ALGEBRAIC SURGERY AND THE LINKS OF COMPLEX HYPERSURFACE SINGULARITIES

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The BNR project

- Since 2011 have joined András Némethi (Budapest) and Maciej Borodzik (Warsaw) in a project on the algebraic invariants of the links of complex hypersurface singularities, using algebraic surgery as an organizing principle.
- Morse theory decomposes cobordisms of manifolds into elementary operations called surgeries.
- Algebraic surgery does the same for cobordisms of chain complexes with Poincaré duality – generalized quadratic forms.
- High-dimensional knot theory. Algebraic surgery in codimension 2 (Springer, 1998) Book described algebraic and geometric surgery for high-dimensional knots and links.
- The spectrum of a singularity is an analytic invariant, defined using Hodge theory. In the first instance the project deals with the topological parts of the spectrum for isolated singularities, and the relationship between the local singularities and the singularity at infinity.

Links

- An m-**dimensional link** is a codimension 2 submanifold

$$L^m \subset S^{m+2}$$

with trivial normal bundle $L \times D^2 \subset S^{m+2}$.

- The link is **spherical** if

$$L = S^m \cup S^m \cup \dots \cup S^m .$$

- An m-**dimensional knot** is a spherical link with

$$L = S^m \subset S^{m+2} .$$

- Classical knots and links are the case $m = 1$

$$L^1 = S^1 \cup S^1 \cup \dots \cup S^1 \subset S^3 .$$

- *Of course, from the point of view of the rest of mathematics, knots in higher-dimensional space deserve just as much attention as knots in 3-space* (Frank Adams, 1976).

Some of what Milnor did in the 1950's and 1960's

- (1956) Paper **On manifolds homeomorphic to the 7-sphere**. Exotic spheres: differentiable manifolds homeomorphic but not diffeomorphic to S^7.
- (1961) Paper **A procedure for killing the homotopy groups of differentiable manifolds** initiated the systematic study of manifolds M using surgery theory
- (1963) Paper **Groups of homotopy spheres I.** (with Kervaire). Surgery classification of exotic spheres in dimensions $\geqslant 5$.
- (1965) Book **Morse theory** The systematic study of manifolds M using Morse functions $f : M \to \mathbb{R}$.
- (1966) Paper **Singularities of 2-spheres in 4-space and cobordism of knots** (with Fox) Cobordism extended to knots.
- (1968) Book **Singular points of complex hypersurfaces**. Motivated by the Brieskorn construction of certain exotic spheres as links of isolated hypersurface singularities.

The link of an isolated hypersurface singularity

- Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be the germ of an analytic function such that the complex hypersurface
$$X \ = \ f^{-1}(0) \subset \mathbb{C}^{n+1}$$
has an isolated singularity at $x \in X$, with
$$\frac{\partial f}{\partial z_k}(x) \ = \ 0 \text{ for } k = 1, 2, \ldots, n+1 \ .$$
- For $\epsilon > 0$ let
$$\begin{aligned} D_\epsilon(x) \ &= \ \{y \in \mathbb{C}^{n+1} \mid \|y - x\| \leqslant \epsilon\} \ \cong \ D^{2n+2} \ , \\ S_\epsilon(x) \ &= \ \{y \in \mathbb{C}^{n+1} \mid \|y - x\| = \epsilon\} \ \cong \ S^{2n+1} \ . \end{aligned}$$
- For $\epsilon > 0$ sufficiently small, the subset
$$L(x)^{2n-1} \ = \ X \cap S_\epsilon(x) \subset S_\epsilon(x)^{2n+1}$$
is a closed $(2n-1)$-dimensional submanifold, the **link of the singularity** of f at x.

2 historical references

- Brauner (1928) Die Verzweigungsstellen einer algebraischen Funktion. For coprime integers $p, q \geqslant 2$ define

$$f \ : \ (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \ ; \ (z_1, z_2) \mapsto z_1^p - z_2^q \ .$$

  The link of the singularity at 0 is the (p, q)-**torus knot**

$$L(0) \ = \ S^1 \subset S^1 \times S^1 \subset S^3$$

  with

$$S^1 \to S^1 \times S^1 \ ; \ z \mapsto (z^q, z^p) \ .$$

- **Example** For $(p, q) = (2, 3)$ this is the trefoil knot.
- Epple (1995) Branch points of algebraic functions and the beginnings of modern knot theory. From 19th century to mid-20th.

