THE BROUWER FIXED POINT THEOREM AND THE DEGREE
(with apologies to J.Milnor)

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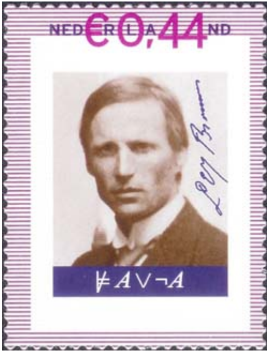
http://www.maths.ed.ac.uk/~aar/slides/brouwer.pdf

SMSTC Lecture 10th February 2011

Luitzen Egbertus Jan Brouwer, 1881-1966



- The stamp was issued by the Dutch Post Office in 2007 to celebrate the 100th anniversary of Brouwer's Ph.D. thesis. You can read the story of the stamp here.
- The MacTutor entry for Brouwer is here.
- There is a crater on the Moon named after Brouwer.

Topology from the differentiable viewpoint

- Brouwer proved the fixed point theorem and defined degree using simplicial complexes.
- Milnor's 1965 book is an excellent introduction to both differentiable and algebraic topology.
- From the introduction:
  We present some topics from the beginnings of topology, centering about L. E. J. Brouwer's definition, in 1912, of the degree of a mapping. The methods used, however, are those of differential topology, rather than the combinatorial methods of Brouwer. The concept of regular value and the theorem of Sard and Brown, which asserts that every smooth mapping has regular values, play a central role.
- Will freely use text from the Milnor book!

The Brouwer Fixed Point Theorem

- **Theorem** Every continuous function $g : D^n \to D^n$ has a fixed point, $x \in D^n$ such that $g(x) = x$.
- Will only give proof for smooth g, although the Milnor book explains how to extend this case to continuous g.
- Original statement:

  Satz 4: ***Eine eindeutige und stetige Transformation eines n-dimensionalen Elementes in sich besitzt sicher einen Fixpunkt.***

  Amsterdam, Juli 1910.

- This is the last sentence of Über Abbildung von Mannigfaltigkeiten, Mathematische Annalen 171, 97-115 (1912)
- The Wikipedia article on the Brouwer fixed point theorem is very informative!
- The theorem has applications in algebraic topology, differential equations, functional analysis, game theory, economics, ...
- There are 139,000 results on Google for "Brouwer fixed point theorem".

The natural degree

- Let $f : M \to N$ be a smooth map of compact n-dimensional manifolds. If $y \in N$ is a regular value of f, the set $f^{-1}(y) \subset N$ is a compact 0-dimensional manifold, i.e. a finite set of points, possibly empty.
- **Definition** The **natural degree** of f at a regular value $y \in N$ is the natural number of solutions $x \in M$ of $f(x) = y \in N$

$$\deg_{\#}(f;y) \ = \ |f^{-1}(y)| \in \mathbb{N} \ = \ \{0,1,2,\dots\} \ .$$

- **Key property** The natural degree function

$$\{\text{regular values of } f\} = N\backslash\{\text{critical values of } f\} \to \mathbb{N}; y \mapsto \deg_{\#}(f;y)$$

is locally constant, meaning that for every regular value $y \in N$ there exists an open subset $U \subseteq \{\text{regular values of } f\}$ with $y \in U$ and

$$\deg_{\#}(f;y) = \deg_{\#}(f;z) \in \mathbb{N} \text{ for all } z \in U \ .$$

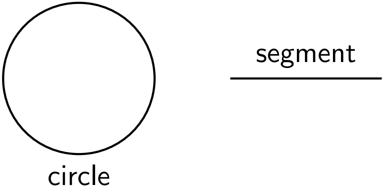
Proof in §1 of the Milnor book.

The classification of 1-dimensional manifolds

- **Theorem** A compact 1-dimensional manifold X is a finite disjoint unions of circles and segments. The boundary ∂X consists of an even number of points

$$\partial X \ = \ \{1, 2, \ldots, 2k\}$$

with $k \geqslant 0$ the number of segments.



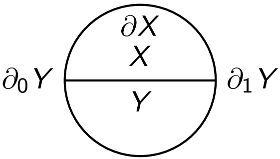
- **Proof** In the Appendix of the Milnor book.

