité de Bott* (2008) gives a modern take on the algebraic connections between Sturm sequences, the signatures of tridiagonal matrices and Bott periodicity.

Tridiagonal matrices and continued fractions

- A vector $\chi = (\chi_1, \chi_2, \dots, \chi_n) \in \mathbb{R}^n$ is **regular** if

$$\chi_k \neq 0 \ , \ \mu_k(\text{Tri}(\chi)) \neq 0 \ (k = 1, 2, \dots, n)$$

so that the tridiagonal symmetric matrix $\text{Tri}(\chi)$ is regular.

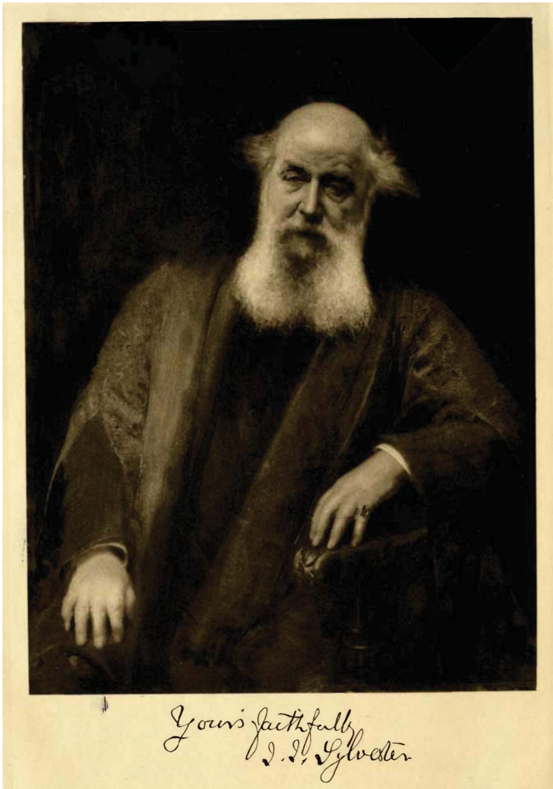
- **Theorem** (Sylvester, 1853) A tridiagonal matrix $\text{Tri}(\chi)$ for a regular $\chi \in \mathbb{R}^n$ is linearly congruent to the diagonal matrix with entries the continued fractions

$$\begin{aligned} \lambda_k(\text{Tri}(\chi)) &= \mu_k(\text{Tri}(\chi))/\mu_{k-1}(\text{Tri}(\chi)) = [\chi_k, \chi_{k-1}, \dots, \chi_1] \\ &= \chi_k - \cfrac{1}{\chi_{k-1} - \cfrac{1}{\chi_{k-2} - \ddots - \cfrac{1}{\chi_1}}} \end{aligned}$$

- The signature of $\text{Tri}(\chi)$ is

$$\tau(\text{Tri}(\chi)) = \sum_{k=1}^{n} \text{sign}(\lambda_k(\text{Tri}(\chi))) \in \{-n, -n+1, \dots, n\} \ .$$

"Aspiring to these wide generalizations, the analysis of quadratic functions soars to a pitch from whence it may look proudly down on the feeble and vain attempts of geometry proper to rise to its level or to emulate it in its flights." (1850)



Savilian Professor of Geometry, Oxford, 1883-1894

Matrices and forms

- Let $\epsilon = 1$ or -1. An **ϵ-symmetric form** (K, ϕ) is a finite-dimensional real vector space K together with a bilinear pairing
$$\phi \ : \ K \times K \to \mathbb{R} \ ; \ (x, y) \mapsto \phi(x, y)$$
such that $\phi(y, x) = \epsilon\phi(x, y) \in \mathbb{R}$ for all $x, y \in K$.
- 1-symmetric = **symmetric**, -1-symmetric = **symplectic**.
- Linear congruence classes of ϵ-symmetric $n \times n$ matrices $S = \epsilon S^*$ $\Longleftrightarrow$ isomorphism classes of ϵ-symmetric forms (K, ϕ) with $\dim(K) = n$.
- A form (K, ϕ) is **nonsingular** if the adjoint linear map
$$\phi \ : \ K \to K^* \ = \ \mathrm{Hom}_{\mathbb{R}}(K, \mathbb{R}) \ ; \ x \mapsto (y \mapsto \phi(x, y))$$
is an isomorphism. Nonsingular forms correspond to invertible matrices.
- The **hyperbolic ϵ-symmetric form** $H_\epsilon(L) = (L \oplus L^*, \phi)$ is nonsingular, with
$$\phi \ = \ \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix} \ : \ L \oplus L^* \to (L \oplus L^*)^* \ = \ L^* \oplus L \ ,$$
$$\phi((x, f), (y, g)) \ = \ g(x) + \epsilon f(y) \ .$$

From a 2ℓ-manifold to a $(-)^\ell$-symmetric form

- Will only consider oriented manifolds. The **intersection form** of a 2ℓ-manifold with boundary $(M, \partial M)$ is the $(-)^\ell$-symmetric form

$$\phi_M \ : \ H_\ell(M;\mathbb{R}) \times H_\ell(M;\mathbb{R}) \to \mathbb{R} \ ; \ (a[P], b[Q]) \mapsto ab[P \cap Q] \ (a, b \in \mathbb{R})$$

  with $[P \cap Q] \in \mathbb{Z}$ the intersection number of transverse closed ℓ-submanifolds $P^\ell, Q^\ell \subset M$.
- The adjoint linear map

$$\phi_M \ : \ H_\ell(M;\mathbb{R}) \to H_\ell(M;\mathbb{R})^* \ ; \ x \mapsto (y \mapsto \phi_M(x, y))$$

  has

$$\ker(\phi_M) \ = \ \operatorname{im}(H_\ell(\partial M;\mathbb{R})) \subseteq H_\ell(M;\mathbb{R}) \ .$$

- If $\partial M = \emptyset$ or $S^{2\ell-1}$ then ϕ_M is the **Poincaré duality isomorphism**, noting that $H_\ell(M;\mathbb{R})^* \cong H^\ell(M;\mathbb{R})$ by the universal coefficient theorem.

The signature

- An **intersection matrix** for a 2ℓ-manifold with boundary $(M, \partial M)$ is the $(-)^\ell$-symmetric $n \times n$ matrix

$$S_M \ = \ \big(\, \phi_M([P_i],[P_j]) \in \mathbb{Z}\,\big)$$

for a basis $\{[P_1],[P_2],\dots,[P_n]\} \subset H_\ell(M;\mathbb{R})$ of ℓ-submanifolds $P_i^\ell \subset M$.

- Weyl (1923) The **signature** of a $4k$-manifold with boundary $(M, \partial M)$ is the signature of the intersection symmetric $n \times n$ matrix S_M

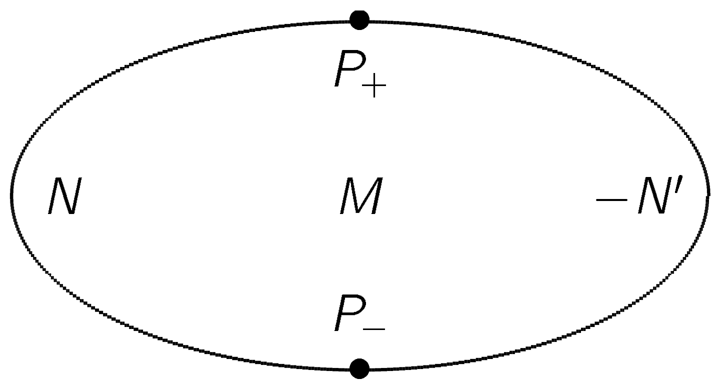
$$\tau(M) \ = \ \tau(S_M) \in \mathbb{Z}\ .$$

- **Standard examples**

$$\tau(S^{2k} \times S^{2k}) \ = \ 0\ ,\ \tau(\mathbb{CP}^{2k}) \ = \ 1\ .$$

Cobordism

- An m-**dimensional cobordism** $(M;N,N';P)$ is an m-manifold M with the boundary decomposed as $\partial M = N \cup_P -N'$ for $(m-1)$-manifolds N, N' with the same boundary $\partial N = \partial N' = P$, and $-N' = N'$ with the opposite orientation. In the diagram $P = P_+ \sqcup P_-$.



