# ALGEBRAIC TRANSVERSALITY

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## What is algebraic transversality?

- Geometric transversality is one of the most important properties of manifolds, dealing with the construction of submanifolds.
- Easy to establish for smooth manifolds (Thom, 1954)
- Hard to establish for topological manifolds (Kirby-Siebenmann, 1970), and that only for dimensions $\geqslant 5$.
- Algebraic transversality deals with the construction of subcomplexes of chain complexes over group rings.
- Algebraic transversality is needed to quantify geometric transversality, to control the algebraic topology of the submanifolds created by the geometric construction.

## Codimension $k$ subspaces

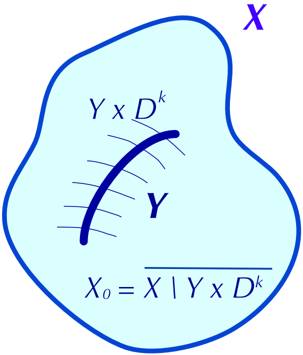
- **Definition** A **framed codimension $k$ subspace** of a space $X$ is a closed subspace $Y \subset X$ such that $X$ has a decomposition

$$X \;=\; X_0 \cup_{Y \times S^{k-1}} Y \times D^k \;,$$

with the complement

$$X_0 \;=\; \mathrm{cl.}(X \backslash Y \times D^k) \subset X$$

a closed subspace homotopy equivalent to $X \backslash Y$.



## Geometric transversality

- **Theorem** (Thom, 1954) Every map $f : M^m \to X$ from a smooth $m$-dimensional manifold to a space $X$ with a framed codimension $k$ subspace $Y \subset X$ is homotopic to a smooth map (also denoted $f$) which is transverse regular at $Y \subset X$, so that

$$N^{m-k} = f^{-1}(Y) \subset M$$

is a framed codimension $k$ submanifold with

$$f = f_0 \cup g \times 1_{D^k} : M = M_0 \cup N \times D^k \to X = X_0 \cup Y \times D^k .$$

- Algebraic transversality studies analogous decompositions of chain complexes! Particularly concerned with homotopy equivalences and contractible chain complexes.

## The infinite cyclic cover example of geometric transversality I.

- $X = S^1$ has framed codimension 1 subspace $Y = \{*\} \subset S^1$ with complement $X_0 = I$

$$S^1 \ = \ I \cup_{\{*\}\times S^0} \{*\} \times D^1 \ .$$

- By geometric transversality every map $f : M^m \to S^1$ is homotopic to a map transverse regular at $\{*\} \subset S^1$, with

$$N^{m-1} = f^{-1}(*) \subset M$$

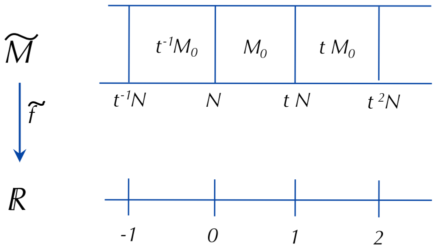
a framed codimension 1 submanifold with complement $M_0 = f^{-1}(I)$

$$M \ = \ M_0 \cup_{N\times S^0} N \times D^1 \ .$$

## The infinite cyclic cover example of geometric transversality II.

- The pullback infinite cyclic cover of $M$ has fundamental domain $(M_0; N, tN)$

$$\widetilde{M} = f^*\mathbb{R} = \bigcup_{j=-\infty}^{\infty} t^j(M_0; N, tN) .$$



## The infinite cyclic cover example of algebraic transversality

- **Proposition** (Higman, Waldhausen, R.)
  For every finite f.g. free $\mathbb{Z}[t,t^{-1}]$-module chain complex $C$ there exists a finite f.g. free $\mathbb{Z}$-module subcomplex $C_0 \subset C$ with $D = C_0 \cap tC_0$ a finite f.g. free $\mathbb{Z}$-module chain complex, and the $\mathbb{Z}$-module chain maps

$$\begin{aligned} i_0 &: D \to C_0 \; ; \; x \mapsto x \; , \\ i_1 &: D \to C_0 \; ; \; x \mapsto t^{-1}x \end{aligned}$$

  such that there is defined a short exact sequence of finite f.g. free $\mathbb{Z}[t,t^{-1}]$-module chain complexes

$$0 \longrightarrow D[t,t^{-1}] \xrightarrow{i_0 - ti_1} C_0[t,t^{-1}] \longrightarrow C \longrightarrow 0$$

- Note that if $C$ is contractible then $C_0, D$ need not be contractible.
- Can replace $\mathbb{Z}$ by any ring $A$.