The boundary of a manifold is not a smooth retract

- **Smooth Non-Retraction Lemma** If $(X, \partial X)$ is a compact manifold with non-empty boundary, there is no smooth map $f : X \to \partial X$ such that

$$f(x) \ = \ x \in \partial X \text{ for all } x \in \partial X \ .$$

- **Proof** (following M. Hirsch). Suppose there were such a map f. Let $y \in \partial X$ be a regular value for f. The inverse image $Y = f^{-1}(y) \subset X$ is a smooth 1-dimensional manifold, with boundary $\partial Y = \{y\}$ consisting of the single point, contradicting the classification of 1-dimensional manifolds.

- 

$$\partial Y = \partial_0 Y \cup \partial_1 Y$$

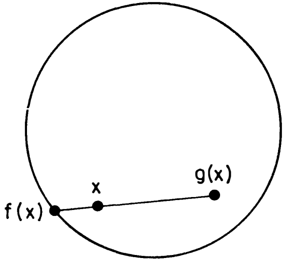
- There is also a continuous version, but rather harder to prove.

S^{n-1} is not a smooth retract of D^n.
$\{0\}$ is a smooth retract of $\mathbb{R}^+$.

- **Smooth Non-Retraction for (D^n, S^{n-1}).** The application of the Smooth Non-Retraction Lemma to the smooth compact n-dimensional manifold with non-empty boundary
$$(X, \partial X) \ = \ (D^n, S^{n-1})$$
gives that the identity map $I : S^{n-1} \to S^{n-1}$ cannot be extended to a smooth map $D^n \to S^{n-1}$.
- In fact, there is no extension of $I : S^{n-1} \to S^{n-1}$ to a continuous map $D^n \to S^{n-1}$, but this is harder to prove.
- Continuous Non-Retraction for (D^1, S^0) is equivalent to D^1 being connected.
- **$\{0\}$ is a smooth retract of $\mathbb{R}^+$.** The Smooth Non-Retraction Lemma is false for non-compact $(X, \partial X)$, e.g. for
$$(X, \partial X) \ = \ (\mathbb{R}^+, \{0\}) \ , \ \text{with } f : \mathbb{R}^+ = [0, \infty) \to \{0\}; x \mapsto 0 \ .$$

The smooth Brouwer Fixed Point Theorem

- **Theorem** Every smooth map $g : D^n \to D^n$ has a fixed point.
- **Proof** Suppose g has no fixed point. For $x \in D^n$, let $f(x) \in S^{n-1}$ be the point nearer x than $g(x)$ on the line through x and $g(x)$, as in the figure



- Then $f : D^n \to S^{n-1}$ is a smooth map with $f(x) = x$ for $x \in S^{n-1}$ is impossible by the Non-Retraction Lemma.

Smooth homotopy

- Given $X \subset \mathbb{R}^k$ let

$$X \times [0,1] \;=\; \{(x,t) \in \mathbb{R}^{k+1} \,|\, x \in X,\, 0 \leqslant t \leqslant 1\} \subset \mathbb{R}^{k+1} \;.$$

- Two mappings $f, g : X \to Y$ are called **smoothly homotopic** (abbreviated $f \simeq g$) if there exists a smooth map

$$F \;:\; X \times [0,1] \to Y$$

  such that

$$F(x,0) \;=\; f(x) \;,\; F(x,1) \;=\; g(x) \in Y \;(x \in X) \;.$$

- Smooth homotopy is an equivalence relation: see §4 of the Milnor book for proof.

The mod 2 degree is a smooth homotopy invariant

- A manifold is **closed** if it is compact and has empty boundary.
- Consider a smooth map $f : M^n \to N^n$ of closed n-dimensional manifolds. If $y \in N$ is a regular value and N is connected, we will prove that the residue class modulo 2 of the natural degree

$$\deg_{\#}(f;y) \;=\; |\{x \in M \,|\, f(x) = y\}| \in \mathbb{Z}_2 \;=\; \{0,1\}$$

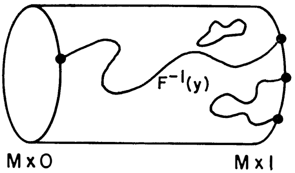
  only depends on the smooth homotopy class of f.
- In particular, the mod 2 degree does not depend on the choice of the regular value y.