- **Proposition** (Thom 1952 for $P=\emptyset$, Novikov 1967 in general)
For $m = 4k+1$ the signature is an cobordism invariant:

$$\tau(N) - \tau(N') \; = \; \tau(\partial M) \; = \; 0 \in \mathbb{Z}$$

with $L = \ker(H_{2k}(\partial M;\mathbb{R}) \to H_{2k}(M;\mathbb{R}))$ a lagrangian of the intersection symmetric form $(H_{2k}(\partial M;\mathbb{R}),\phi_{\partial M})$.

The signature theorem

- Hirzebruch, *On Steenrod's reduced powers, the index of inertia and the Todd genus* (1953). The signature of a closed $4k$-manifold M is expressed in terms of characteristic classes by

$$\tau(M) \;=\; \int_M L(M) \;=\; \langle L_k(p_1, p_2, \ldots, p_k), [M]\rangle \in \mathbb{Z} \subset \mathbb{R}$$

with $L(M) \in H^{4k}(M;\mathbb{Q})$ the L-genus, a rational polynomial in the Pontrjagin classes $p_j = p_j(\tau_M) \in H^{4j}(M;\mathbb{Z})$ of the tangent bundle τ_M.
- Ida's Hirzebruch signature dish:

The index theorem

- Atiyah and Singer, *The index of elliptic operators* (1968)
Index theorem expressing the analytic index of an elliptic operator on a closed manifold in terms of characteristic classes.
- The signature is the index of the signature operator: the Atiyah-Singer index theorem in this case recovers the Hirzebruch signature theorem.
- The proof of the index theorem is a piece of cake:

The signature defect

- The **signature defect** of a $4k$-manifold with boundary $(M, \partial M)$ measures the extent to which the Hirzebruch signature theorem holds

$$\mathrm{def}(M) \ = \ \int_M L(M) - \tau(M) \in \mathbb{R} \ ,$$

defined whenever there is given $L(M) \in H^{4k}(M, \partial M; \mathbb{R})$ with image $L(M) \in H^{4k}(M; \mathbb{R})$.
- Exotic spheres of Milnor (1956) detected by signature defect.
- Computed by Hirzebruch and Zagier in particular cases (60's,70's).
- Atiyah, Patodi and Singer, *Spectral asymmetry and Riemannian geometry* (1974). Index theorem identifying $\mathrm{def}(M) = \eta(\partial M)$ with a spectral invariant depending on the Riemannian structure of ∂M. Generalization of the Hirzebruch signature theorem for closed manifolds.
- Atiyah, Donnelly and Singer, *η-invariants, signature defects of cusps, and values of L-functions* (1983) Topological proof of Hirzebruch's conjecture on the values of L-functions of totally real number fields.

Geometric plumbing: from a $(-)^\ell$-symmetric form to a 2ℓ-manifold

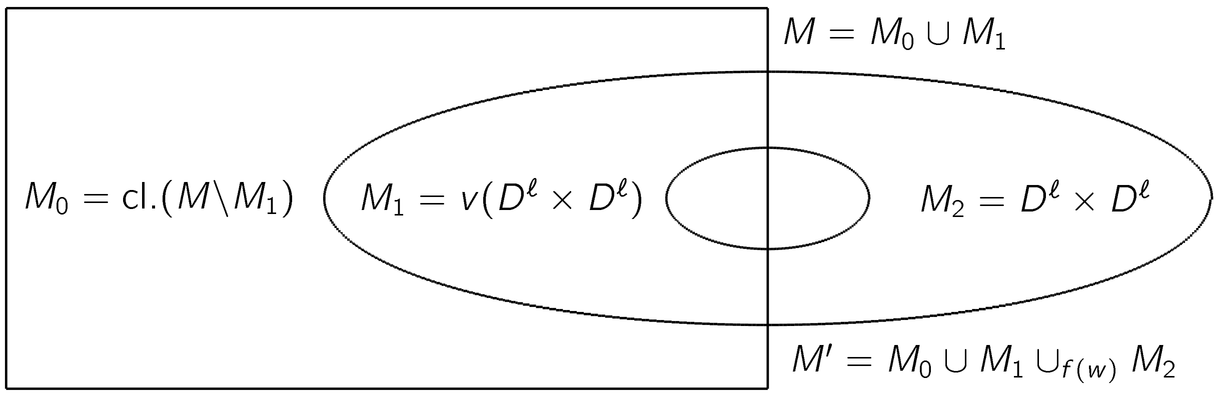
- $SO(\ell) = \{$orthogonal $\ell \times \ell$ matrices A with $\det(A) = 1\}$.
- **Input** (i) An 2ℓ-manifold with boundary $(M, \partial M)$,
  (ii) an embedding $v : (D^\ell \times D^\ell, S^{\ell-1} \times D^\ell) \subseteq (M, \partial M)$,
  (iii) a map $w : S^{\ell-1} \to SO(\ell)$, the clutching map of the ℓ-plane bundle over S^ℓ classified by $w \in \pi_{\ell-1}(SO(\ell)) = \pi_\ell(BSO(\ell))$

$$\mathbb{R}^\ell \to E(w) \to S^\ell \ = \ D^\ell \cup_{S^{\ell-1}} D^\ell \ .$$

- **Output** The **plumbed** 2ℓ-manifold with boundary

$$(M', \partial M') \ = \ (M \cup_{f(w)} D^\ell \times D^\ell, \mathrm{cl.}(\partial M \backslash S^{\ell-1} \times D^\ell) \cup D^\ell \times S^{\ell-1}) \ ,$$

$$f(w) \ : \ S^{\ell-1} \times D^\ell \to S^{\ell-1} \times D^\ell \ ; \ (x, y) \mapsto (x, w(x)(y))$$



$M_1 \cup_{f(w)} M_2 = D^\ell$-bundle of w over S^ℓ

The algebraic effect of geometric plumbing

- **Proposition** If $(M, \partial M)$ has $(-)^\ell$-symmetric intersection matrix S the geometric plumbing $(M', \partial M')$ has the $(-)^\ell$-symmetric intersection matrix given by algebraic plumbing

$$S' = \begin{pmatrix} S & v^* \\ v & \chi(w) \end{pmatrix}$$

with

$$v = v[D^\ell \times D^\ell] \in H_\ell(M, \partial M; \mathbb{R}) = H_\ell(M; \mathbb{R})^* ,$$

$$\chi(w) = \text{degree}(S^{\ell-1} \to^w SO(\ell) \to S^{\ell-1}) \in \mathbb{Z} ,$$

$$SO(\ell) \to S^{\ell-1} ; \; A \mapsto A(0, \dots, 0, 1) .$$

- $\chi(w) \in \mathbb{Z}$ is the Euler number ($= 0$ for ℓ odd) of the ℓ-plane vector bundle $w \in \pi_{\ell-1}(SO(\ell)) = \pi_\ell(BSO(\ell))$ over S^ℓ.
- If S is invertible the signatures of $(M, \partial M)$, $(M', \partial M')$ are related by

