THE MOD 8 SIGNATURE OF A SURFACE BUNDLE

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Report on a joint project with

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Dedicated to the memory of Fritz Hirzebruch

on his 89th birthday

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Introduction

- The **signature** of an oriented m-dimensional manifold with boundary $(M, \partial M)$ is

$$\sigma(M) = \begin{cases} \text{signature } (H^{m/2}(M), \phi) & \text{if } m \equiv 0 (\text{mod} 4) \\ 0 & \text{otherwise,} \end{cases}$$

  $\phi : H^{m/2}(M) \times H^{m/2}(M) \to \mathbb{Z}$ symmetric intersection form.
- The non-multiplicativity of σ for a fibre bundle $F \to E \to B$

$$\sigma(E) - \sigma(B)\sigma(F) \in \mathbb{Z}$$

  has been studied for 60 years: Chern, Hirzebruch and Serre (1956), Kodaira (1969), Atiyah (1970), Hirzebruch (1970), Meyer (1972), Hambleton, Korzeniewski and R. (2005) ...
- Particularly interesting for a surface bundle

$$F = \Sigma_g = \mathop{\#}_{g} S^1 \times S^1 \to E \to B = \Sigma_h$$

  with $\sigma(\Sigma_g) = 0$ by definition. In general, $\sigma(E) \neq 0 \in \mathbb{Z}$.

The Meyer signature class

- In his 1972 Bonn thesis Werner Meyer (a student of Hirzebruch) constructed the **signature class**

$$\tau \in H^2(\mathrm{Sp}(2g,\mathbb{Z});\mathbb{Z}) \ .$$

- The signature of a surface bundle $\Sigma_g \to E \to \Sigma_h$ is the evaluation

$$\sigma(E) \ = \ \langle f^*\tau, [\Sigma_h]\rangle \in \mathbb{Z}$$

with

$$f : \pi_1(\Sigma_h) \to \mathrm{Sp}(2g,\mathbb{Z}) = \mathrm{Aut}_{\mathbb{Z}}(H^1(\Sigma_g),\phi)$$

the monodromy action, and

$$\phi \ : \ H^1(\Sigma_g) \times H^1(\Sigma_g) \to \mathbb{Z};\ (x,y) \mapsto \langle x \cup y, [\Sigma_g]\rangle \ .$$

the nonsingular symplectic intersection form over $\mathbb{Z}$.

Divisibility by 4, but not by 8 in general

- Meyer also constructed an explicit cocycle for the signature class τ, and computed

$$\tau \;=\; 4 \in H^2(\mathrm{Sp}(2g,\mathbb{Z});\mathbb{Z}) \;=\; \begin{cases} \mathbb{Z}_{12} & \text{if } g=1 \\ \mathbb{Z}\oplus\mathbb{Z}_2 & \text{if } g=2 \\ \mathbb{Z} & \text{if } g \geqslant 3. \end{cases}$$

- The signature of $\Sigma_g \to E \to \Sigma_h$ is divisible by 4

$$\sigma(E) \in 4\mathbb{Z} \subset \mathbb{Z}$$

- Every multiple of 4 arises as $\sigma(E)$ for some E.
- The image of $\tau/4$ in $H^2(\mathrm{Sp}(2g,\mathbb{Z});\mathbb{Z}_2) = \mathbb{Z}_2$ $(g \geqslant 4)$ determines the mod 8 signature

$$\sigma(E) = \langle f^*\tau, [\Sigma_h]\rangle \in 4\mathbb{Z}/8\mathbb{Z} \;=\; \mathbb{Z}_2 \;.$$

- Carmen Rovi (Edinburgh Ph.D. thesis, 2015) identified $\sigma(E)/4 \in \mathbb{Z}_2$ with an Arf-Kervaire invariant.

The mod 8 signature and group cohomology

- **Problem** Does there exist a class $\tau_k \in H^2(\mathrm{Sp}(2g,\mathbb{Z}_k);\mathbb{Z}_8)$ for the mod 8 signature for some $k \geqslant 2$, such that

$$\tau_k \ = \ p_k^*[\tau] \ = \ 4 \in H^2(\mathrm{Sp}(2g,\mathbb{Z}_k);\mathbb{Z}_8) \ = \ \mathbb{Z}_8 \ ?$$

  with $p_k = \text{projection} : \mathbb{Z} \to \mathbb{Z}_k$. Posed for $k=2$ by Klaus and Teichner.