The natural degree is a homotopy invariant mod 2

- **Homotopy Lemma** Let $f, g : M \to N$ be smoothly homotopic maps between n-dimensional manifolds, with M closed. If $y \in N$ is a regular value for both f and g then

$$\deg_{\#}(f; y) \;\equiv\; \deg_{\#}(g; y) \pmod 2 \;.$$

- **Proof** Let $F : M \times [0,1] \to N$ be a smooth homotopy. First suppose that y is also a regular value of F. Then $X = F^{-1}(y)$ is a compact 1-manifold with boundary $\partial X = f^{-1}(y) \times \{0\} \cup g^{-1}(y) \times \{1\}$



- The total number of points in ∂X is $\deg_{\#}(f; y) + \deg_{\#}(g; y)$, which is even by the classification of 1-dimensional manifolds.
- See §4 of the Milnor book for proof in the case when y is not regular.

Smooth isotopy and the Homogeneity Lemma

- Diffeomorphisms $f, g : X \to Y$ are **smoothly isotopic** if there exists a smooth homotopy $F : f \simeq g$ such that for each $t \in [0, 1]$ the smooth map
$$F(-, t) \ : \ X \to Y \ ; \ x \mapsto F(x, t)$$
is a diffeomorphism.
- **Homogeneity Lemma** Let y and z be arbitrary interior points of the smooth, connected manifold N. Then there exists a diffeomorphism $h : N \to N$ that is smoothly isotopic to the identity and such that $h(y) = z$.
- **Proof for** $N = S^n$ Choose h to be the rotation which carries y into z and leaves fixed all vectors orthogonal to the plane through y and z.
- See §4 of the Milnor book for the proof in the general case.

The mod 2 degree is independent of y

- **Proposition** Let $f : M^n \to N^n$ be a smooth map of n-dimensional manifolds with M closed and N connected. If $y, z \in N$ are regular values of f then

$$\deg_{\#}(f;y) \;\equiv\; \deg_{\#}(f;z) \pmod 2 \;.$$

- **Proof** Given regular values y and z, let h be a diffeomorphism from N to N which is isotopic to the identity and which carries y to z. Then z is a regular value of the composition $h \circ f$. Since $h \circ f$ is homotopic to f, the Homotopy Lemma asserts that

$$\deg_{\#}(h \circ f;z) \;=\; \deg_{\#}(f;z) \pmod 2 \;.$$

But

$$(h \circ f)^{-1}(z) \;=\; f^{-1} \circ h^{-1}(z) \;=\; f^{-1}(y) \;,$$

so that $\deg_{\#}(h \circ f;z) = \deg_{\#}(f;y)$ and

$$\deg_{\#}(f;y) \;=\; \deg_{\#}(f;z) \pmod 2 \;.$$

The mod 2 degree is a homotopy invariant

- **Definition** The **mod 2 degree** of a smooth map $f : M^n \to N^n$ is

$$\deg_2(f) \;=\; [\deg_{\#}(f;y)] \in \mathbb{Z}_2$$

  for any regular value $y \in N$ of f.
- **Theorem** The mod 2 degree of a smooth map of closed manifolds $f : M^n \to N^n$ with N connected is a homotopy invariant.
- **Proof** Suppose that f is smoothly homotopic to g. By Sard's theorem, the regular values of f and g are both dense in N, so there exists an element $y \in N$ which is a regular value for both f and g. The congruence

$$\deg_2(f) \;\equiv\; \deg_{\#}(f;y) \;\equiv\; \deg_{\#}(g;y) \;\equiv\; \deg_2(g) \pmod 2$$

  now shows that $\deg_2 f$ is a smooth homotopy invariant, and completes the proof.

An application of the homotopy invariance of the mod 2 degree

- **Example** A constant map $c : M \to M$ has even mod 2 degree. The identity map $I : M \to M$ has odd degree. Hence the identity map of a closed manifold M is not homotopic to a constant map.
- **Non-retraction of $S^{n-1} \subset D^n$** (again) For $M = S^{n-1}$ this result implies the assertion that no smooth map $f : D^n \to S^{n-1}$ such that

$$f(x) \ = x \text{ for all } x \in S^{n-1} \ .$$

  Such a map f would give rise to a smooth homotopy

$$F \ : \ S^{n-1} \times [0,1] \to S^{n-1} \ ; \ (x,t) \mapsto f(tx)$$

  between a constant map and the identity.