Seifert surfaces

- A **Seifert surface** for a link $L^m \subset S^{m+2}$ is a codimension 1 submanifold $F^{m+1} \subset S^{m+2}$ such that

$$\partial F \ = \ L \subset S^{m+2}$$

  with a trivial normal bundle $F \times D^1 \subset S^{m+2}$.
- Every link $L \subset S^{m+2}$ admits a Seifert surface F: extend the projection $\partial C = L \times S^1 \to S^1$ to a map

$$p \ : \ C \ = \ \mathrm{cl.}(S^{m+2} \backslash L \times D^2) \to S^1$$

  and let $F = p^{-1}(*) \subset S^{m+2}$ be the transverse inverse image.
- In general, Seifert surfaces are not canonical.
- A fibre of a fibred link $L \subset S^{m+2}$ is a Seifert surface F.

Fibred links

- The **complement** of a link $L^m \subset S^{m+2}$ is the $(m+2)$-dimensional manifold with boundary

$$(C, \partial C) \ = \ (\text{cl.}(S^{m+2} \backslash L \times D^2), L \times S^1)$$

  such that

$$S^{m+2} \ = \ L \times D^2 \cup_{L \times S^1} C \ .$$

- The link is **fibred** if the projection $\partial C = L \times S^1 \to S^1$ can be extended to the projection of a fibre bundle $p : C \to S^1$, and there is given a particular choice of extension.
- A fibred link with fibre F has a **monodromy** automorphism $(h, \partial h) : (F, \partial F) \to (F, \partial F)$ with $\partial h = \text{id.} : \partial F = L \to L$ and

$$C \ = \ T(h) \ = \ F \times [0,1]/\{(y,0) \sim (h(y),1) \,|\, y \in F\} \ .$$

The Milnor fibration

- **Proposition** (M, 1968) The link of an isolated hypersurface singularity is fibred.
- The complement $C(x)$ of $L(x) \subset S_\epsilon(x)^{2n+1}$ is such that
$$p \ : \ C(x) \to S^1 \ ; \ y \mapsto \frac{f(y)}{|f(y)|}$$
is the projection of a fibre bundle.
- The **Milnor fibre** is a canonical Seifert surface
$$(F(x), \partial F(x)) \ = \ (p, \partial p)^{-1}(*) \subset (C(x), \partial C(x))$$
with
$$\partial F(x) \ = \ L(x) \subset S(x)^{2n+1} \ .$$
- The fibre $F(x)$ is $(n-1)$-connected, and
$$F(x) \ \simeq \ \bigvee_{\mu} S^n \ , \ H_n(F(x)) \ = \ \mathbb{Z}^\mu$$
with $\mu = b_n(F(x)) \geqslant 0$ the **Milnor number**.

Pushing the Seifert surface from S^{2n+1} into D^{2n+2}

- For any link $L^{2n-1} \subset S^{2n+1}$ and Seifert surface $F^{2n} \subset S^{2n+1}$ can push F rel ∂ into a codimension 2 submanifold $F' \subset D^{2n+2}$ with

$$F' \cap S^{2n+1} = \partial F' = \partial F = L \subset S^{2n+1} .$$

- Push-in of the Milnor fibre $F(x)$ of the link $L(x) \subset S_\epsilon(x)^{2n+1}$ of a hypersurface singularity can be realized analytically.
- For $c \neq 0 \in \mathbb{C}$ with $|c| > 0$ sufficiently small the **smoothing**

$$X' = f^{-1}(c) \subset \mathbb{C}^{n+1}$$

is a variety with no singularities in $D_\epsilon(x)$, and

$$X' \cap (D_\epsilon(x), S_\epsilon(x)) \cong (F(x), \partial F(x)) .$$

The intersection form

- Let $(F, \partial F)$ be a $2n$-dimensional manifold with boundary, such as a Seifert surface. Denote $H_n(F)$/torsion by $H_n(F)$.
- The **intersection form** is the $(-1)^n$-symmetric bilinear pairing