- If there exists such a class τ_k then the mod 8 signature

$$\sigma(E) \ = \ \langle f_k^*\tau_k, [\Sigma_h]\rangle \in 4\mathbb{Z}_8 \subset \mathbb{Z}_8$$

  depends only on the mod k monodromy action

$$f_k \ : \ \pi_1(\Sigma_h) \to \mathrm{Sp}(2g,\mathbb{Z}) \to \mathrm{Sp}(2g,\mathbb{Z}_k) \ .$$

- $k=2$ **will not do**, since $H^2(\mathrm{Sp}(2g,\mathbb{Z}_2);\mathbb{Z}_8) = 0$ $(g \geqslant 4)$.

The mod 8 signature class

- **Theorem 1** (BCRR, 2016) $k = 4$ **will do.** The **mod 8 signature class**

$$\tau_4 \ = \ 4 \in H^2(\mathrm{Sp}(2g,\mathbb{Z}_4);\mathbb{Z}_8) \ = \ \mathbb{Z}_8$$

is such that

$$\sigma(E) \ = \ \langle f_4^* \tau_4, [\Sigma_h] \rangle \in 4\mathbb{Z}_8 \subset \mathbb{Z}_8$$

with $f_4 : \pi_1(\Sigma_h) \xrightarrow{f} \mathrm{Sp}(2g,\mathbb{Z}) \longrightarrow \mathrm{Sp}(2g,\mathbb{Z}_4)$.

- **Proof** It is enough to show that

$$\tau \in H^2(\mathrm{Sp}(2g,\mathbb{Z});\mathbb{Z}) = \mathbb{Z} \longrightarrow H^2(\mathrm{Sp}(2g,\mathbb{Z});\mathbb{Z}_8) = \mathbb{Z}_8 \ ,$$

$$\tau_4 \in H^2(\mathrm{Sp}(2g,\mathbb{Z}_4);\mathbb{Z}_8) = \mathbb{Z}_8 \xrightarrow{\cong} H^2(\mathrm{Sp}(2g,\mathbb{Z});\mathbb{Z}_8) = \mathbb{Z}_8$$

have the same images.

- **Easy**, but no cocycle and no geometry!

The mapping torus $T(\alpha)$

- The **mapping class group** of Σ_g is defined as usual by

$$\mathrm{Mod}_g \;=\; \pi_0(\mathrm{Homeo}^+(\Sigma_g))$$

  with $\mathrm{Homeo}^+(\Sigma_g)$ the group of orientation-preserving homeomorphisms $\alpha : \Sigma_g \to \Sigma_g$.
- The **mapping torus** of $\alpha \in \mathrm{Mod}_g$ is the closed oriented 3-manifold

$$T(\alpha) \;=\; \Sigma_g \times I/\{(x,0) \sim (\alpha(x),1)\,|\,x \in \Sigma_g\}$$

  Total space of fibre bundle

$$\Sigma_g \to T(\alpha) \to S^1 \ .$$

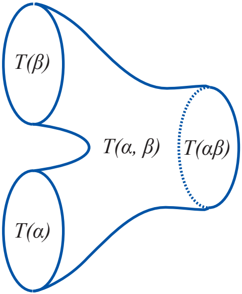
The double mapping torus $T(\alpha, \beta)$

- The **double mapping torus** $T(\alpha, \beta)$ of $\alpha, \beta \in \mathrm{Mod}_g$ is the total space of the fibre bundle

$$\Sigma_g \to T(\alpha, \beta) \to P \ = \ \text{pair of pants} \ ,$$

an oriented 4-manifold with boundary

$$\partial T(\alpha, \beta) \ = \ T(\alpha) \sqcup T(\beta) \sqcup -T(\alpha\beta)$$



A cocycle for $\tau \in H^2(\mathrm{Sp}(2g,\mathbb{Z});\mathbb{Z})$

- **Theorem** (Meyer, 1972)
  The Wall non-additivity of the signature formula gives

$$\sigma(T(\alpha,\beta)) = \sigma(\ker((1-\alpha^{-1}\ 1-\beta) : H \oplus H \to H), \Phi)$$
$$H = H^1(\Sigma_g)\ ,\ \Phi((x_1,y_1),(x_2,y_2)) = \phi(x_1+y_1,(1-\beta)(y_2))\ .$$