## Split homotopy equivalences

- **Definition** A homotopy equivalence $f : M \to X$ from a smooth $m$-dimensional manifold **splits** at a framed codimension $k$ subspace $Y \subset X$ if $f$ is transverse regular at $Y \subset X$, and the restrictions

$$\begin{aligned} g &= f| : N = f^{-1}(Y) \to Y \ , \\ f_0 &= f| : M_0 = M \backslash N \to X_0 = X \backslash Y \end{aligned}$$

  also homotopy equivalences.
- **Definition** $f$ **splits up to homotopy** if it is homotopic to a homotopy equivalence (also denoted by $f$) which is split.
- In general, homotopy equivalences do not split up to homotopy. Surgery theory provides splitting obstructions.

## The uniqueness of smooth manifold structures

- **Surgery Theory Question** Is a homotopy equivalence $f : M \to X$ of smooth $m$-dimensional manifolds homotopic to a diffeomorphism?
- **Answer** No, in general. The Browder-Novikov-Sullivan-Wall theory (1960's) provides obstructions in homotopy theory and algebra, and systematic construction of counterexamples. For $X = S^m$ this is the Kervaire-Milnor classification of exotic spheres.
- **Example** Diffeomorphisms are split. If $f$ is homotopic to a diffeomorphism then $f$ splits up to homotopy at every submanifold $Y \subset X$.
- **Contrapositive** If $f$ does not split up to homotopy at a submanifold $Y \subset X$ then $f$ is not homotopic to a diffeomorphism.

## The uniqueness of topological manifold structures

- **Surgery Theory Question** Is a homotopy equivalence $f : M \to X$ of topological $m$-dimensional manifolds homotopic to a homeomorphism?
- **Answer** No, in general. As in the smooth case, surgery theory provides systematic obstruction theory for $m \geqslant 5$. Need Kirby-Siebenmann (1970) structure theory for topological manifolds.
- **Example** Homeomorphisms are split. If $f$ is homotopic to a homeomorphism then $f$ splits up to homotopy at every submanifold $Y \subset X$.
- **Contrapositive** If $f$ does not split up to homotopy at a submanifold $Y \subset X$ then $f$ is not homotopic to a homeomorphism.

## The Borel Conjecture

- **BC** (1953) Every homotopy equivalence $f : M \to X$ of smooth $m$-dimensional aspherical manifolds is homotopic to a homeomorphism.
- http://www.maths.ed.ac.uk/ aar/surgery/borel.pdf
  Birth of the Borel rigidity conjecture.
- In the last 30 years the conjecture has been verified in many cases, using surgery theory, geometric group theory and differential geometry (Farrell-Jones, Lück).

## The existence of smooth manifold structures

- A smooth $m$-dimensional manifold $M$ is a finite $CW$ complex with $m$-dimensional Poincaré duality $H^{m-*}(M) \cong H_*(M)$
- **Surgery Theory Question** If $X$ is a finite $CW$ complex with $m$-dimensional Poincaré duality isomorphisms

  $$H^{m-*}(X) \ \cong \ H_*(X) \text{ (with } \mathbb{Z}[\pi_1(X)]\text{-coefficients)}$$

  is $X$ homotopy equivalent to a smooth $m$-dimensional manifold?
- The Browder-Novikov-Sullivan-Wall surgery theory deals with both existence and uniqueness, providing obstructions in terms of homotopy theory and algebra.
- Various examples of Poincaré duality spaces not of the homotopy type of smooth manifolds

## The existence of topological manifold structures

- **Surgery Theory Question** If $X$ is a finite $CW$ complex with $m$-dimensional Poincaré duality isomorphisms is $X$ homotopy equivalent to a topological $m$-dimensional manifold?
- For $m \geqslant 5$ the Browder-Novikov-Sullivan-Wall surgery theory provides algebraic obstructions. The reduction to pure algebra makes use of algebraic transversality and codimension 1 splitting obstructions (R.,1992).