$$b \ : \ H_n(F) \times H_n(F) \to \mathbb{Z} \ ; \ (y,z) \mapsto \langle y^* \cup z^*, [F] \rangle$$

  with $y^*, z^* \in H^n(F, \partial F)$ the Poincaré-Lefschetz duals of $y, z \in H_n(F)$ and $[F] \in H_{2n}(F, \partial F)$ the fundamental class.
- The intersection pairing is $(-1)^n$-symmetric

$$b(y,z) \ = \ (-1)^n b(z,y) \in \mathbb{Z} \ .$$

- The adjoint $\mathbb{Z}$-module morphism

$$\begin{aligned} b = (-1)^n b^* : H_n(F) \to H_n(F)^* = \mathrm{Hom}_{\mathbb{Z}}(H_n(F), \mathbb{Z}) \ ; \\ y \mapsto (z \mapsto b(y,z)) \ . \end{aligned}$$

  is an isomorphism if ∂F and F have the same number of components.

The monodromy theorem

- The monodromy induces an automorphism of the intersection form
$$h_* \ : \ (H_n(F), b) \to (H_n(F), b) \ ,$$
or equivalently $h^* : (H^n(F), b^{-1}) \to (H^n(F), b^{-1})$.
- **Monodromy theorem** (Brieskorn, 1970)
For the fibred link $L \subset S^{2n+1}$ of an isolated hypersurface the $\mu = b_n(F)$ eigenvalues of the monodromy automorphism
$$h^* : H^n(F;\mathbb{C}) = \mathbb{C}^\mu \to H^n(F;\mathbb{C}) = \mathbb{C}^\mu$$
are roots of 1
$$\lambda_k \ = \ e^{2\pi i \alpha_k} \in S^1 \subset \mathbb{C} \ (1 \leqslant k \leqslant \mu)$$
for some $\{\alpha_1, \alpha_2, \dots, \alpha_\mu\} \in \mathbb{Q}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z}$. Furthermore, h^* is such that for some $N \geqslant 1$
$$((h^*)^N - \mathrm{id.})^{n+1} \ = \ 0 \ : \ H^n(F;\mathbb{C}) \to H^n(F;\mathbb{C}) \ .$$

Spectral pairs I.

- Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ have an isolated singularity at $x \in f^{-1}(0)$, with Milnor fibre $F^{2n} = F(x)$. If F were a complex nonsingular projective variety then $H^n(F;\mathbb{C})$ would carry a pure Hodge structure, with only one filtration.
- Steenbrink (1976) used analysis to construct a mixed Hodge structure on $H^n(F;\mathbb{C})$, with both a Hodge and a weight filtration. Invariant under h^* and polarized by b.
- The **Hodge numbers** are the dimensions of the generalized eigenspaces of h^*, with $\lambda = e^{2\pi i\alpha} \in S^1$ ($\alpha \in \mathbb{Q}$) the eigenvalues,

$$h_\lambda^{p,q} = \dim_{\mathbb{C}}(\ker((h^* - \lambda\,\mathrm{id.})^\infty : H^{p,q}(F;\mathbb{C}) \to H^{p,q}(F;\mathbb{C}))) \in \mathbb{N}$$

with weight

$$w = \begin{cases} p+q & \text{if } \lambda \neq 1, \\ p+q-1 & \text{if } \lambda = 1 \,. \end{cases}$$

Spectral pairs II.

- The Hodge numbers encoded in the **spectral pairs** of f at x

$$\mathrm{Spp}(f) \;=\; \sum_{k=1}^{\mu}(\alpha_k, w_k) \in \mathbb{N}[\mathbb{Q}\times\mathbb{N}]$$

  with $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_\mu$ such that $\{\lambda_k = e^{2\pi i\alpha_k} \in S^1\}$ are the eigenvalues of h^*, $[\alpha_k] = n - p$, and $w_k \in \mathbb{N}$ the weights.
- A spectral pair (α, w) includes a lift to $\alpha \in \mathbb{Q}$ of the eigenvalue $\lambda = e^{2\pi i\alpha} \in S^1$ of the monodromy automorphism

$$h^* \;:\; H^n(F;\mathbb{C}) \to H^n(F;\mathbb{C}) \;.$$

- w is determined by the unipotent part of the monodromy (upper-triangular with respect to some basis), giving information about the size of the Jordan blocks of

$$(\;\ker((h^* - \lambda\,\mathrm{id.})^\infty : H^n(F;\mathbb{C}) \to H^n(F;\mathbb{C}))\;,\;\; h^*|\;).$$