- The function

$$\tau\ :\ \mathrm{Sp}(2g,\mathbb{Z}) \times \mathrm{Sp}(2g,\mathbb{Z}) \to \mathbb{Z}\ ;\ (\alpha,\beta) \mapsto \sigma(T(\alpha,\beta))$$

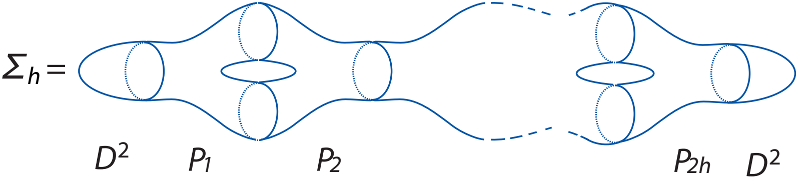
  is a cocycle for the signature class $\tau \in H^2(\mathrm{Sp}(2g,\mathbb{Z});\mathbb{Z})$.

The idea of proof of Meyer's Theorem

- For a surface bundle $\Sigma_g \to E \to \Sigma_h$ with monodromy

$$\pi_1(\Sigma_h) = \langle \alpha_1, \beta_1, \ldots, \alpha_h, \beta_h | [\alpha_1, \beta_1] \ldots [\alpha_h, \beta_h] \rangle \to \mathrm{Mod}_g$$

lift the decomposition



to
$$E \; = \; D^2 \times \Sigma_g \cup \bigcup_{i=1}^{4h} T(\widetilde{\omega}_{i-1}, \omega_i) \cup D^2 \times \Sigma_g \qquad \text{(simplified)}$$

with $\widetilde{\omega}_i$ the ith factor in $[\alpha_1, \beta_1] \ldots [\alpha_h, \beta_h]$ and ω_i the product of the first i factors.

- By Novikov additivity $\sigma(E) = -\sum_{i=1}^{4h} \sigma(T(\widetilde{\omega}_{i-1}, \omega_i)) \in \mathbb{Z}$.

The Brown-Kervaire invariant $BK(V, b, q) \in \mathbb{Z}_8$

- Defined by E.H.Brown (1972) for a nonsingular symmetric form (V, b) over $\mathbb{Z}_2$ with $\mathbb{Z}_4$-valued quadratic refinement q

  (f.g. free $\mathbb{Z}_2$-module $V, b : V \times V \to \mathbb{Z}_2, q : V \to \mathbb{Z}_4$)

  by the Gauss sum

$$\sum_{x \in V} e^{2\pi i q(x)/4} \;=\; \sqrt{2}^{\dim_{\mathbb{Z}_2} V} e^{2\pi i BK(V,b,q)/8} \in \mathbb{C}$$

- The mod 8 signature of a nonsingular symmetric form (H, ϕ) over $\mathbb{Z}$ is

$$\sigma(H, \phi) \;=\; BK(H/2H, b, q) \in \mathbb{Z}_8$$

  with

$$b(x, y) \;=\; [\phi(x, y)] \;,\;\; q(x) \;=\; [\phi(x, x)] \;.$$

A cocycle for $\tau_4 \in H^2(\mathrm{Sp}(2g,\mathbb{Z}_4);\mathbb{Z}_2)$

- The verification that Meyer's function

$$\tau \ : \ \mathrm{Sp}(2g,\mathbb{Z}) \times \mathrm{Sp}(2g,\mathbb{Z}) \to \mathbb{Z}$$

  is a cocycle used the Novikov additivity for the signature of the union of manifolds with boundary

$$\sigma(M \cup_{\partial M = -\partial M'} M') \ = \ \sigma(M) + \sigma(M') \in \mathbb{Z} \ .$$

- Our cocycle

$$\tau_4 \ : \ \mathrm{Sp}(2g,\mathbb{Z}_4) \times \mathrm{Sp}(2g,\mathbb{Z}_4) \to \mathbb{Z}_2$$

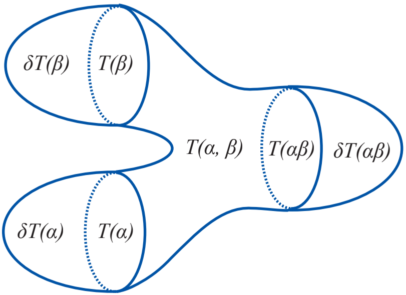
  is constructed using the $\mathbb{Z}_8$-valued Brown-Kervaire invariant, for which there is **no analogue of Novikov additivity**.