$$\tau(M') = \tau(M) + \text{sign}(\chi(w) - vS^{-1}v^*) \in \mathbb{Z} .$$

Graph manifolds

- A **graph manifold** is a 2ℓ-manifold with boundary constructed from $D^\ell \times D^\ell$ by the geometric plumbing of n ℓ-plane bundles over S^ℓ, using a graph with vertices $j = 1, 2, \ldots, n$ and weights $\chi_j \in \pi_{\ell-1}(SO(\ell))$. The weights are ℓ-plane bundles χ_j over S^ℓ.
- (Milnor 1959, Hirzebruch 1961) For $\ell \geqslant 2$ every $(-)^\ell$-symmetric $n \times n$ matrix $S = (s_{ij} \in \mathbb{Z})$ is realized by a graph 2ℓ-manifold with boundary $(M, \partial M)$ such that

  $$(H_\ell(M;\mathbb{R}), \phi_M) \;=\; (\mathbb{R}^n, S)\ .$$

  For $\ell = 2k$ and $k \neq 1, 2, 4$ need the diagonal entries $s_{jj} \in \mathbb{Z}$ to be even, since the Hopf invariant of any $S^{4k-1} \to S^{2k}$ is even (Adams).
- If the graph is a tree then for $\ell \geqslant 2$ M is $(\ell-1)$-connected, and for $\ell \geqslant 3$ M and ∂M are both $(\ell-1)$-connected.
- Rabo von Randow, *Zur Topologie von dreidimensionalen Baummannigfaltigkeiten* (1962) and Alois Scharf, *Zur Faserung von Graphenmannigfaltigkeiten* (1975)

From a $(2\ell+1)$-manifold with boundary to a lagrangian

- A **lagrangian** of an ϵ-symmetric form (K,ϕ) is a subspace $L \subseteq K$ such that $L = L^{\perp}$, i.e.

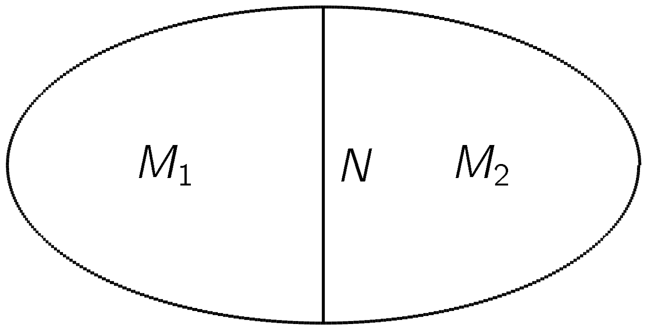
$$\phi(L,L) \;=\; \{0\} \text{ and } L \;=\; \{x \in K \,|\, \phi(x,y) = 0 \in \mathbb{R} \text{ for all } y \in L\}\ .$$

- A nonsingular ϵ-symmetric form (K,ϕ) is isomorphic to the hyperbolic form $H_\epsilon(L)$ if and only if it admits a lagrangian L.
- A nonsingular symmetric form (K,ϕ) admits a lagrangian if and only if it has signature $\tau(K,\phi) = 0 \in \mathbb{Z}$, if and only if it is isomorphic to $H_+(\mathbb{R}^n) = (\mathbb{R}^n \oplus \mathbb{R}^n, \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix})$ with $n = \dim_{\mathbb{R}}(K)/2$.
- Every nonsingular symplectic form (K,ϕ) admits a lagrangian, and is isomorphic to $H_-(\mathbb{R}^n) = (\mathbb{R}^n \oplus \mathbb{R}^n, \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix})$ with $n = \dim_{\mathbb{R}}(K)/2$.
- A $(2\ell+1)$-manifold with boundary $(M,\partial M)$ determines a lagrangian $L = \ker(H_\ell(\partial M;\mathbb{R}) \to H_\ell(M;\mathbb{R}))$ of the $(-)^\ell$-symmetric intersection form $(H_\ell(\partial M;\mathbb{R}),\phi_{\partial M})$.

From a closed $(2\ell+1)$-manifold to a formation

- An ϵ-**symmetric formation** $(K,\phi;L_1,L_2)$ is a nonsingular ϵ-symmetric form (K,ϕ) together with an ordered pair of lagrangians L_1,L_2
- For any formation $(K,\phi;L_1,L_2)$ there exists an automorphism $A:(K,\phi)\to(K,\phi)$ such that $A(L_1)=L_2$.
- A decomposition of a closed $(2\ell+1)$-manifold M

$M^{2\ell+1}=M_1\cup_N M_2$



$N^{2\ell}=M_1\cap M_2=\partial M_1=\partial M_2$

determines a $(-)^\ell$-symmetric formation $(H_\ell(N;\mathbb{R}),\phi_N;L_1,L_2)$ with lagrangians

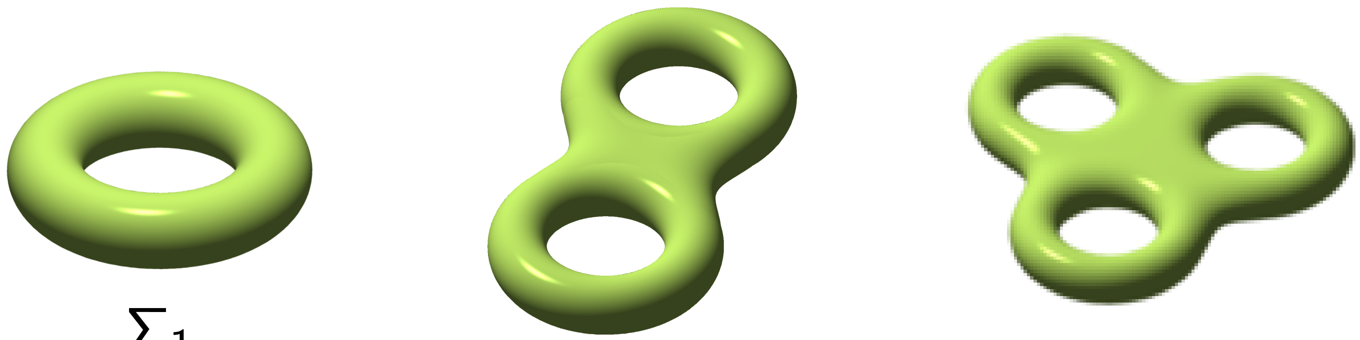
$$L_j \;=\; \ker(H_\ell(N;\mathbb{R})\to H_\ell(M_j;\mathbb{R}))\ (j=1,2)\ .$$

If $H_\ell(M_j,N;\mathbb{R})=0$ and $H_{\ell+1}(M_j;\mathbb{R})=0$ then

$$L_1\cap L_2 \;=\; H_{\ell+1}(M;\mathbb{R})\ ,\ H_\ell(N;\mathbb{R})/(L_1+L_2) \;=\; H_\ell(M;\mathbb{R})\ .$$

The symplectic group $Sp(2n)$ and automorphisms of the surfaces Σ_n

- The **symplectic group** $Sp(2n) = \mathrm{Aut}(H_-(\mathbb{R}^n))$ $(n \geqslant 1)$ consists of the invertible $2n \times 2n$ matrices A such that $A^* \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} A = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.
  Similarly for $Sp(2n;\mathbb{Z}) \subset Sp(2n)$.
- The **surface of genus** n is $\Sigma_n = \underset{n}{\#}\, S^1 \times S^1$.