The semicontinuity of the singularity spectrum

- The **spectrum** of f at an isolated hypersurface singularity x is

$$\mathrm{Sp}(f) \;=\; \sum_{j=1}^{\mu} \alpha_j \in \mathbb{N}[\mathbb{Q}]$$

  with $\mu = b_n(F)$ the Milnor number.
- **Arnold semicontinuity conjecture** (1981)
  The spectrum is semicontinuous in the following sense: if (f, x) is adjacent to (f', x') with $\mu' < \mu$ then $\alpha_k \leqslant \alpha'_k$ for $k = 1, 2, \ldots, \mu'$.
- Varchenko (1983) and Steenbrink (1985) proved a reformulation of the conjecture in terms of semicontinuity domains, using Hodge theoretic methods.

The mod 2 spectral pairs and the mod 2 spectrum

- The real Seifert form and the spectral pairs of isolated hypersurface singularities (Némethi, Comp. Math. 1995)
  Introduced the **mod 2 spectral pairs** and **mod 2 spectrum** of f at an isolated hypersurface singularity

$$\begin{aligned} \mathrm{Spp}_2(f) &= \sum_{k=1}^{\mu} (\alpha_k, w_k) \in \mathbb{N}[\mathbb{Q}/2\mathbb{Z} \times \mathbb{N}] \ , \\ \mathrm{Sp}_2(f) &= \sum_{k=1}^{\mu} \alpha_k \in \mathbb{N}[\mathbb{Q}/2\mathbb{Z}] \ . \end{aligned}$$

- The spectrum is an analytic invariant, and the semicontinuity is analytic. How much of it is purely topological?

The BNR programme

- Borodzik+Némethi The spectrum of plane curves via knot theory (2011, to appear in J. LMS) applied the cobordism theory of links, **Murasugi-type inequalities** for the **Tristram-Levine signatures** to give a topological proof of the semicontinuity of the mod 2 spectrum of the links of isolated singularities of $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$.
- BNR (2012) 3 papers in preparation, including the use of **algebraic surgery** to prove more general Murasugi-type inequalities, giving a topological proof for semicontinuity of the mod 2 spectrum of the links of isolated singularities of $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ for all $n \geqslant 1$.

Seifert forms

- For any link $L^{2n-1} \subset S^{2n+1}$ and Seifert surface $F^{2n} \subset S^{2n+1}$ the intersection form has a **Seifert form** refinement

$$S \ : \ H_n(F) \times H_n(F) \to \mathbb{Z}$$

  such that

$$b(y,z) \ = \ S(y,z) + (-1)^n S(z,y) \in \mathbb{Z} \ .$$

- Seifert (for $n=1$, 1934) and Kervaire (for $n \geqslant 2$, 1965) defined S geometrically using the linking of n-cycles in $L, L' \subset S^{2n+1}$, with L' a copy of L pushed away.
- In terms of adjoints

$$b \ = \ S + (-1)^n S^* \ : \ H_n(F) \to H^n(F) \ = \ H_n(F)^* \ .$$

The variation map of a fibred link

- The **variation map** of a fibred link $L^{2n-1} \subset S^{2n+1}$ is an isomorphism
$$V \ : \ H_n(F, \partial F) \to H_n(F)$$
satisfying the **Picard-Lefschetz** relation
$$h - \text{id.} \ = \ V \circ b \ : \ H_n(F) \to H_n(F) \ .$$
- The Seifert form of a fibred link $L^{2n-1} \subset S^{2n+1}$ with respect to the fibre Seifert surface $F^{2n} \subset S^{2n+1}$ is an isomorphism
$$S \ = \ V^{-1} \circ b \ : \ H_n(F) \to H^n(F) \ \cong \ H_n(F)^* \ .$$