## Obstructions to splitting homotopy equivalences up to homotopy

- In general, homotopy equivalences of manifolds are not split up to homotopy, in both the smooth and topological categories.
- There are algebraic $K$ and $L$-theory obstructions to splitting homotopy equivalences up to homotopy for $m - k \geqslant 5$ (Browder, Wall, Cappell 1960's and 1970's).
- Waldhausen (1970's) dealt with the case $m = 3$, $k = 1$, motivated by the Haken theory of 3-manifolds.
- Cappell (1974) constructed homotopy equivalences

  $$f \ : \ M^m \to X \ = \ \mathbb{R}\mathbb{P}^m \# \mathbb{R}\mathbb{P}^m$$

  which cannot be split up to homotopy, for $m \equiv 1 (\bmod 4)$ with $m \geqslant 5$, and $Y = S^{m-1} \subset X$.
- Same algebraic $K$- and $L$-theory obstructions to decomposing Poincaré duality space as $X = X_0 \cup Y \times D^k$, with $Y$ codimension $k$ Poincaré. (R.)

## *CW* complexes and chain complexes I.

- Given a *CW* complex $X$ and a regular cover $\widetilde{X}$ with group of covering translations $\pi$ let $C(\widetilde{X})$ be the cellular chain complex, a free $\mathbb{Z}[\pi]$-module chain complex with one generator for each cell of $X$.subcomplex
- A map $f : M \to X$ from a *CW* complex induces a $\pi$-equivariant map $\widetilde{f} : \widetilde{M} = f^*\widetilde{X} \to \widetilde{X}$ of the covers, and hence a $\mathbb{Z}[\pi]$-module chain map $\widetilde{f} : C(\widetilde{M}) \to C(\widetilde{X})$.
- **Theorem** (J.H.C. Whitehead) A map $f : M \to X$ is a homotopy equivalence if and only if $f_* : \pi_1(M) \to \pi_1(X)$ is an isomorphism and the algebraic mapping cone $\mathcal{C}(\widetilde{f})$ is chain contractible, with $\widetilde{X}$ the universal cover of $X$ and $\pi = \pi_1(X)$.

## $CW$ complexes and chain complexes II.

- If $i : Y \subset X$ is the inclusion of a framed codimension $k$ subcomplex the decomposition $X = X_0 \cup_{Y \times S^{k-1}} Y \times D^k$ lifts to a $\pi$-equivariant decomposition
$$\widetilde{X} \;=\; \widetilde{X}_0 \cup_{\widetilde{Y} \times S^{k-1}} \widetilde{Y} \times D^k$$
with $\widetilde{Y} = i^*\widetilde{X}$ the pullback cover of $Y$, a framed codimension $k$ subcomplex of $\widetilde{X}$.
- The $\mathbb{Z}[\pi]$-module chain complex of $\widetilde{X}$ has an algebraic decomposition
$$C(\widetilde{X}) \;=\; C(\widetilde{X}_0) \cup_{C(\widetilde{Y}) \otimes C(S^{k-1})} C(\widetilde{Y}) \otimes C(D^k)\ .$$
- Algebraic transversality studies $\mathbb{Z}[\pi]$-module chain complexes with such decompositions.
- If $f : M \to X$ is transverse at $Y \subset X$ the algebraic mapping cone of $\widetilde{f} : \widetilde{M} \to \widetilde{X}$ has such a decomposition
$$\mathcal{C}(\widetilde{f}) \;=\; C(\widetilde{f}_0) \cup_{C(\widetilde{g}) \otimes C(S^{k-1})} C(\widetilde{g}) \otimes C(D^k)\ .$$

## The fundamental groups in codimension 1

- If $X$ is a connected $CW$ complex and $Y \subset X$ is a connected framed codimension 1 subcomplex then

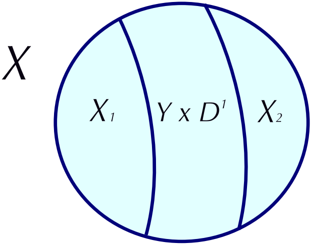
$$X = \begin{cases} X_1 \cup_{Y \times D^1} X_2 & \text{if } X_0 = X_1 \sqcup X_2 \text{ is disconnected} \\ X_0 \cup_{Y \times S^0} Y \times D^1 & \text{if } X_0 \text{ is connected} \end{cases}$$

  according as to whether $Y$ separates $X$ or not.
- The fundamental group $\pi_1(X)$ is given by the Seifert-van Kampen theorem to be the $\begin{cases} \text{amalgamated free product} \\ HNN \text{ extension} \end{cases}$