Σ_1 Σ_2 Σ_3

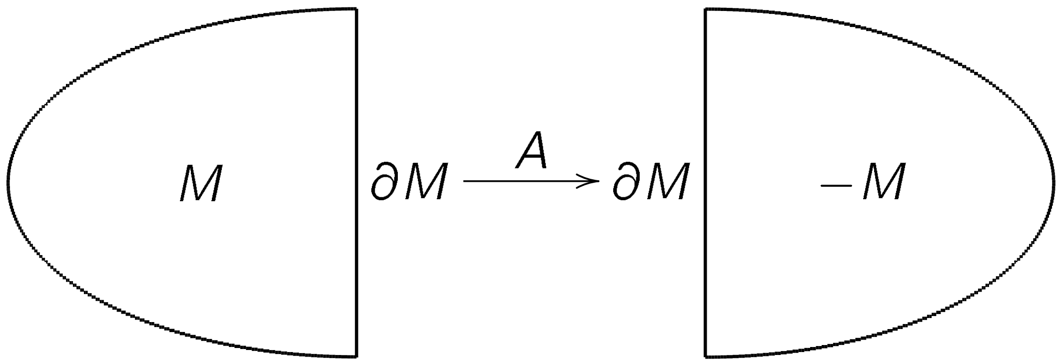
- The **mapping class group** $\Gamma_n = \pi_0(\mathrm{Aut}(\Sigma_n))$ is the group of automorphisms of Σ_n, modulo isotopy. Canonical group morphism

$$\gamma_n \ : \ \Gamma_n \to Sp(2n;\mathbb{Z}) \ ; \ (A : \Sigma_n \to \Sigma_n) \mapsto (A_* : H_1(\Sigma_n) \to H_1(\Sigma_n)) \ .$$

  Isomorphism for $n = 1$. Surjection for $n \geqslant 2$.

Twisted doubles

- A **twisted double** of an m-manifold with boundary $(M, \partial M)$ with respect to an automorphism $A : \partial M \to \partial M$ is the closed m-manifold $D(M, A) = M \cup_A -M$.



- Every closed $(2\ell+1)$-manifold is a twisted double $D(M, A)$ (non-uniquely), with an induced automorphism $A : (K, \phi) \to (K, \phi)$ of the nonsingular $(-)^\ell$-symmetric form $(K, \phi) = (H_\ell(\partial M; \mathbb{R}), \phi_{\partial M})$. The corresponding $(-)^\ell$-symmetric formation is $(K, \phi; L, A(L))$ with

$$L = \ker(K \to H_\ell(\partial M; \mathbb{R})) \subset K .$$

The Heegaard decompositions of a 3-manifold

- Heegaard (1898) Every closed 3-manifold M is a twisted double

$$M \ = \ D(\#_n S^1 \times D^2, A)$$

for some automorphism $A : \Sigma_n \to \Sigma_n$. Non-unique.

- A induces the symplectic automorphism

$$\gamma_n(A) \ = \ A_* \ : \ (H_1(\Sigma_n;\mathbb{R}),\phi_{\Sigma_n}) \ = \ H_-(\mathbb{R}^n) \to H_-(\mathbb{R}^n) \ .$$

- The symplectic formation of M with respect to the Heegaard decomposition is

$$(\, H_-(\mathbb{R}^n)\, ;\, \mathbb{R}^n \oplus \{0\}\, ,\, A_*(\mathbb{R}^n \oplus \{0\})\,) \ .$$

The modular group $SL_2(\mathbb{Z})$

- Introduced by Dedekind, *Erläuterungen zu den vorstehenden Fragmenten*, 1876. Commentary on Riemann's work on elliptic functions.
- The **modular group** $SL_2(\mathbb{Z}) = Sp(2;\mathbb{Z})$ is the group of 2×2 integer matrices
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
such that
$$\det(A) = ad - bc = 1 \in \mathbb{Z} .$$
- Every element $A \in SL_2(\mathbb{Z})$ is induced by an automorphism of the torus
$$A : \Sigma_1 = S^1 \times S^1 \to S^1 \times S^1 ; \ (e^{ix}, e^{iy}) \mapsto (e^{i(ax+by)}, e^{i(cx+dy)}) .$$
- $SL_2(\mathbb{Z}) = \Gamma_1$ is the mapping class group of the torus Σ_1.

The lens spaces

- Tietze, *Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten* (1908)
  The **lens space** is the closed, parallelizable 3-manifold defined for coprime $a, c \in \mathbb{Z}$ with $c > 0$ by
  $$L(c, a) \;=\; S^3/\mathbb{Z}_c$$
  with $S^3 = \{(u, v) \in \mathbb{C}^2 \,|\, |u|^2 + |v|^2 = 1\}$ and
  $$\mathbb{Z}_c \times \mathbb{C}^2 \to \mathbb{C}^2 \;;\; (m, (u, v)) \mapsto (\zeta^{am} u, \zeta^m v) \text{ with } \zeta = e^{2\pi i/c} \;.$$
- $\pi_1(L(c, a)) = \mathbb{Z}_c$, $H_*(L(c, a); \mathbb{R}) = H_*(S^3; \mathbb{R})$.
- The lens space has a genus 1 Heegaard decomposition
  $$L(c, a) \;=\; S^1 \times D^2 \cup_A S^1 \times D^2$$
  for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, corresponding to the symplectic formation $(H_-(\mathbb{R}); \mathbb{R} \oplus \{0\}, L)$ with
  $$L \;=\; A(\mathbb{R} \oplus \{0\}) \;=\; \{(ax, cx) \,|\, x \in \mathbb{R}\} \subset \mathbb{R} \oplus \mathbb{R} \;.$$

The Hirzebruch-Jung resolution of cyclic surface singularities I.

- For $A \in SL_2(\mathbb{Z})$ with $c \neq 0$ the Euclidean algorithm gives a regular $\chi \in \mathbb{Z}^n$ with $|\chi_k| > 1$, such that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \chi_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \chi_2 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} \chi_n & -1 \\ 1 & 0 \end{pmatrix},$$

$$a/c = [\chi_1, \chi_2, \ldots, \chi_n] = \chi_1 - \cfrac{1}{\chi_2 - \cfrac{1}{\chi_3 - \ddots - \cfrac{1}{\chi_n}}}$$

- The factorization is realized by a graph 4-manifold $M(\chi)$ with $\partial M(\chi) = L(c, a)$, intersection matrix $\mathrm{Tri}(\chi)$. The plumbing tree is the graph A_n weighted by $\chi_k \in \pi_1(SO(2)) = \pi_1(S^1) = \mathbb{Z}$

$$A_n : \overset{\chi_1}{\bullet} \text{———} \overset{\chi_2}{\bullet} \text{———} \overset{\chi_3}{\bullet} \text{———} \cdots \text{———} \overset{\chi_{n-1}}{\bullet} \text{———} \overset{\chi_n}{\bullet}$$

noting the diffeomorphism $S^1 \to SO(2); e^{i\theta} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

The Hirzebruch-Jung resolution of cyclic surface singularities II.

