SURGERY THEORY

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- *Surgery on compact manifolds*

  C.T.C. Wall, Academic Press, 1970, A.M.S. 1999
- *Surgery on simply-connected manifolds*

  W. Browder, Springer, 1972
- *Characteristic classes*

  J. Milnor and J. Stasheff, Princeton, 1976
- *Algebraic and geometric surgery*

  A.R., Oxford, 2002
- *Surgery for amateurs*

  A.R. and J.Roe, to appear

Time scale

1885 Classification of surfaces, $n = 2$

1905 n-manifolds, duality (Poincaré)

1925 Morse theory

1940 Embeddings (Whitney)

1950 Transversality, cobordism (Thom)

1954 Signature theorem (Hirzebruch)

1956 Exotic spheres (Milnor)

1962 h-cobordism theorem for $n \geqslant 5$ (Smale)

1962–1970 Browder-Novikov-Sullivan-Wall surgery theory for $n \geqslant 5$

1965 Topological invariance of rational Pontrjagin classes (Novikov)

1970 Kirby-Siebenmann topological manifold surgery theory for $n \geqslant 5$

1970– Applications of surgery theory to n-manifold classifications for $n \geqslant 5$

1980– Surgery theory for $n = 3, 4$: under construction

Manifolds and homotopy theory

- Surgery theory considers the following questions:

  When is a space homotopy equivalent to a manifold?

  When is a homotopy equivalence of manifolds homotopic to a diffeomorphism?

- Initially developed for differentiable manifolds, the theory also has PL (= piecewise linear) and topological versions.

- Surgery theory works best for $n \geqslant 5$, when "topology = algebra".

  Much harder for $n = 3, 4$.

  Much easier for $n = 0, 1, 2$.

Some results of surgery theory, and two conjectures

• (Milnor, 1956) S^7 has 28 differentiable structures.

• (Kervaire, 1960) There exists a topological 10-manifold without a differentiable structure.

• (Novikov, 1962) For $n \geqslant 5$ a topological n-manifold M which is simply-connected ($\pi_1(M) = \{1\}$) has only a finite number of differentiable structures.

• Borel Rigidity Conjecture (1950's) Any homotopy equivalence $M' \simeq M$ of n-manifolds with $\pi_i(M) = 0$ for $i \geqslant 2$ is homotopic to a homeomorphism.

• Novikov conjecture (1969) Homotopy invariance of the higher signatures.

Surgery

- Given a differentiable n-manifold M^n and an embedding $S^i \times D^{n-i} \subset M$ $(-1 \leqslant i \leqslant n)$ define the n-manifold

  $$M' \;=\; (M - S^i \times D^{n-i}) \cup D^{i+1} \times S^{n-i-1}$$

  obtained from M by <u>surgery</u>.

- <u>Example</u> Let K, L be disjoint n-manifolds, and let $D^n \subset K$, $D^n \subset L$. The effect of surgery on $S^0 \times D^n \subset M = K \sqcup L$ is the <u>connected sum</u> n-manifold $M' = K \# L$ defined by

  $$K \# L \;=\; (K - D^n) \cup [0,1] \times S^{n-1} \cup (L - D^n)$$

- Given that surgery is such a drastic topological operation (e.g. effect on connectivity), it is surprising that it can be used to distinguish manifold structures within a homotopy type.

Attaching handles

- Let L be an $(n+1)$-manifold with boundary ∂L. Given an embedding
$$S^i \times D^{n-i} \subset \partial L$$
define the $(n+1)$-manifold
$$L' = L \cup_{S^i \times D^{n-i}} h^{i+1}$$
obtained from L by attaching an $(i+1)$-handle
$$h^{i+1} = D^{i+1} \times D^{n-i} .$$

- Proposition The boundary $\partial L'$ is obtained from ∂L by surgery on $S^i \times D^{n-i} \subset \partial L$.

- Proposition There is a homotopy equivalence $L' \simeq L \cup_{S^i} D^{i+1}$, i.e. the homotopy theoretic effect of attaching an $(i+1)$-handle is to attach an $(i+1)$-cell.

The trace

- The trace of the surgery on $S^i \times D^{n-i} \subset M^n$ is the elementary $(n+1)$-dimensional cobordism $(W; M, M')$ obtained from $M \times [0,1]$ by attaching an $(i+1)$-handle

$$W = (M \times [0,1]) \cup_{S^i \times D^{n-i} \times \{1\}} h^{i+1}$$

- Proposition The trace cobordism admits a Morse function $(W; M, M') \to ([0,1]; \{0\}, \{1\})$ with a single critical value of index $i+1$.

- Proposition If an $(n+1)$-dimensional cobordism $(W; M, M')$ admits a Morse function $(W; M, M') \to ([0,1]; \{0\}, \{1\})$ with a single critical value of index $i+1$ then $(W; M, M')$ is the trace of a surgery on an embedding $S^i \times D^{n-i} \subset M$.