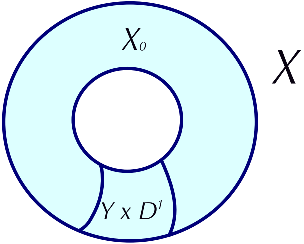
$$\pi_1(X) = \begin{cases} \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) \\ \pi_1(X_0) *_{\pi_1(Y)} \{t\}. \end{cases}$$ determined by the morphisms

$$\begin{cases} \pi_1(Y) \to \pi_1(X_1)\ ,\ \pi_1(Y) \to \pi_1(X_2) \\ \pi_1(Y \times \{0\}) \to \pi_1(X_0)\ ,\ \pi_1(Y \times \{1\}) \to \pi_1(X_0)\ . \end{cases}$$

**Separating and non-separating codimension 1 subspaces**




*Y separates X*

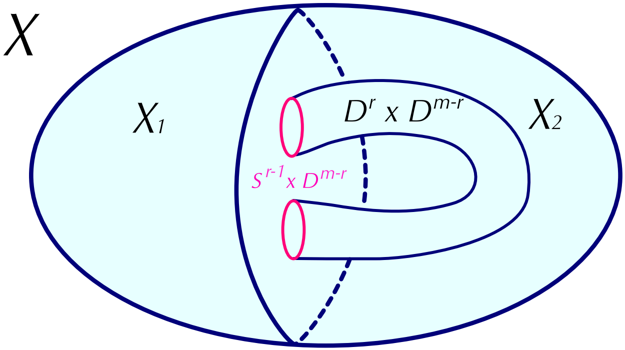



*Y does not separate X*

## Handle exchanges I.

- Will only deal with the separating case.
- Let $M$ be an $m$-dimensional manifold with a separating framed codimension 1 submanifold $N^{m-1} \subset M$, so that
$$M = M_1 \cup_N M_2 .$$
- A **handle exchange** uses an embedding
$$(D^r \times D^{m-r}, S^{r-1} \times D^{m-r}) \subset (M_i, N) \ (i = 1 \text{ or } 2)$$
to obtain a new decomposition
$$M = M_1' \cup_{N'} M_2'$$
with
$$\begin{aligned} N' &= \text{cl.}(N \backslash S^{r-1} \times D^{m-r}) \cup D^r \times S^{m-r-1} , \\ M_i' &= \text{cl.}(M_i \backslash D^r \times D^{m-r}) , \\ M_{2-i}' &= M_{2-i} \cup D^r \times D^{m-r} . \end{aligned}$$
- Initiated by Stallings ($m = 3$) and Levine in the 1960's.

**Handle exchanges II.**

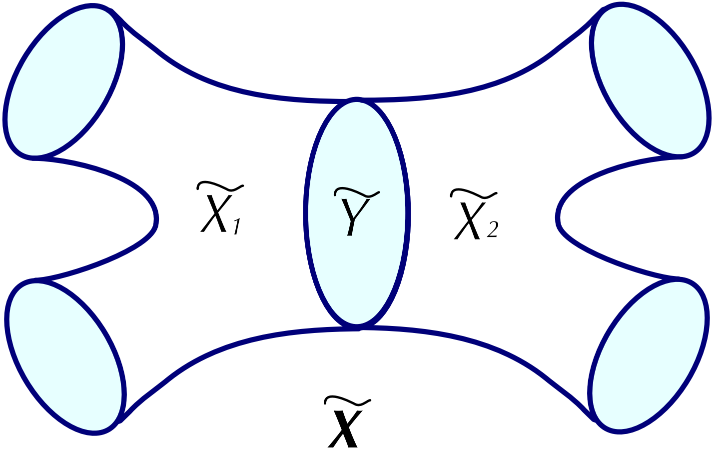



$$X_1' = X_1 \cup D^r \times D^{m-r} \ , \ X_2' = \mathrm{cl.}(X_2 \backslash D^r \times D^{m-r}) \ .$$

## Codimension 1 geometric transversality I.

- Let $X = X_1 \cup_Y X_2$ be a connected $CW$ complex with a separating connected framed codimension 1 subspace $Y \subset X$ such that $\pi_1(Y) \to \pi_1(X)$ is injective. Then
$$\pi_1(X) \;=\; \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) \;=\; \pi$$
with injective morphisms
$$\pi_1(Y) \;=\; \rho \to \pi_1(X_1) \;=\; \pi_1 \;,\; \pi_1(Y) \;=\; \rho \to \pi_1(X_2) \;=\; \pi_2 \;.$$
- The Bass-Serre tree $T$ is a contractible non-free $\pi$-space with
$$T^{(0)} \;=\; [\pi : \pi_1] \cup [\pi : \pi_2] \;,\; T^{(1)} \;=\; [\pi : \rho] \;,\; T/\pi \;=\; I \;.$$
- The universal cover of $X$ decomposes as
$$\widetilde{X} \;=\; [\pi : \pi_1] \times \widetilde{X}_1 \cup_{[\pi:\rho]\times\widetilde{Y}} [\pi : \pi_2] \times \widetilde{X}_2$$
with $\widetilde{X}_1, \widetilde{X}_2, \widetilde{Y}$ the universal covers of $X_1, X_2, Y$, and
$$\widetilde{Y} \;=\; \widetilde{X}_1 \cap \widetilde{X}_2 \subset \widetilde{X} \;.$$