Oriented vector spaces

- In order to define the degree as an integer (rather than an integer modulo 2) we must introduce orientations, first for vector spaces and then for manifolds.
- **Definition** An **orientation** for a finite dimensional real vector space is an equivalence class of ordered bases as follows : the ordered basis $(b_1, \ldots, b_n)$ determines the same orientation as the basis $(b'_1, \ldots, b'_n)$ if

$$b'_i \ = \ \sum_j a_{ij} b_j \text{ with } \det(a_{ij}) > 0 \ .$$

  It determines the opposite orientation if $\det(a_{ij}) < 0$. Thus each positive dimensional vector space has precisely two orientations.
- The vector space $\mathbb{R}^n$ has a standard orientation corresponding to the basis

$$(1, 0, \ldots, 0) \ , \ (0, 1, 0, \ldots., 0) \ , \ \ \ldots \ \ , \ (0, \ldots, 0, 1) \ .$$

- In the case of the zero dimensional vector space it is convenient to define an "orientation" as the symbol $+\ 1$ or -1.

Oriented manifolds I.

- **Definition** An **oriented smooth manifold** consists of an m-dimensional manifold M together with a choice of orientation for each tangent space TM_x $(x \in M)$. If $m \geqslant 1$ these are required to fit together as follows :
  - For each point in M there should exist a neighborhood $U \subset M$ and a diffeomorphism h mapping U onto an open subset of $\mathbb{R}^m$ or H^m (closed upper half-space) which is orientation preserving, in the sense that for each $x \in U$ the isomorphism dh_x carries the specified orientation for TM_x into the standard orientation for $\mathbb{R}^m$.
- If M is connected and orientable, then it has precisely two orientations.
- If M has a boundary, we can distinguish three kinds of vectors in the tangent space TM_x at a boundary point :
  - (i) there are the vectors tangent to the boundary, forming an $(m-1)$ dimensional subspace $T(\partial M)_x \subset TM_x$,
  - (ii) there are the "outward" vectors, forming an open half space bounded by $T(\partial M)_x$
  - (iii) there are the "inward" vectors forming a complementary half space.

Oriented manifolds II.

- Each orientation for M determines an orientation for ∂M as follows :
- For $x \in \partial M$ choose a positively oriented basis $(v_1, v_2, \ldots, v_m)$ for TM_x in such a way that $v_2, \ldots, v_m$ are tangent to the boundary (assuming that $m \geqslant 2$) and that v_1 is an "outward" vector.
- Then $(v_2, \ldots, v_m)$ determines the required orientation for ∂M at x.
- If the dimension of M is 1, then each boundary point x is assigned the orientation -1 or $+1$ according as a positively oriented vector at x points inward or outward .
- **Example** The unit sphere $S^{m-1} \subset \mathbb{R}^m$ can be oriented as the boundary of the disk D^m.

The degree at a regular value

- Now let M and N be closed oriented n-dimensional manifolds and let $f : M \to N$ be a smooth map.
- **Definition** Let $x \in M$ be a regular point of f, so $df_x : TM_x \to TN_{f(x)}$ is a linear isomorphism between oriented vector spaces. The **sign** of df_x is $+1$ or -1 according as df_x preserves or reverses orientation.
- **Definition** The **degree** of f at a regular value $y \in N$ is

$$\deg(f;y) \ = \ \sum_{x \in f^{-1}(y)} \operatorname{sign} df_x \in \mathbb{Z} \ .$$

- Related to natural degree by

$$-\deg_{\#}(f;y) \leqslant \deg(f;y) \leqslant \deg_{\#}(f;y) \ .$$

- The degree function

$$\{\text{regular values of } f\} = N \backslash \{\text{critical values of } f\} \to \mathbb{Z}; y \mapsto \deg(f;y)$$

is locally constant, defined on a dense open subset of N.

