ALGEBRAIC TRANSVERSALITY

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- How does one build $(n+1)$-dimensional manifolds from n-dimensional manifolds?
  - Fibre bundles over S^1.
  - Open books.

- How does one find n-dimensional submanifolds inside $(n+1)$-dimensional manifolds?
  - Geometric transversality.

- What is algebraic transversality?

Time scale

- 2- and 3-dimensional manifolds : 1900 –

- Algebraic varieties : 1920 –

- Algebraic K- and L-theory : 1940 –

- High-dimensional manifolds : 1960 –

- 3- and 4-dimensional manifolds,
  TQFT : 1980 –

Aims

- 1 : Give homological criterion on a high-dimensional manifold M which is necessary and sufficient to decompose M as a fibre bundle over S^1 .
  - Algebraic K-theory of chain complexes.

- 2 : Likewise for open book decomposition.
  - Algebraic L-theory of chain complexes with Poincaré duality.

Fibre bundles over S^1

- The mapping torus of a map $h : F \to F$ is the identification space

$$T(h) = (F \times [0,1])/\sim$$

with $(x,0) \sim (h(x),1)$.

- If F is a closed n-dimensional manifold and h is an automorphism then $T(h)$ is a closed $(n+1)$-dimensional manifold which is a fibre bundle over S^1, with projection

$$T(h) \to [0,1]/(0 \sim 1) = S^1; [x,t] \to [t]\,.$$

Brief history of fibre bundles over S^1

- Stallings (1961) : a sufficient group- and homotopy-theoretic criterion for a 3-dimensional manifold M to fibre over S^1.

- Browder and Levine (1964) ($\pi_1(M) = \mathbb{Z}$) and Farrell (1970) (any $\pi_1(M)$) : necessary and sufficient conditions for an n-dimensional manifold M to fibre over S^1, for $n \geq 6$:
  - a finitely dominated infinite cyclic cover $\overline{M}$,
  - the vanishing of the Whitehead torsion
  $$\tau = \tau(M \to T(\zeta)) \in Wh(\pi_1(M))$$
  with $\zeta : \overline{M} \to \overline{M}$ generating covering translation.

- Novikov, Farber and Pazhitnov (1981–) : S^1-valued Morse theory.

Fredholm localization

- A = ring.

- $A[z, z^{-1}]$ = Laurent polynomial extension.

- Definition : A square matrix ω in $A[z, z^{-1}]$ is Fredholm if coker(ω) is a f.g. projective A-module.

- Example : $\omega = 1 - z$ is Fredholm.

- Definition : Ω = Fredholm matrices in $A[z, z^{-1}]$. $\Omega^{-1}A[z, z^{-1}]$ = the noncommutative localization of $A[z, z^{-1}]$ inverting each $\omega \in \Omega$.

- Example : if $A = K$ is a field then $\Omega^{-1}A[z, z^{-1}] = K(z)$ is the function field.

- $K_1(\Omega^{-1}A[z, z^{-1}]) = K_1(A[z, z^{-1}]) \oplus \mathrm{Aut}_0(A)$.

Recognizing fibre bundles homologically

- M = compact n-dimensional manifold with infinite cyclic cover $\overline{M}$, such that $\pi_1(M) = \pi \times \mathbb{Z}$, $\pi_1(\overline{M}) = \pi$.

- $A = \mathbb{Z}[\pi]$, $A[z, z^{-1}] = \mathbb{Z}[\pi \times \mathbb{Z}]$.

- Theorem 1 $\overline{M}$ is finitely dominated if and only if $H_*(M; \Omega^{-1}A[z, z^{-1}]) = 0$.

- Theorem 2 If $n \geq 6$ then M is a fibre bundle over S^1 if and only if $\overline{M}$ is finitely dominated with $\Omega^{-1}A[z, z^{-1}]$-coefficient Reidemeister-Whitehead torsion (= Farrell obstruction) is 0, that is

$$\begin{aligned}&\tau(M; \Omega^{-1}A[z, z^{-1}]) = 0\\&\in K_1(\Omega^{-1}A[z, z^{-1}])/(\{\pm(\pi \times \mathbb{Z})\} \oplus \mathrm{Aut}_0(A))\\&\qquad = Wh(\pi \times \mathbb{Z}).\end{aligned}$$

Open books

- The relative mapping torus of an automorphism of an n-dimensional manifold with boundary

$$h : (F, \partial F) \to (F, \partial F)$$

with $h|_{\partial F} =$ id. is the closed $(n+1)$-dimensional manifold

$$t(h) = T(h) \cup_{\partial F \times S^1} \partial F \times D^2 .$$

- A closed $(n+1)$-dimensional manifold M has an open book decomposition if $M = t(h)$ for some h.

- Example: for a fibred knot $k : S^{n-1} \subset S^{n+1}$ $S^{n+1} = t(h)$, with h the monodromy of the Seifert surface $F^n \subset S^{n+1}$, $\partial F = k(S^{n-1})$.

- Note: open book = fibre bundle over S^1 if $\partial F = \emptyset$.

Brief history of open books

- Alexander (1923): every 3-dimensional manifold has an open book decomposition.