Hermitian variation structures

- Use the conjugation involution

$$\mathbb{C} \to \mathbb{C} \; ; \; z = x + iy \mapsto \bar{z} = x - iy$$

to define the **dual** vector space of a finite-dimensional complex vector space U

$$U^* \; = \; \mathrm{Hom}_{\mathbb{C}}(U, \mathbb{C}) \; , \; (z, f)(y) = f(y)\bar{z} \; .$$

Identify $U^{**} = U$ using the isomorphism

$$U \to U^{**} \; ; \; y \mapsto (f \mapsto \overline{f(y)}) \; .$$

- For $\epsilon = \pm 1$ an ϵ-**hermitian variation structure** $(U; b, h, V)$ consists of
  1. an isomorphism $b : U \to U^*$ with $b(y, z) = \epsilon\overline{b(z, y)} \in \mathbb{C}$, so that $(U, b = \epsilon b^*)$ is a nonsingular ϵ-hermitian form,
  2. an automorphism $h : (U, b) \to (U, b)$, with $h^* bh = b$,
  3. an isomorphism $V : U^* \to U$ such that

$$V^* \; = \; -\epsilon V \circ h^* \; , \; V \circ b \; = \; h - I \; .$$

What does the hermitian variation structure of an isolated hypersurface singularity tell us?

- **Theorem** (Némethi, 1995) The mod 2 spectral pairs of an isolated hypersurface singularity

$$\mathrm{Spp}_2(f) = \sum_{k=1}^{\mu} (\alpha_k, w_k) \in \mathbb{N}[(\mathbb{Q}/2\mathbb{Z}) \times \mathbb{N}]$$

carry exactly as much information as the real Seifert form

$$1 \otimes S \ : \ H_n(F;\mathbb{R}) \to H_n(F;\mathbb{R})^* = H^n(F, \partial F; \mathbb{R}) \ .$$

- $1 \otimes S$ induces a $(-1)^n$-hermitian variation structure over $\mathbb{C}$.
- The mod 2 spectrum $\mathrm{Sp}_2(f)$ is determined by the $(-1)^n$-hermitian variation structure $(H^n(F;\mathbb{C}); b, h^*, V)$

$$\mathrm{Sp}_2(f) = \sum_{\alpha \in (0,2]} s(\alpha)\alpha \in \mathbb{N}[\mathbb{Q}/2\mathbb{Z}]$$

- The multiplicity $s(\alpha)$ is determined by h polarized by b for $\lambda \neq 1$, and by S for $\lambda = 1$, with $\lambda = e^{2\pi i \alpha} \in S^1$.

The automorphism multisignature

- The **automorphism multisignature** (40.16 of High-dimensional knot theory) of a $(-1)^n$-hermitian variation structure $(U;b,h,V)$ is

$$\sigma_*^{Aut}(U;b,h,V) \ = \ \sum_{\beta\in(0,1]} t(\beta)\beta \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$$

with $\lambda = e^{2\pi i\beta} \in S^1$ the eigenvalues of h on $S^1 \subset \mathbb{C}$ and

$$t(\beta) \ = \ \text{signature}(\ker((h-\lambda\,\text{id.})^\infty : U \to U), -i^n b|) \in \mathbb{Z} \ .$$

- The mod 2 spectrum and the automorphism multisignature determine each other: the morphism

$$\mathbb{N}[\mathbb{Q}/2\mathbb{Z}] \to \mathbb{Z}[\mathbb{Q}/\mathbb{Z}] \ ; \ \alpha \mapsto (-1)^{\lfloor\alpha\rfloor}\beta \ , \ \beta = \alpha - \lfloor\alpha\rfloor \ .$$

sends $\text{Sp}_2(f)$ to $\sigma_*^{Aut}(H^n(F;\mathbb{C});b,h^*,V)$.

The cobordism of links

- A **cobordism of links** is a codimension 2 submanifold

$$(K^{2n}; L_0, L_1) \subset S^{2n+1} \times ([0,1]; \{0\}, \{1\})$$

with trivial normal bundle $K \times D^2 \subset S^{2n+1} \times [0,1]$.