Mapping tori are boundaries

- $\Omega_3 = 0$: every closed oriented 3-dimensional manifold is the boundary of an oriented 4-manifold, so there exists a function

  $$\delta T \ : \ \text{Mod}_g \to \{\text{oriented 4-manifolds with boundary}\} \ ; \ \alpha \mapsto \delta T(\alpha)$$

  such that $\partial \delta T(\alpha) = T(\alpha)$.
- So for any $\alpha, \beta \in \text{Mod}_g$ have closed oriented 4-dimensional manifold $T(\alpha, \beta) \cup (\delta T(\alpha) \sqcup \delta T(\beta) \sqcup \delta T(\alpha\beta))$



The mod 8 signature cocycle

- **Theorem 2** (BCRR, 2016)
  For any δT the function
  $$\mathrm{Mod}_g \times \mathrm{Mod}_g \to \mathbb{Z}_8 \ ;$$
  $$(\alpha,\beta) \mapsto BK(T(\alpha,\beta) \cup \delta T(\alpha) \cup \delta T(\beta) \cup -\delta T(\alpha\beta))$$
  is a cocycle for the pullback of
  $$4\tau_4 \ = \ p_4^*[\tau] \in H^2(\mathrm{Sp}(2g,\mathbb{Z}_4);\mathbb{Z}_8)$$
  along the $\mathbb{Z}_4$-coefficient monodromy $\mathrm{Mod}_g \to \mathrm{Sp}(2g,\mathbb{Z}_4)$.
- Very implicit, since it relies on the choice of bounding 4-manifolds $\delta T(\alpha)$. In general, not divisible by 4.
- **Algebraic Poincaré cobordism to the rescue**.

Algebraic Poincaré cobordism

- (R., 1980-...) For any ring with involution A

$$\begin{cases} L^n(A) \\ L_n(A) \end{cases} = \text{cobordism groups of } n\text{-dimensional f.g. free } A\text{-module}$$

$$\text{chain complexes with a } \begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases} \text{ chain equivalence } C^{n-*} \to C$$

- $1+T : L_n(A) =$ Wall surgery obstruction group $\to L^n(A)$.
- $L^0(A)$ (resp. $L_0(A)$) $=$ Witt group of nonsingular symmetric (resp. quadratic) forms over A
- For $A = \mathbb{Z}$ signature $\sigma : L^0(\mathbb{Z}) \cong \mathbb{Z}$ with

$$1+T = 8 : L_0(\mathbb{Z}) = \mathbb{Z} \to L^0(\mathbb{Z}) = \mathbb{Z} \ .$$

- For $A = \mathbb{Z}_4$ Brown-Kervaire invariant $BK : L^0(\mathbb{Z}_4) \cong \mathbb{Z}_8$ with

$$1+T = 4 : L_0(\mathbb{Z}_4) = \mathbb{Z}_2 \to L^0(\mathbb{Z}_4) = \mathbb{Z}_8 \ .$$

- Symmetric signature $\Omega_n \to L^n(\mathbb{Z}) \to L^n(\mathbb{Z}_4)$.

Generalized signature cocycle and class via algebra

- Manifolds with boundary, union, mapping torus and double mapping torus all have analogues in the world of algebraic Poincaré cobordism, for any ring A.
- The **algebraic** mapping torus gives morphism

$$T \;:\; \mathrm{Sp}(2g, A) \to L^3(A) \;;\; \alpha \mapsto T(\alpha) \;.$$

- **Theorem 3** (BCRR, 2016) If $L^3(A) = 0$ the **algebraic** double mapping torus gives a class $\tau^A \in H^2(\mathrm{Sp}(2g, A); L^4(A))$ with cocycle

$$\tau^A \;:\; \mathrm{Sp}(2g, A) \times \mathrm{Sp}(2g, A) \to L^4(A) \;;$$
$$(\alpha, \beta) \mapsto \tau^A(\alpha, \beta) = T(\alpha, \beta) \cup \delta T(\alpha) \cup \delta T(\beta) \cup -\delta T(\alpha\beta)$$

for any choice of $\alpha \mapsto \delta T(\alpha)$ with $\partial \delta T(\alpha) = T(\alpha)$.