- Winkelnkemper (1972): for $n \geq 7$ a simply-connected n-dimensional manifold has an open book decomposition if and only if

$$\text{signature}(M) = 0 \in \mathbb{Z}\,.$$

- Quinn (1979): non-simply-connected obstruction theory in dimensions ≥ 5 to the existence and uniqueness of open book decompositions:

  – asymmetric Witt obstruction in even dimensions,

  – no obstruction in odd dimensions.

Recognizing open books homologically

- M = compact n-dimensional manifold.

- $A = \mathbb{Z}[\pi_1(M)]$, $A[z,z^{-1}] = \mathbb{Z}[\pi_1(M \times S^1)]$.

- <u>Theorem 3</u> If $n \geq 6$ then M has an open book decomposition if and only if the $\Omega^{-1}A[z,z^{-1}]$-coefficient symmetric signature (= Quinn open book obstruction) is 0

  $$\sigma^*(M;\Omega^{-1}A[z,z^{-1}]) = 0 \in L^n(\Omega^{-1}A[z,z^{-1}]) .$$

- A manifold of dimension ≥ 6 has an open book decomposition if and only if it is π_1-bordant to a fibre bundle over S^1.

Algebraic K-theory transversality

- Codimension 1 geometric transversality :
  - every infinite cyclic cover of a compact manifold has a compact fundamental domain.

- Codimension 1 algebraic transversality :
  - every finite f.g. free $A[z, z^{-1}]$-module chain complex C has an algebraic fundamental domain, with a chain equivalence

$$C(f - zg : D[z, z^{-1}] \to E[z, z^{-1}])$$

    for A-module chain maps $f, g : D \to E$. (Higman (1940), Waldhausen (1972))

- $K_1(A[z, z^{-1}]) = K_1(A) \oplus K_0(A) \oplus 2\widetilde{\mathrm{Nil}}_0(A)$. (Bass, Heller & Swan (1965))

Algebraic Poincaré complexes

- A = ring with involution $a \to \overline{a}$.

- An n-dimensional algebraic Poincaré complex over A is an A-module chain complex C with a symmetric chain equivalence

  $$\phi \simeq \phi^* : C^{n-*} = \mathrm{Hom}_A(C, A)_{n-*} \to C$$

  inducing duality $H^{n-*}(C) \cong H_*(C)$.

- Similarly for pairs, cobordism.

- $L^n(A)$ = cobordism group of n-dimensional algebraic Poincaré complexes over A.

- Similar description of Wall surgery obstruction groups $L_n(A)$, using extra quadratic structure. Difference in 2-primary torsion only.

Symmetric signature

- The symmetric signature of an n-dimensional Poincaré space M is

$$\sigma^*(M) = (C(\widetilde{M}), \phi) \in L^n(\mathbb{Z}[\pi_1(M)])$$

with $\widetilde{M}$ the universal cover of M and $\phi = [M] \cap -$ (Mishchenko (1974)).

- Homotopy invariant.

- Bordism invariant

$$\sigma^* : \Omega_n(K) \to L^n(\mathbb{Z}[\pi_1(K)])\,;\ M \to \sigma^*(M)\,.$$

- Example : for $n = 4k$

$$\sigma^*(M) = \text{signature}(M) \in L^{4k}(\mathbb{Z}) = \mathbb{Z}\,.$$

Algebraic L-theory transversality

- The asymmetric L-groups of a ring with involution A are the cobordism groups of pairs (C, λ) with C an n-dimensional f.g. free A-module chain complex and $\lambda : C^{n-*} \to C$ a chain equivalence.

- The asymmetric L-groups are 0 for odd n.

- Theorem 4 The symmetric L-groups $L^n(\Omega^{-1}A[z, z^{-1}])$ of Fredholm localization (with $\overline{z} = z^{-1}$) are isomorphic to the asymmetric L-groups of A.

Manifold transversality

- True.

- The bordism group $\Omega_n(X)$ of maps

  $$M^n = n\text{-dimensional manifold} \to X$$

  is a generalized homology theory.

- Künneth formula

  $$\Omega_n(K \times S^1) \ = \ \Omega_n(K) \oplus \Omega_{n-1}(K)$$

- Codimension 1 transversality: every map $f : M^n \to K \times S^1$ homotopic to one with

  $$N^{n-1} = f^{-1}(K \times \text{pt.}) \subset M^n$$

  a codimension 1 submanifold.

- An infinite cyclic cover of a compact manifold has a compact manifold fundamental domain.

Poincaré space transversality

- False in general – same obstructions as for algebraic Poincaré complex transversality.

- A finite n-dimensional Poincaré space P is a finite CW complex with
$$H^{n-*}(P) \cong H_*(P)\,.$$

- The bordism group $\Omega^h_n(X)$ of maps $P \to X$ is not a generalized homology theory.

- $$\Omega^h_n(K \times S^1) \;=\; \Omega^h_n(K) \oplus \Omega^p_{n-1}(K)$$
with $\Omega^p_*(K)$ the bordism group of finitely dominated Poincaré spaces with map to K.

- An infinite cyclic cover of a finite Poincaré space has a finitely dominated fundamental domain.

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