- The 4-manifold $M(\chi)$ resolves the singularity at $(0,0,0)$ of the 2-dimensional complex space
$$\{(w, z_1, z_2) \in \mathbb{C}^3 \mid w^c = z_1(z_2)^{c-a}\} \ .$$
- Jung, *Darstellung der Funktionen eines algebraischen Körpers zweier unabhängigen Veränderlichen* x, y *in der Umgebung* $x = a, y = b$ (1909)
- Hirzebruch, *Über vierdimensionale Riemannsche Flächen mehrdeutiger analytischer Funktionen von zwei komplexen Veränderlichen* (1952).
- The signature of $M(\chi)$ is
$$\tau(M(\chi)) = \tau(\text{Tri}(\chi)) = \sum_{k=1}^{n} \text{sign}([\chi_k, \chi_{k-1}, \ldots, \chi_1]) = \sum_{k=1}^{n} \text{sign}(\chi_k) \in \mathbb{Z}.$$
- Hirzebruch and Mayer, *O*(n)*-Mannigfaltigkeiten, exotische Sphären und Singularitäten* (1968)
- Hirzebruch, Neumann and Koh, *Differentiable manifolds and quadratic forms* (1971)

The sawtooth function (( ))

- Used by Dedekind (1876) to count $\pm 2\pi$ jumps in the complex logarithm

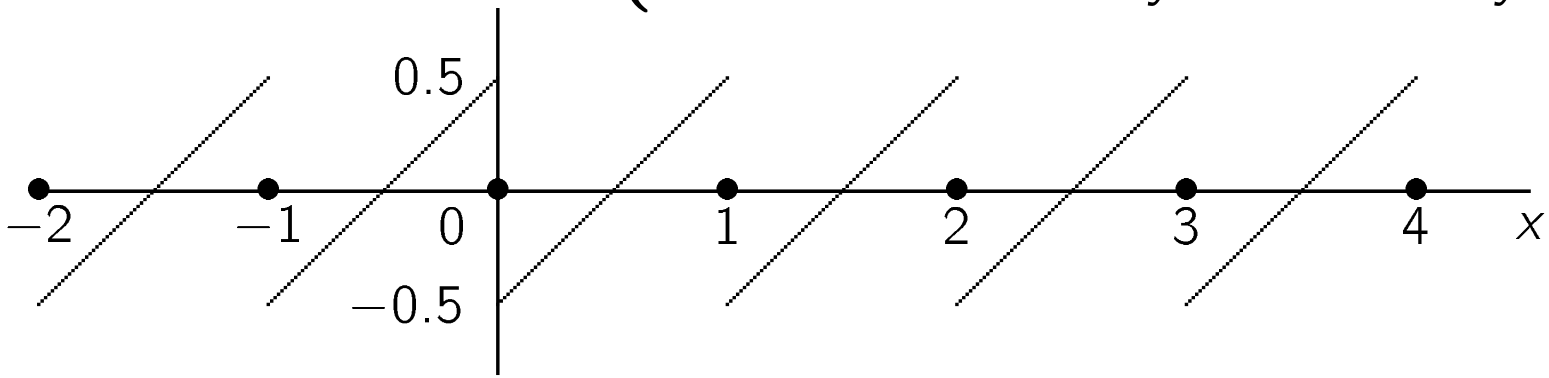
$$\log(re^{i\theta}) \; = \; \log(r) + i(\theta + 2n\pi) \in \mathbb{C} \;\; (n \in \mathbb{Z}) \; .$$

- The **sawtooth function** $((\;)) : \mathbb{R} \to [-1/2, 0)$ is defined by

$$((x)) \; = \; \begin{cases} \{x\} - 1/2 & \text{if } x \in \mathbb{R} \backslash \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$

with $\{x\} \in [0, 1)$ the fractional part of $x \in \mathbb{R}$. Nonadditive:

$$((x)) + ((y)) - ((x+y)) \; = \; \begin{cases} -1/2 & \text{if } 0 < \{x\} + \{y\} < 1 \\ 1/2 & \text{if } 1 < \{x\} + \{y\} < 2 \\ 0 & \text{if } x \in \mathbb{Z} \text{ or } y \in \mathbb{Z} \text{ or } x + y \in \mathbb{Z} \; . \end{cases}$$



Dedekind sums and signatures

- The **Dedekind sum** for $a, c \in \mathbb{Z}$ with $c \neq 0$ is

$$s(a,c) \;=\; \sum_{k=1}^{|c|-1} ((\frac{k}{c}))((\frac{ka}{c})) \;=\; \frac{1}{4|c|}\sum_{k=1}^{|c|-1} \cot(\frac{k\pi}{c})\cot(\frac{ka\pi}{c}) \in \mathbb{Q}\ .$$

- Hirzebruch, *The signature theorem: reminiscences and recreations* (1971) and *Hilbert modular surfaces* (1973)
- Hirzebruch and Zagier, *The Atiyah-Singer theorem and elementary number theory* (1974)
- Kirby and Melvin, *Dedekind sums, μ-invariants and the signature cocycle* (1994) For any regular sequence $\chi = (\chi_1, \chi_2, \dots, \chi_n) \in \mathbb{Z}^n$ the signature defect of $M(\chi)$ is

$$\tau(\mathrm{Tri}(\chi)) - (\sum_{j=1}^{n} \chi_j)/3 \;=\; \begin{cases} b/3d & \text{if } c = 0 \\ (a+d)/3 - 4\mathrm{sign}(c)s(a,c) & \text{if } c \neq 0\ . \end{cases}$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \chi_1 & -1 \\ 1 & 0 \end{pmatrix}\dots\begin{pmatrix} \chi_n & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$.

The tailoring of topological pants I.

- **Input**: Three n-manifolds N_0, N_1, N_2 with the same boundary

$$\partial N_0 \;=\; \partial N_1 \;=\; \partial N_2 \;=\; P\ .$$

  The diffeomorphisms $f_j : \partial N_j \to \partial N_{j-1}$ ($j(\bmod 3)$) satisfy $f_1 f_2 f_3 = \mathrm{Id}$.
- **Output**: The **pair of pants** $(n+1)$-manifold

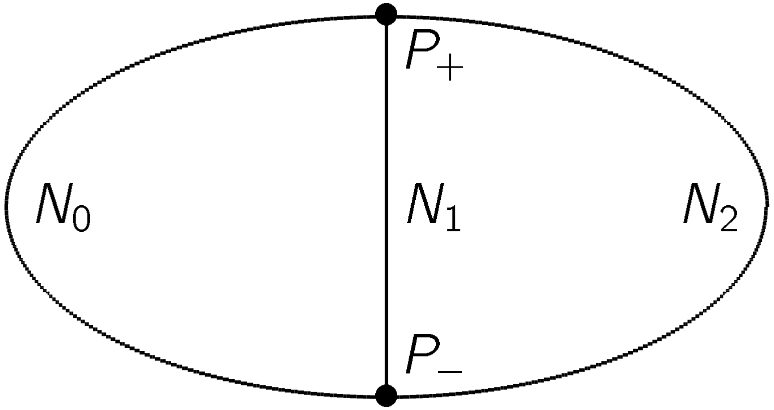
$$Q \;=\; Q(P, N_0, N_1, N_2) \;=\; (N_0 \times I \sqcup N_1 \times I \sqcup N_2 \times I)/\sim\ ,$$

$$(a_j, b_j) \sim (f_j(a_j), 1 - b_j)\ (a_j \in \partial N_j, b_j \in [0, 1/2])$$

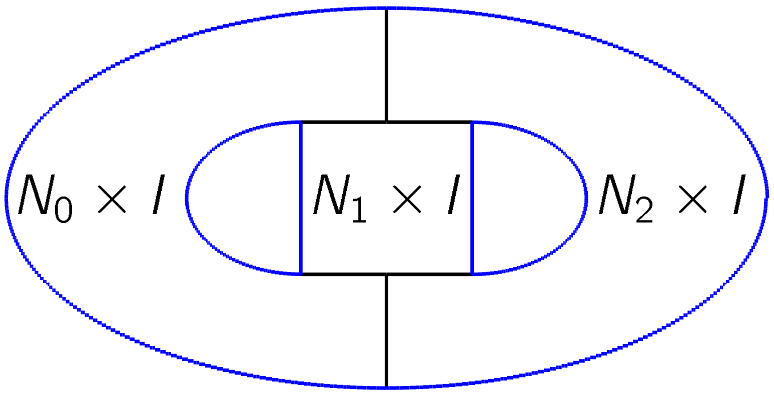
  with boundary $\partial Q = (N_0 \cup_P N_1) \sqcup (N_1 \cup_P N_2) \sqcup (N_2 \cup_P N_0)$.
- Ordinary pair of 2-dimensional pants is the special case $n = 1$, $N_j = D^1$, $P = S^0$ used in Atiyah, *Topological quantum field theory* (1988).