## The universal cover $\widetilde{X}$ of $X = X_1 \cup_Y X_2$



**Codimension 1 geometric transversality II.**

- If $X$ is finite the cellular f.g. free $\mathbb{Z}[\pi]$-module chain complex $C(\widetilde{X})$ has f.g. free $\mathbb{Z}[\pi_i]$-module chain subcomplexes $C(\widetilde{X}_i) \subset C(\widetilde{X})$ and a f.g. free $\mathbb{Z}[\rho]$-module chain subcomplex

$$C(\widetilde{Y}) \;=\; C(\widetilde{X}_1) \cap C(\widetilde{X}_2) \subset C(\widetilde{X})$$

  with a short exact Mayer-Vietoris sequence of f.g. free $\mathbb{Z}[\pi]$-module chain complexes

$$0 \longrightarrow \mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\rho]} C(\widetilde{Y}) \longrightarrow \mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\pi_1]} C(\widetilde{X}_1) \oplus \mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\pi_2]} C(\widetilde{X}_2) \longrightarrow C(\widetilde{X}) \longrightarrow 0 \;\; .$$

- If $f : M \to X$ is a homotopy equivalence of $m$-dimensional manifolds there is no obstruction to making $f$ transverse regular at $Y \subset X$, but there are algebraic $K$- and $L$-theory obstructions to splitting $f$ up to homotopy, involving the MV sequence of the contractible $\mathbb{Z}[\pi]$-module chain complex $\mathcal{C}(\widetilde{f} : \widetilde{M} \to \widetilde{X})$ and algebraic handle exchanges.

## Codimension 1 algebraic transversality

- Let $\pi = \pi_1 *_\rho \pi_2$ be an injective amalgamated free product.
- **Proposition** (Waldhausen, R.) For any f.g. free $\mathbb{Z}[\pi]$-module chain complex $C$ there exist f.g. free $\mathbb{Z}[\pi_i]$-module chain subcomplexes $D_i \subset C$ and a f.g. free $\mathbb{Z}[\rho]$-module chain subcomplex $E = D_1 \cap D_2 \subset C$ with a short exact MV sequence of f.g. free $\mathbb{Z}[\pi]$-module chain complexes

$$0 \to \mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\rho]} E \to \mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\pi_1]} D_1 \oplus \mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\pi_2]} D_2 \to C \to 0 \quad .$$

  Any two such choices $(D_1, D_2, E)$ are related by a finite sequence of algebraic handle exchanges. If $C$ is contractible there is an algebraic $K$-theory obstruction to choosing $D_1, D_2, E$ to be contractible.
- **Corollary** (Cappell, R.) If $C$ has $m$-dimensional Poincaré duality there is an algebraic $L$-theory obstruction to choosing $(D_i, \mathbb{Z}[\pi_i] \otimes_{\mathbb{Z}[\rho]} E)$ to have $m$-dimensional Poincaré-Lefschetz duality and $E$ to have $(m-1)$-dimensional Poincaré duality.

## Universal transversality

- Let $X$ be a finite simplicial complex, with barycentric subdivision $X'$ and dual cells
$$D(\sigma) \;=\; \{\widehat{\sigma}_0\widehat{\sigma}_1\dots\widehat{\sigma}_r \,|\, \sigma \leqslant \sigma_0 < \sigma_1 < \cdots < \sigma_r\}\;.$$
- A map $f : M \to |X| = |X'|$ from an $m$-dimensional manifold is **universally transverse** if each inverse image
$$M(\sigma) \;=\; f^{-1}(D(\sigma)) \subset M$$
is a framed codimension $|\sigma|$ submanifold with boundary
$$\partial M(\sigma) \;=\; \bigcup_{\tau > \sigma} M(\tau)\;.$$
- The algebraic obstruction theory for the existence and uniqueness of topological manifold structures in a homotopy type uses algebraic universal tranvsersality.

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