- An *h*-**cobordism** of links is a cobordism such that the inclusions $L_0, L_1 \subset K$ are homotopy equivalences, e.g. if

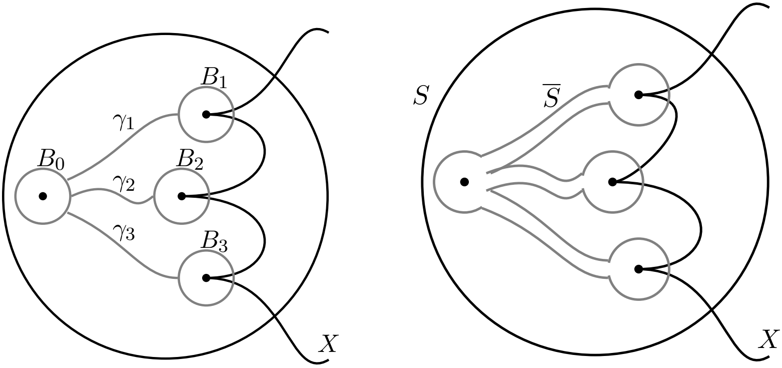
$$(K; L_0, L_1) \cong L_0 \times ([0,1]; \{0\}, \{1\}) .$$

- The *h*-cobordism theory of knots was initiated by Milnor (with Fox) in the 1950's. In the last 50 years the *h*-cobordism theory of knots and links has been much studied by topologists, both for its own sake and for the applications to singularity theory.

The cobordism of singularities of links I.

- Suppose that $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ has only isolated singularities $x_1, x_2, \ldots, x_k \in X = f^{-1}(0)$ with $\|x_j\| < 1$. Let $B_j \subset D^{2n+2}$ be small balls around the x_j's, with links
$$L(x_j) \;=\; X \cap \partial B_j \subset \partial B_j \;\cong\; S^{2n+1} \;.$$
- Assume that $S = S^{2n+1}$ is transverse to X, with $L = X \cap S \subset S$ the **link at infinity**.
- Choose disjoint ball $B_0 \subset B$, and paths γ_j inside D^{2n+2} from ∂B_0 to ∂B_j, with neighbourhoods U_j. The union
$$U \;=\; B_0 \cup \bigcup_{j=1}^{k} (B_j \cup U_j)$$
is diffeomorphic to D^{2n+2}. Will construct cobordism between the links
$$L \;,\; \overline{L} \;=\; \coprod_{j=1}^{k} L(x_j) \subset \partial U = \overline{S} \cong S^{2n+1} \;.$$

The cobordism of links of singularities II.



The cobordism of links of singularities III.

- The $2n$-dimensional submanifold

$$K^{2n} \;=\; X \cap \mathrm{cl.}(D^{2n+2} \backslash \bigcup_{j=1}^{k} B_j) \subset \mathrm{cl.}(D^{2n+2} \backslash U) \;\cong\; S^{2n+1} \times [0,1]$$

  defines a cobordism of links

$$(K; L, \overline{L}) \subset S^{2n+1} \times ([0,1]; \{0\}, \{1\}) \;.$$

- The Milnor fibres $F, \overline{F}$ for the links $L, \overline{L}$ are such that

$$F \cup_L K \cup_{\overline{L}} \overline{F} \;\cong\; F \cup_L X'$$

  with $X' \subset D^{2n}$ the smoothing of X inside D^{2n+2} such that $X' \cap B_j = F(x_j)$ is a push-in of the Milnor fibre of $L(x_j)$, and $\overline{F} = F(x_1) \cup \cdots \cup F(x_k)$.
- $(K; L, \overline{L})$ is not an h-cobordism of links in general.

The Tristram-Levine signatures $\sigma_\xi(F)$

- **Definition** (1969) The **Tristram-Levine signatures** of a link $L^{2n-1} \subset S^{2n+1}$ with respect to a Seifert surface F and $\xi \in S^1$

  $$\sigma_\xi(F) \ = \ \text{signature}(H_n(F;\mathbb{C}), (1-\xi)S+(-1)^{n+1}(1-\bar{\xi})S^*) \in \mathbb{Z}\,.$$

- The $(-1)^{n+1}$-hermitian form related to the complement cl.$(D^{2n+2}\backslash F' \times D^2)$ of push-in $F' \subset D^{2n+2}$.
- Tristram and Levine studied how $\sigma_\xi(F)$ behave under
  1. change of Seifert surface,
  2. change of ξ,
  3. the h-cobordism of links.
- **Theorem** (Levine, 1970) For $n > 1$ the signatures $\sigma_\xi(F) \in \mathbb{Z}$ determine the h-cobordism class of a knot $S^{2n-1} \subset S^{2n+1}$ modulo torsion.
- For the BNR project need to also consider how $\sigma_\xi(F)$ behaves under
  4. the cobordism of links.