The tailoring of topological pants II.

$$P = P_+ \cup P_- = \partial N_0 = \partial N_1 = \partial N_2$$



$$\text{Tailor's dummy} = N = (N_0 \sqcup N_1 \sqcup N_2)/\sim$$



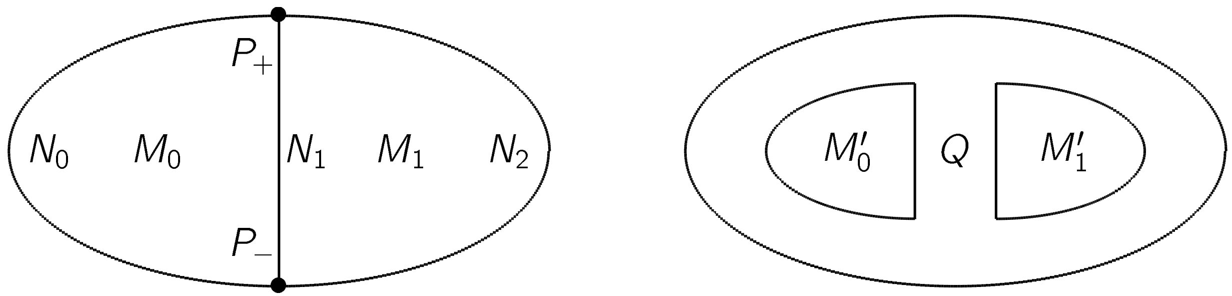
$$\text{Pair of pants } Q = ((N_0 \times I) \sqcup (N_1 \times I) \sqcup (N_2 \times I))/\sim$$

$$\partial Q = (N_0 \cup_P N_1) \sqcup (N_1 \cup_P N_2) \sqcup (N_2 \cup_P N_0) \text{ in blue}$$

The Wall non-additivity of the signature I.

- Wall *The non-additivity of the signature* (1969).
  The signature of the union $M = M_0 \cup M_1$ of $4k$-dimensional cobordisms $(M_0; N_0, N_1; P)$, $(M_1; N_1, N_2; P)$ is

$$\tau(M) \ = \ \tau(M_0) + \tau(M_1) + \tau(P; N_0, N_1, N_2) \in \mathbb{Z} \ .$$



$M = M_0 \cup M_1 = M_0' \cup Q \cup M_1'$

- M is a manifold. The union $N = N_0 \cup N_1 \cup N_2 \subset M$ is a stratified set, not a manifold. The non-additivity term is the signature

$$\tau(P; N_0, N_1, N_2) \ = \ \tau(Q) \in \mathbb{Z}$$

  of the pair of pants $Q = Q(P; N_0, N_1, N_2)$, a neighbourhood of $N \subset M$. The complement $\mathrm{cl.}(M \backslash Q) = M_0' \cup M_1'$ is the union of disjoint copies M_0', M_1' of M_0, M_1.

The Wall non-additivity of the signature II.

- $\tau(P;N_0,N_1,N_2)=\tau(Q)$ is the triple signature $\tau(K,\phi;L_0 by,L_1,L_2)$ of the nonsingular symplectic form $(K,\phi)=(H_{2k-1}(P;\mathbb{R}),\phi_P)$ with respect to the three lagrangians

$$L_j \ = \ \ker(K\to H_{2k-1}(N_j;\mathbb{R}))\ (j=0,1,2)\ .$$

- The **triple signature** $\tau(K,\phi;L_0,L_1,L_2)=\tau(V,\psi)\in\mathbb{Z}$ is the signature of the nonsingular symmetric form (V,ψ) defined by

$$V \ = \ \frac{\{(x,y,z)\in L_0\oplus L_1\oplus L_2\,|\,x+y+z=0\in K\}}{\{(a-b,b-c,c-a)\,|\,a\in L_2\cap L_0, b\in L_0\cap L_1, c\in L_1\cap L_2\}}\ ,$$

$$\psi(x,y,z)(x',y',z') \ = \ \phi(x,y')\in\mathbb{R}\ .$$

- **Example** The lagrangians of $H_-(\mathbb{R})$ are the 1-dimensional subspaces

$$L(\theta) \ = \ \{(r\cos\theta,r\sin\theta)\,|\,r\in\mathbb{R}\}\subset\mathbb{R}^2\ (\theta\in[0,\pi))\ .$$

The triple signature jumps by ±1 at $\theta_j-\theta_{j+1}\in\pi\mathbb{Z}$, for $j(\bmod 3)$

$\tau(H_-(\mathbb{R});L(\theta_0),L(\theta_1),L(\theta_2))=\mathrm{sign}(\sin(\theta_0-\theta_1)\sin(\theta_1-\theta_2)\sin(\theta_2-\theta_0))$.

The Maslov index

- The Wall non-additivity invariant has been identified with the universal jump-counting invariant

$$\textbf{Maslov index} \in \mathbb{Z}$$

  of quantum mechanics, symplectic geometry, dynamical systems and knot theory. Related to spectral flow.
- Arnold *On a characteristic class entering into conditions of quantization* (1969)
- Leray *Lagrangian analysis and quantum mechanics* (1981)
- Arnold *Sturm theorems and symplectic geometry* (1985)
- Kashiwara and Schapira *Sheaves on manifolds* (1994)
- The Maslov index website

  http://www.maths.ed.ac.uk/~aar/maslov.htm

  has many more references.

The multiplicativity of the signature

- The tensor product $S \otimes T = (s_{ij}t_{k\ell})$ of symmetric matrices $S = (s_{ij})$, $T = (t_{k\ell})$ has signature

$$\tau(S \otimes T) \;=\; \tau(S)\tau(T) \in \mathbb{Z}\ .$$

- The signature of a product of closed manifolds is

$$\tau(X \times F) \;=\; \tau(X)\tau(F) \in \mathbb{Z}$$

  with $\tau = 0$ if $\dim \not\equiv 0 \,(\mathrm{mod}\, 4)$.
- Since $\tau(M) \equiv \chi(M)(\mathrm{mod}\, 2)$, for any fibre bundle $F \to M^{4k} \to X$

$$\tau(M) \;\equiv\; \tau(X)\tau(F)\ (\mathrm{mod}\, 2)\ .$$

- Chern, Hirzebruch and Serre, *On the index of a fibered manifold* (1957). If $\pi_1(X)$ acts trivially on $H_*(F;\mathbb{R})$ then $\tau(M) = \tau(X)\tau(F) \in \mathbb{Z}$.
- Hambleton, Korzeniewski and Ranicki, *The signature of a fibre bundle is multiplicative mod 4* (2007) For any fibre bundle $F \to M^{4k} \to X$