The algebraic Poincaré cobordism of $A = \mathbb{Z}$

- $L^3(\mathbb{Z}) = 0$. Canonical null-cobordism $\delta T(\alpha)$ for algebraic $T(\alpha)$ with Euler characteristic

$$\chi(\alpha) \; = \; \dim_{\mathbb{Z}} \ker(1 - \alpha : \mathbb{Z}^{2g} \to \mathbb{Z}^{2g}) \; (\alpha \in \mathrm{Sp}(2g, \mathbb{Z})) \; .$$

- Isomorphism

$$\sigma \; : \; L^4(\mathbb{Z}) \to \mathbb{Z} \; ; \; (C, \phi) \mapsto \sigma(H^2(C), \phi_0) \; .$$

- (Turaev 1985) The cocycle

$$\begin{aligned} \tau \; : \; & \mathrm{Sp}(2g, \mathbb{Z}) \times \mathrm{Sp}(2g, \mathbb{Z}) \to \mathbb{Z} \; ; \\ & (\alpha, \beta) \mapsto \sigma\tau^{\mathbb{Z}}(\alpha, \beta) - (\chi(\alpha) + \chi(\beta) - \chi(\alpha\beta)) \end{aligned}$$

  is divisible by 4, representing the Meyer signature class

$$\tau \; = \; 4 \in H^2(\mathrm{Sp}(2g, \mathbb{Z}); \mathbb{Z}) \; = \; \mathbb{Z} \; .$$

The algebraic Poincaré cobordism of $A = \mathbb{Z}_4$

- $L^3(\mathbb{Z}_4) = 0$. Canonical null-cobordism $\delta T(\alpha)$ for algebraic $T(\alpha)$ with Euler characteristic

  $$\chi_4(\alpha) \ = \ \dim_{\mathbb{Z}_2} \ker(1-\alpha : \mathbb{Z}_2^{2g} \to \mathbb{Z}_2^{2g}) \ (\alpha \in \mathrm{Sp}(2g, \mathbb{Z}_4)) \ .$$

  Need to use $\mathbb{Z}_2$-coefficients, since the $\mathbb{Z}_4$-module $\mathbb{Z}_2$ is not free!
- Split surjection

  $$BK \ : \ L^4(\mathbb{Z}_4) \to \mathbb{Z}_8 \ ; \ (C, \phi) \mapsto BK(H^2(C; \mathbb{Z}_2), \phi_0, \mathcal{P}(\phi))$$

  with $\mathcal{P}(\phi) =$ Pontrjagin square : $H^2(C; \mathbb{Z}_2) \to \mathbb{Z}_4$.
- **Theorem 4** (BCRR, 2016) The cocycle

  $$\tau_4 \ : \ \mathrm{Sp}(2g, \mathbb{Z}_4) \times \mathrm{Sp}(2g, \mathbb{Z}_4) \to \mathbb{Z}_8 \ ;$$
  $$(\alpha, \beta) \mapsto BK\tau^{\mathbb{Z}_4}(\alpha, \beta) - (\chi_4(\alpha) + \chi_4(\beta) - \chi_4(\alpha\beta))$$

  is divisible by 4, representing the mod 8 signature class

  $$\tau_4 \ = \ 4 \in H^2(\mathrm{Sp}(2g, \mathbb{Z}_4); \mathbb{Z}_8) \ = \ \mathbb{Z}_8 \ .$$

The non-additivity of the Brown-Kervaire invariant

- There are two ways of glueing together two copies of the singular symmetric form $(\mathbb{Z}_4, 2)$ over $\mathbb{Z}_4$.
- $(\mathbb{Z}_4, 2) \cup_1 (\mathbb{Z}_4, 2) = (\mathbb{Z}_4 \oplus \mathbb{Z}_4, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) = 0 \in L^0(\mathbb{Z}_4) = \mathbb{Z}_8$.
- $(\mathbb{Z}_4, 2) \cup_{-1} (\mathbb{Z}_4, 2) = (\mathbb{Z}_4 \oplus \mathbb{Z}_4, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}) = 1 \in L_0(\mathbb{Z}_4) = \mathbb{Z}_2$, the Arf-Kervaire invariant of the trefoil knot $K : S^1 \subset S^3$.
- Therefore **cannot** define a Brown-Kervaire invariant for singular symmetric forms over $\mathbb{Z}_4$ with Novikov-style additivity.
- $(\mathbb{Z} \oplus \mathbb{Z}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix})$ is the intersection symmetric form of the 4-manifold M given by the A_2-plumbing of two copies of τ_{S^2}, with boundary the lens space $\partial M = L(3, 2)$ a 2-fold branched cover of S^3 along K.