The relation between $\mathbf{Sp}_2(f)$ and $\sigma_\xi(F(x))$

- Borodzik+Némethi (2010), Hodge-type structures as link invariants (to appear in Ann. Inst. Fourier).
- Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ have isolated singularity at $x \in f^{-1}(0)$ with link $L(x) \subset S^{2n+1}$ and the mod 2 spectrum $\mathrm{Sp}_2(f)$, where $|\mathrm{Sp}_2(f)| = \mu = b_n(F(x))$.
- If $\alpha \in [0,1)$ is such that $\xi = e^{2\pi i \alpha}$ is not an eigenvalue of the monodromy

$$h^* \ : \ H^n(F(x); \mathbb{C}) = \mathbb{C}^\mu \to H^n(F(x); \mathbb{C}) = \mathbb{C}^\mu$$

then

$$|\mathrm{Sp}_2(f) \cap (\alpha, \alpha+1)| \ = \ \big(\, \mu - \sigma_\xi(F(x))\,\big)/2 \ ,$$
$$|\mathrm{Sp}_2(f) \backslash (\alpha, \alpha+1)| \ = \ \big(\, \mu + \sigma_\xi(F(x))\,\big)/2 \ .$$

Surgery and Morse theory

- **Surgery** on an m-dimensional manifold F uses an embedding $S^n \times D^{m-n} \subset F$ to construct a new manifold

$$F' = \mathrm{cl.}(F \backslash S^n \times D^{m-n}) \cup D^{n+1} \times S^{m-n-1} .$$

- The **trace** of the surgery is the elementary cobordism $(E; F, F')$ defined by **attaching a** $(n+1)$**-handle**

$$E = F \times [0,1] \cup D^{n+1} \times D^{m-n} .$$

- **Theorem** (Thom, Milnor 1960) Every cobordism $(E; F_0, F_1)$ admits a Morse function $(E; F_0, F_1) \to ([0,1]; \{0\}, \{1\})$. The closed manifold F_1 is obtained from F_0 by a sequence of surgeries, and $(E; F_0, F_1)$ is a union of their traces.

The effect of surgery on the intersection form

- For a $(4k+1)$-dimensional cobordism $(E;F,F')$ the intersection form $(H_{2k}(F_1),b_1)$ is obtained from $(H_{2k}(F_0),b_0)$ by adding and subtracting forms of the type

$$(\mathbb{Z}\oplus\mathbb{Z},\begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix})$$

  for $c\in\mathbb{Z}$.
- In particular

$$\text{signature}(H_{2k}(F_0),b_0) \;=\; \text{signature}(H_{2k}(F_1),b_1)\in\mathbb{Z}\ .$$

Relative cobordisms

- An $(m+2)$-dimensional **relative cobordism**
$$(E; F_0, F_1; K; L_0, L_1)$$
is an $(m+2)$-dimensional manifold E with boundary
$$\partial E \;=\; F_0 \cup_{L_0} K \cup_{L_1} F_1$$
with F_0, F_1, K $(m+1)$-dimensional manifolds with boundaries
$$\partial F_0 \;=\; L_0 \;,\; \partial F_1 \;=\; L_1 \;,\; \partial K \;=\; L_0 \sqcup L_1 \;.$$
- **Absolute example** An absolute cobordism $(E; F_0, F_1)$ with $K = L_0 = L_1 = \emptyset$.
- **Singularity example** For every cobordism of links
$$(K^{m+1}; L_0, L_1) \subset S^{m+2} \times ([0,1]; \{0\}, \{1\})$$
there exists a relative cobordism of the Seifert surfaces
$$(E^{m+2}; F_0, F_1; K; L_0, L_1) \subset S^{m+2} \times ([0,1]; \{0\}, \{1\})$$
of Seifert surfaces.