$$\tau(M) \;\equiv\; \tau(X)\tau(F)\ (\mathrm{mod}\, 4)\ .$$

The non-multiplicativity of the signature for fibre bundles

- Kodaira, *A certain type of irregular algebraic surfaces* (1967)
  Fibre bundles $F^2 \to M^4 \to X^2$ with
  $$\tau(M) - \tau(X)\tau(F) \neq 0 \in 4\mathbb{Z} \subset \mathbb{Z} .$$
- Hirzebruch, *The signature of ramified coverings* (1969)
  Analysis of non-multiplicativity using the signature of branched covers, and the Atiyah-Singer index theorem.
- Atiyah, *The signature of fibre-bundles* (1969) A characteristic class formula for the signature of a fibre bundle $F^{2\ell} \to M^{4k} \to X$
  $$\tau(M) \ = \ \langle \mathrm{ch}(\mathrm{Sign}) \cup \widetilde{L}(X), [X] \rangle \in \mathbb{Z} \subset \mathbb{R}$$
  with $\mathrm{Sign} = \{\tau_K(H_\ell(F_x;\mathbb{C}), \phi_{F_x}) \,|\, x \in X\}$ the virtual bundle of the topological K-theory signatures of hermitian forms, such that $(H_*(M;\mathbb{C}), \phi_M) = (H_*(X;\mathrm{Sign}), \phi_X)$ with $\mathrm{ch}(\mathrm{Sign}) \in H^{2*}(X;\mathbb{C})$ the Chern character, and $\widetilde{L}(X) \in H^{4*}(X;\mathbb{Q})$ a modified $\mathcal{L}$-genus.

Central extensions

- A **central extension** of a group E is an exact sequence

$$\{1\} \to C \to D \to E \to \{1\}$$

with $cd = dc \in D$ for all $c \in C$, $d \in D$. In particular, C is abelian.
- Central extensions with prescribed C, E are classified by the **cohomology group**

$$H^2(E; C) \;=\; Z^2(E; C)/B^2(E; C) \;.$$

- A **cocycle** $f \in Z^2(E; C)$ is a function $f : E \times E \to C$ such that

$$f(x, y) - f(y, z) \;=\; f(xy, z) - f(x, yz) \in C \;(x, y, z \in E)$$

classifying $D = C \times_f E$, $(a, x)(b, y) = (a + b + f(x, y), xy)$.
- The **coboundary** $\delta g \in B^2(E; C)$ of a function $g : E \to C$ is the cocycle

$$\delta g \;:\; E \to C \;;\; x \mapsto g(x) + g(y) - g(xy) \;.$$

- Central extensions with $C = \mathbb{Z}$ are of central importance in both the algebraic and geometric aspects of the signature.

Infinite cyclic covers

- $1 \in H^1(S^1;\mathbb{Z}) = \mathbb{Z}$ classifies the central extension $\mathbb{Z} \to \mathbb{R} \to S^1$ with $p : \mathbb{R} \to S^1; x \mapsto e^{2\pi ix}$ the universal infinite cyclic cover,
- Let $f : G \to S^1$ be a morphism of topological groups. The pullback is the central extension $\mathbb{Z} \to \overline{G} \to G$ classified by $f^*(1) \in H^2(G;\mathbb{Z})$ with

$$q : \overline{G} = f^*\mathbb{R} = \{(x,y) \in \mathbb{R}\times G \,|\, p(x) = f(y) \in S^1\} \to G \; ; \; (x,y) \mapsto y$$

with $\overline{G}$ also a topological group. A section $s : G \to \overline{G}$ of q gives a cocycle for $f^*(1) \in H^2(G;\mathbb{Z})$

$$c_s \; : \; G \times G \to \mathbb{Z} \; ; \; (x,y) \mapsto s(x)s(y)s(xy)^{-1} \; .$$

- The section of p

$$s \; : \; S^1 \to \mathbb{R} \; ; \; e^{2\pi ix} \mapsto \log(e^{2\pi ix})/2\pi i \; = \; \{x\}$$

determines the cocycle $c_s : S^1 \times S^1 \to \mathbb{Z}$ for $1 \in H^2(S^1;\mathbb{Z}) = \mathbb{Z}$ with

$$c_s(e^{2\pi ix}, e^{2\pi iy}) = \{x\} + \{y\} - \{x+y\} = \begin{cases} 0 & \text{if } 0 \leqslant \{x\} + \{y\} < 1 \\ 1 & \text{if } 1 \leqslant \{x\} + \{y\} < 2 \; . \end{cases}$$

The Meyer signature cocycle

- Let $(K,\phi) = H_-(\mathbb{R}^n)$. For $A, B \in \text{Aut}(K,\phi) = Sp(2n)$ let

  $\tau(A,B) \ = \ \tau(K\oplus K, \phi\oplus -\phi; (1\oplus A)\Delta_K, (1\oplus B)\Delta_K, (1\oplus AB)\Delta_K) \in \mathbb{Z}$.

- W.Meyer *Die Signatur von lokalen Koeffizientensystem und Faserbündeln* (1972) The triple signature function

  $$c_n \ : \ Sp(2n)\times Sp(2n) \to \mathbb{Z} \ ; \ (A,B) \mapsto \tau(A,B)$$

  is a cocycle $c_n \in Z^2(Sp(2n);\mathbb{Z})$.

- The signature of the total space of a surface bundle $\Sigma_n \to M^4 \to X^2$ with $(H_1(\Sigma_n;\mathbb{R}), \phi_{\Sigma_n}) = H_-(\mathbb{R}^n)$ is

  $$\tau(M) \ = \ -\langle \Gamma^*[c_n], [X]\rangle \in \mathbb{Z}$$

  with $[c_n] \in H^2(Sp(2n);\mathbb{Z})$ the signature class, and $\Gamma : \pi_1(X) \to Sp(2n)$ the characteristic map.

- The pullback of c_n generates $H^2(\text{mapping class group of } \Sigma_n;\mathbb{Q}) = \mathbb{Q}$.

The Atiyah signature cocycle I.

- Atiyah, *The logarithm of the Dedekind η-function* (1987).
- The Lie group defined for $p, q \geqslant 0$ by

$$U(p,q) \;=\; \{\text{automorphisms of the hermitian form } (\mathbb{C}^p, I_p) \oplus (\mathbb{C}^q, -I_q)\}$$

  consists of the invertible $(p+q)\times(p+q)$ matrices $A = (a_{jk} \in \mathbb{C})$ such that $A^*(I_p \oplus -I_q)A = I_p \oplus -I_q$, with $A^* = (\overline{a}_{kj})$.
- Given a surface with boundary (X, Y) and a group morphism $\Gamma : \pi_1(X) \to U(p,q)$ there is a signature (Lusztig)

$$\tau(X,\Gamma) \;=\; \tau(H_1(X;\Gamma),\phi_X) \in \mathbb{Z}\ .$$

- Let $(X_2, Y_2) = (\text{cl.}(S^2 \backslash (\bigsqcup_3 D^2)), \bigsqcup_3 S^1)$ be the pair of pants, with $\pi_1(X_2) = \mathbb{Z} * \mathbb{Z}$. The cocycle $c_{p,q} \in Z^2(U(p,q);\mathbb{Z})$ defined by

$$c_{p,q} \;:\; U(p,q)\times U(p,q) \to \mathbb{Z}\ ;\ (A,B) \mapsto \tau(H_1(X_2;(A,B)),\phi_{X_2})$$

  is such that $\tau(X,\Gamma) = -\langle \Gamma^*(c_{p,q}), [X]\rangle \in \mathbb{Z}$ for any (X, Y), Γ.
- $c_{n,n}$ restricts on $Sp(2n) \subset U(n,n)$ to the Meyer cocycle $c_n \in Z^2(Sp(2n);\mathbb{Z})$.

The Atiyah signature cocycle II.