The Brouwer degree of a smooth map $f : M^n \to N^n$

- **Theorem A.** For a smooth map $f : M^n \to N^n$ of closed manifolds with N connected, the integer $\deg(f; y) \in \mathbb{Z}$ does not depend on the choice of regular value $y \in N$.
- **Definition** The **degree** of f is
$$\deg(f) = \deg(f; y) \in \mathbb{Z} \text{ for any regular } y \in N .$$
- **Theorem B.** If f is smoothly homotopic to g, then
$$\deg(f) = \deg(g) \in \mathbb{Z} .$$
- The proofs of theorems A and B are essentially the same as the proofs of the corresponding results for the natural degree. It is only necessary to keep careful control of orientations and the signs.
- **Composition property** The composite of smooth maps $f : M^n \to N^n$, $g : M^n \to P^n$ is a smooth map $g \circ f : M \to P$ with
$$\deg(g \circ f) = \deg(f)\deg(g) \in \mathbb{Z} .$$
- If either M or N is given the opposite orientation $\deg(f)$ changes sign.

The degree of a map $f : S^1 \to S^1$

- **Example** For $n \in \mathbb{Z}$ the loop going around $S^1 \subset \mathbb{C}$ n times
$$f_n \ : \ S^1 \to S^1 \ ; \ z \mapsto z^n \ .$$
has $\deg(f_n; y) = n$ for each $y \in S^1$, since y has n distinct nth roots $z \in S^1$
$$z^n \ = \ y \ .$$
and f_n is orientation-preserving if and only if $n \geqslant 1$.
- Every map $f : S^1 \to S^1$ is homotopic to f_n for a unique $n \in \mathbb{Z}$, and the degree function is an isomorphism
$$\text{degree} \ : \ \pi_1(S^1) = [S^1, S^1] \xrightarrow{\cong} \mathbb{Z} \ ; \ [f] = [f_n] \mapsto n \ .$$
- By Cauchy's theorem, for analytic $f : S^1 \to S^1$
$$\deg(f) \ = \ \frac{1}{2\pi i} \oint \frac{dz}{z} \in \mathbb{Z} \ .$$

Varying degrees

- **Proposition 1** An orientation-reversing diffeomorphism of a closed manifold is not smoothly homotopic to the identity.
- **Proof** A diffeomorphism $f : M \to N$ has

$$\deg(f) \ = \ \begin{cases} +1 & \text{if } f \text{ is orientation-preserving} \\ -1 & \text{if } f \text{ is orientation-reversing} \ . \end{cases}$$

- **Proposition 2** The reflection

$$r_i \ : \ S^n \to S^n \ ; \ (x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, -x_i, \dots, x_{n+1})$$

is an orientation-reversing diffeomorphism, with degree

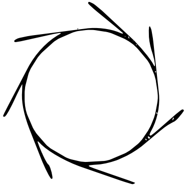
$$\deg(r_i) \ = \ -1 \in \mathbb{Z} \ .$$

Varying degrees II.

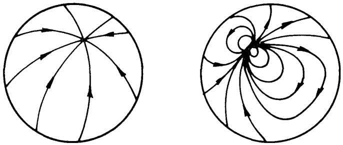
- **Proposition 3** The antipodal map
$$a \ : \ S^n \to S^n \ ; \ x = (x_1, \dots, x_{n+1}) \mapsto -x = (-x_1, \dots, -x_{n+1}) \ .$$
has degree
$$\deg(a) \ = \ (-1)^{n+1} \in \mathbb{Z} \ .$$
- **Proof** Apply the logarithmic property to
$$a = r_1 \circ r_2 \cdots \circ r_{n+1} \ : \ S^n \to S^n \ .$$
- **Proposition 4** If n is even a is not smoothly homotopic to the identity.
- **Proof** By
$$\deg(a) \ = \ -1 \neq \deg(I) \ = \ 1 \in \mathbb{Z} \ .$$
(Not detected by mod 2 degree).

Vector fields I.

- As an application of the degree, following Brouwer, we show that S^n admits a smooth field of nonzero tangent vectors if and only if n is odd.
- A nonzero vector field on S^1



- Attempts for S^2



Vector fields II.