The behaviour of the Tristram-Levine signatures under relative cobordism

- Conventional surgery and Morse theory used to describe the behaviour of the signature under cobordism.
- The BNR project requires a further development of surgery and Morse theory for manifolds with boundary, in order to describe the behaviour of the Tristram-Levine signatures under the relative cobordism of Seifert surfaces of links.
- In fact, only the algebraic surgery version is required for the project.

Relative Morse theory

- Given an $(m+1)$-dimensional manifold with boundary (F,L) and an embedding

$$(D^{n+1}\times D^{m-n}, S^n\times D^{m-n})\subset (F,L)$$

define the **elementary right product** relative cobordism $(E;F,F';K;L,L')$ by

$$\begin{aligned} E &= F\times[0,1]\ ,\ F' \ = \ \mathrm{cl.}(F\backslash D^{n+1}\times D^{m-n})\ ,\\ K &= L\times[0,1]\cup D^{n+1}\times D^{m-n}\ ,\\ L' &= \mathrm{cl.}(L\backslash S^n\times D^{m-n})\cup D^{n+1}\times S^{m-n-1}\ . \end{aligned}$$

- Reversing the ends defines an **elementary left product** relative cobordism $(E;F',F;K;L',L)$.
- **Theorem** (BNR, 2012) Every non-empty relative cobordism $(E;F_0,F_1;K;L_0,L_1)$ is a union of elementary left and right product cobordisms.

The Murasugi-type inequality

- **Theorem** (BNR, 2012) Suppose given a cobordism of $(2n-1)$-dimensional links

$$(K; L_0, L_1) \subset S^{2n+1} \times ([0,1]; \{0\}, \{1\})$$

and Seifert surfaces $F_0, F_1 \subset S^{2n+1}$ for $L_0, L_1 \subset S^{2n+1}$. Then for any $\xi \neq 1 \in S^1$

$$\begin{aligned} &|\sigma_\xi(L_0) - \sigma_\xi(L_1)| \\ &\leqslant b_n(F_0 \cup_{L_0} K \cup_{L_1} F_1) - b_n(F_0) - b_n(F_1) + n_0(\xi) + n_1(\xi) \end{aligned}$$

with b_n the nth Betti number and

$$n_j(\xi) \ = \ \text{nullity}((1-\xi)S_j + (-1)^{n+1}(1-\bar{\xi})S_j^*) \ (j = 0, 1) \ .$$

- Proved by applying previous Theorem to express the relative cobordism as a union of elementary right and left product cobordisms, and working out the effect on σ_ξ.

The semicontinuity of the mod 2 spectrum

- **Theorem** (BNR, 2012) Let $f_t : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ $(t \in \mathbb{C})$ be a family of germs of analytic maps such that $x_0 \in (f_0)^{-1}(0)$ is an isolated singularity. For a small $\epsilon > 0$, $\|t\| > 0$ let $x_1, x_2, \ldots, x_k \in (f_t)^{-1}(0) \cap B_\epsilon(0)$ be all the singularities of f_t in $B_\epsilon(0)$. Let $\alpha \in [0,1]$ be such that $\xi = e^{2\pi i \alpha}$ is not an eigenvalue of the monodromy h_0 of x_0. Then

$$|\mathsf{Sp}_{2,0}(f_0) \cap (\alpha, \alpha+1)| \geqslant \sum_{j=1}^{k} |\mathsf{Sp}_{2,j}(f_t) \cap (\alpha, \alpha+1)| \ ,$$

$$|\mathsf{Sp}_{2,0}(f_0) \backslash [\alpha, \alpha+1]| \geqslant \sum_{j=1}^{k} |\mathsf{Sp}_{2,j}(f_t) \backslash [\alpha, \alpha+1]|$$

  where $\mathsf{Sp}_{2,0}(f_0)$, $\mathsf{Sp}_{2,j}(f_t)$ are the mod 2 spectra of x_0, x_j.
- Proved topologically by applying the Murasugi-type inequality to the singularity construction of the relative cobordism of Seifert surfaces between $F(x_0)$ and $\overline{F} = \coprod_{j=1}^{k} F(x_j)$.