- The signature class $[c_{p,q}] \in H^2(U(p,q);\mathbb{Z}) = \mathrm{Hom}(\pi_1(U(p,q)),\mathbb{Z})$ is given by

  $$\pi_1(U(p,q)) \;=\; \pi_1(U(p))\times\pi_1(U(q)) \;=\; \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}\;;\;(x,y)\mapsto 2x-2y\;.$$

- $c_{1,0} \in Z^2(S^1;\mathbb{Z})$ is the cocycle on $U(1,0) = U(1) = S^1$

  $$c_{1,0}\;:\;S^1\times S^1 \to \mathbb{Z}\;;\;(e^{2\pi ix},e^{2\pi iy})\mapsto 2(((x))+((y))-((x+y)))$$

  classifying the central extension $\mathbb{Z}\to\mathbb{R}\times\mathbb{Z}_2\to S^1$ with

  $$\mathbb{Z}\to\mathbb{R}\times\mathbb{Z}_2\;;\;m\mapsto(m/2,m(\mathrm{mod}\,2))\;,$$
  $$\mathbb{R}\times\mathbb{Z}_2\to S^1\;;\;(x,r)\mapsto e^{2\pi i(x-r/2)}\;(r=0,1)\;.$$

- With real coefficients $-c_{1,0} = d\eta \in B^2(S^1;\mathbb{R})$ is the coboundary of

  $$\eta\;:\;S^1\to\mathbb{R}\;;\;e^{2\pi ix}\mapsto -2((x)) = \begin{cases} 1-2\{x\} & \text{if } x\notin\mathbb{Z} \\ 0 & \text{if } x\in\mathbb{Z}\;. \end{cases}$$

  The simplest evaluation of the Atiyah-Patodi-Singer η-invariant.

The signature extension

- The pullback $\gamma_n^*(c_{n,n}) \in H^2(\Gamma_n;\mathbb{Z})$ classifies the **signature extension**

$$\mathbb{Z} \to \widehat{\Gamma}_n \to \Gamma_n$$

  of the mapping class group Γ_n.
- Atiyah, *On framings of 3-manifolds* (1990)
  Every 3-manifold N has a canonical 2-framing α, i.e. a trivialization of $\tau_N \oplus \tau_N$, characterized by the property that for any 4-manifold M with $\partial M = N$ the signature defect is

$$\operatorname{def}(M) \ = \ 0 \ .$$

- Interpretation of $\widehat{\Gamma}_n$ in terms of the canonical 2-framing.
- The case $n = 1$, $\Gamma_1 = SL_2(\mathbb{Z})$ of particular importance in string theory and Jones-Witten theory.

Complex structures

- The **unitary group** is

  $$U(n) \ = \ U(n,0) \ = \ \{\text{automorphisms of the hermitian form } (\mathbb{C}^n, I_n)\} \ .$$

- A **complex structure** on a nonsingular symplectic form (K,ϕ) is an automorphism $J : K \to K$ such that
  (i) $J^2 = -I : K \to K$,
  (ii) the symmetric form

  $$\phi J \ : \ K \times K \to \mathbb{R} \ ; \ (x,y) \mapsto \phi(x,Jy)$$

  is positive definite.
- A choice of orthonormal basis gives isomorphism $(K,\phi,J) \cong (\mathbb{C}^n, I_n, i)$.
- **Example** The hyperbolic symplectic form $H_-(\mathbb{R}^n)$ has the standard complex structure $J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} : \mathbb{R}^n \oplus \mathbb{R}^n \to \mathbb{R}^n \oplus \mathbb{R}^n$.

  The vector space isomorphism

  $$\mathbb{R}^n \oplus \mathbb{R}^n \to \mathbb{C}^n \ ; \ (x,y) \mapsto x + iy$$

  defines an isomorphism $(H_-(\mathbb{R}^n), J_n) \cong (\mathbb{C}^n, I_n)$.

The space of lagrangians in $H_-(\mathbb{R}^n)$

- Let $\Lambda(n)$ be the space of lagrangians L in $H_-(\mathbb{R}^n)$.
- Arnold, *On a characteristic class entering into conditions of quantization* (1967). The function
$$U(n)/O(n) \to \Lambda(n)\ ;\ A \mapsto A(\mathbb{R}^n \oplus 0)$$
is diffeomorphism, identifying
$$O(n)\ =\ \{A \in U(n)\,|\,A(\mathbb{R}^n \oplus \{0\}) = \mathbb{R}^n \oplus \{0\}\}\ .$$
- The square of the determinant
$$\det{}^2\ :\ \Lambda(n)\ =\ U(n)/O(n) \to S^1\ ;\ A \mapsto \det(A)^2$$
induces an isomorphism of groups
$$\pi_1(\Lambda(n)) \xrightarrow{\cong} \pi_1(S^1) = \mathbb{Z}; (\omega : S^1 \to \Lambda(n)) \mapsto \mathrm{degree}(\det{}^2 \circ \omega : S^1 \to S^1)$$
given geometrically by the **Maslov index**.
- The triple signature $\tau(H_-(\mathbb{R}^n); L_0, L_1, L_2) \in \mathbb{Z}$ is the Maslov index of a loop $S^1 \to \Lambda(n)$ passing through $L_0, L_1, L_2 \in \Lambda(n)$.

The Maslov index (again)

- Cappell, Lee and Miller, *On the Maslov index* (1994)
  (i) For any nonsingular symplectic form (K, ϕ) with complex structure J and lagrangians L_0, L_1 there exists an automorphism $A : (K, \phi) \to (K, \phi)$ such that $AJ = JA$ and $A(L_0) = L_1 \subset K$, with eigenvalues $e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n} \in S^1$ $(0 \leqslant \theta_j < \pi)$. The **Maslov index**

$$\eta_J(K, \phi, L_0, L_1) \;=\; \sum_{j=1}^{n} \eta(e^{2i\theta_j}) \;=\; \sum_{j=1, \theta_j \neq 0}^{n} (1 - 2\{\theta_j/\pi\}) \in \mathbb{R}$$

  is the η-invariant of the first order elliptic operator $-J\dfrac{d}{dt}$ on the functions $f : [0,1] \to K$ such that $f(0) \in L_0$, $f(1) \in L_1$.
- (ii) The Maslov index is a real coboundary for the triple signature

$$\begin{aligned} &\tau(K, \phi; L_0, L_1, L_2) \\ &= \; \eta_J(K, \phi; L_0, L_1) + \eta_J(K, \phi; L_1, L_2) + \eta_J(K, \phi; L_2, L_0) \in \mathbb{Z} \subset \mathbb{R} \end{aligned}$$

  for any complex structure J on (K, ϕ).

The real signature

- **Definition** The **real signature** of a $4k$-dimensional relative cobordism $(M^{4k}; N_0, N_1; P)$ with respect to a complex structure J on $(H_{2k-1}(P;\mathbb{R}), \phi_P)$ is

$$\begin{aligned}&\tau_J(M; N_0, N_1; P)\\ &= \tau(H_{2k}(M;\mathbb{R}), \phi_M) + \tau_J(H_{2k-1}(P;\mathbb{R}), \phi_P); L_0, L_1) \in \mathbb{R}\end{aligned}$$

  with $L_j = \ker(H_{2k-1}(P;\mathbb{R}) \to H_{2k-1}(N_j;\mathbb{R}))$.
- **Proposition** The real signature of the union of $4k$-dimensional relative cobordisms $(M_0; N_0, N_1; P)$, $(M_1; N_1, N_2; P)$ is the sum of the real signatures

$$\tau_J(M_0 \cup_{N_1} M_1; N_0, N_2; P) = \tau_J(M_0; N_0, N_1; P) + \tau_J(M_1; N_1, N_2; P) \in \mathbb{R}\,.$$