- **Definition** A **smooth tangent vector field** on a manifold $M \subset \mathbb{R}^k$ is a smooth map $v : M \to \mathbb{R}^k$ such that $v(x) \in TM_x$ for each $x \in M$.
- In the case of the sphere $M = S^n \subset \mathbb{R}^{n+1}$ this is equivalent to the dot product condition

$$v(x) \bullet x \;=\; 0 \in \mathbb{R} \text{ for all } x \in S^n \;.$$

If $v(x) \neq 0$ for all $x \in S^n$ the map

$$\overline{v} \;:\; S^n \to S^n \;;\; x \mapsto \frac{v(x)}{\|v(x)\|}$$

is smooth. Now define a smooth homotopy

$$F \;:\; S^n \times [0, \pi] \to S^n \;;\; (x, \theta) \mapsto x \cos\theta + \overline{v}(x) \sin\theta \;.$$

Vector fields III.

- Computation shows that

$$F(x,\theta) \bullet F(x,\theta) \;=\; 1\;,$$
$$F(x,0) \;=\; x\;,\; F(x,1) \;=\; -x\;.$$

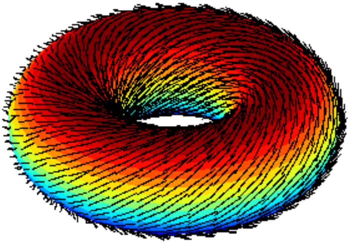
  Thus the antipodal map $a : S^n \to S^n$ is homotopic to the identity. But for n even we have seen that this is impossible.
- On the other hand, if $n = 2k-1$, the explicit formula

$$v : S^{2k-1} \to S^{2k-1}; (x_1, \ldots, x_{2k}) \mapsto (x_2, -x_1, \ldots, x_{2k}, -x_{2k-1})$$

  defines a non-zero tangent vector field on S^{2k-1}.
- The antipodal map $a : S^{2k-1} \to S^{2k-1}$ is homotopic to the identity.

The Hairy Ball Theorem

- **Hairy Ball Theorem** Every vector field $v : S^2 \to \mathbb{R}^3$ has at least one zero.
- **Proof** There is no non-zero vector field on S^2.
- **Comment** A manifold M admits a non-zero vector field (i.e. can be combed) if and only if the Euler characteristic $\chi(M) \in \mathbb{Z}$ is 0. Since $\chi(S^n) = 1 + (-1)^n$ can comb S^{2k-1} and cannot comb S^{2k}. The torus $T^2 = S^1 \times S^1$ has $\chi(T^2) = 0$, so can be combed.
- A hairy torus:



The Brouwer degree of a smooth map $f : M^n \to N^n$

- Theorems A and B imply that for closed oriented n-dimensional manifolds M, N the function

$$\text{degree} \; : \; [M,N] \to \mathbb{Z} \; ; \; [f] \mapsto \deg(f)$$

  is well-defined.
- The function is an isomorphism of abelian groups for $N = S^n$ (Hopf-Whitney theorem).
- **Example** For $M = N = S^n$ degree defines an isomorphism

$$\deg \; : \; \pi_n(S^n) \; = \; [S^n, S^n] \to \mathbb{Z} \; ; \; [f] \mapsto \deg(f) \; .$$

- Proofs in §7 of the Milnor book.

The Brouwer third degree

- Brouwer officiating at a Ph.D. examination in Amsterdam in 1960

- Photo from Volume 2 of the biography Mystic, Geometer, and Intuitionist: The Life of L.E.J. Brouwer by Dirk van Dalen, Oxford University Press (Vol.1, 1999, Vol.2, 2005)

And now for something completely different ...

- The supporters of a Dutch Ph.D. candidate are called paranymphs.
  In the Netherlands a pair of paranymphs (paranimfen) are present at the doctoral thesis defence. This ritual originates from the ancient concept where obtaining a doctorate was seen as a de facto marriage to the university. Furthermore the paranymphs would also act as a physical shield in case the debate became too heated, or as a backup for the doctoral candidate to ask for advice when answering questions. Today their role is symbolic and seen as a position of honor similar to a best man at a wedding.
- 50 years on, Dutch Ph.D. defenses have become somewhat less formal, and the paranymphs can be nymphs. Here is a YouTube film of a 2010 Amsterdam Ph.D. thesis defence of Roland van der Veen on knot theory.