Handle decomposition

- A handle decomposition of an $(n+1)$-dimensional cobordism $(W;M,M')$ is an expression as a union of elementary cobordisms

  $$(W;M,M') \;=\;$$
  $$(W_0;M,M_1)\cup(W_1;M_1,M_2)\cup\cdots\cup(W_k;M_k,M')$$

  such that

  $$W_r \;=\; (M_r\times[0,1])\cup h^{i_r+1}$$

  is the trace of a surgery on $S^{i_r}\times D^{n-i_r}\subset M_r$ with $-1\leqslant i_0\leqslant i_1\leqslant\cdots\leqslant i_k\leqslant n$.
  Note that M or M' (or both) could be empty.

- Handle decompositions non-unique, e.g. handle cancellation

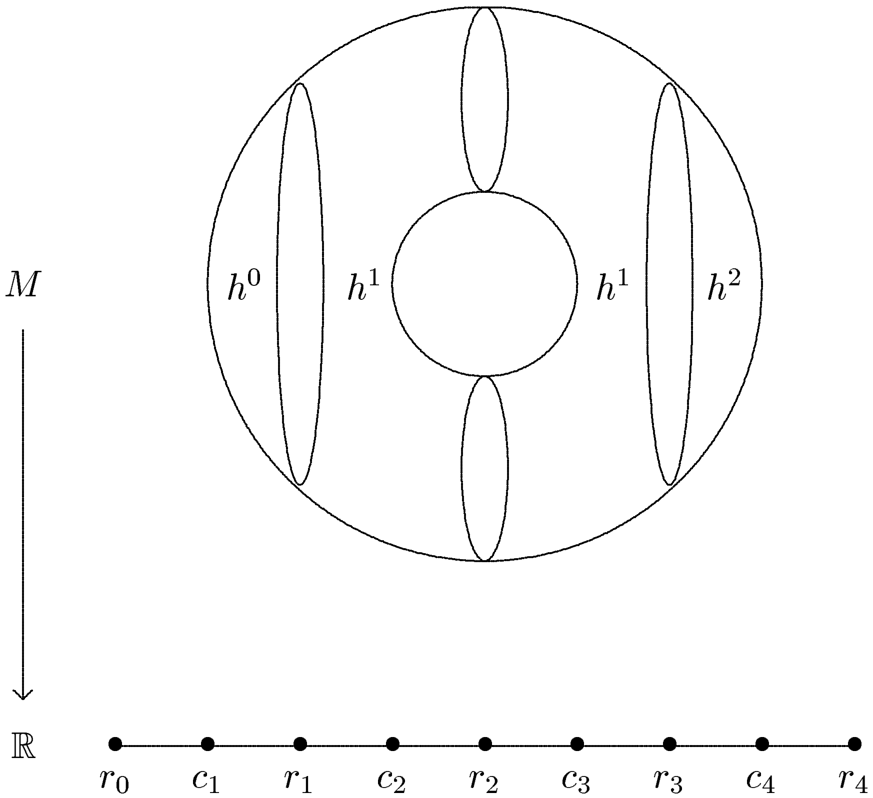
  $$W\cup h^{i+1}\cup h^{i+2} \;=\; W$$

  if one-point intersection

  $$(\{0\}\times S^{m-i-1})\cap(S^{i+1}\times\{0\}) \;=\; \{*\}\subset\partial(W\cup h^{i+1})\;.$$

The standard handle decomposition of the 2-torus

$$M \;=\; T^2 \;=\; S^1 \times S^1 \;=\; h^0 \cup h^1 \cup h^1 \cup h^2 \;.$$



$$M \;=\; S^1 \times S^1 \;=\; h^0 \cup h^1 \cup h^1 \cup h^2$$

$$\mathrm{Ind}(c_1) \;=\; 0\;,\; \mathrm{Ind}(c_2) \;=\; \mathrm{Ind}(c_3) \;=\; 1\;,\; \mathrm{Ind}(c_4) \;=\; 2$$

Cobordism $=$ sequence of surgeries

- Theorem (Thom, Milnor) Every $(n+1)$-dimensional cobordism $(W; M, M')$ admits a handle decomposition,

$$W = (M \times [0,1]) \cup \bigcup_{j=0}^{k} h^{i_j+1}$$

with $-1 \leqslant i_0 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant n$.

- Proof For any cobordism $(W; M, M')$ there exists a Morse function

$$(W; M, M') \to ([0,1]; \{0\}, \{1\})$$

with critical values $c_0 < c_1 < \cdots < c_k$ in $(0,1)$: there is one $(i+1)$-handle for each critical point of index $i+1$.

- Corollary Manifolds M, M' are cobordant if and only if M' can be obtained from M by a sequence of surgeries.

Poincaré duality

- <u>Theorem</u> For any oriented $(n+1)$-dimensional cobordism $(W;M,M')$ cap product with the fundamental class $[W] \in H_{n+1}(W, M\cup -M')$ is a chain equivalence

  $$[W]\cap - \ :\ C(W,M)^{n+1-*} \xrightarrow{\simeq} C(W,M')$$

  inducing isomorphisms

  $$H^{n+1-*}(W,M) \xrightarrow{\cong} H_*(W,M')$$

- <u>Proof</u> Compare the handle decompositions given by any Morse function

  $$f : (W;M,M') \to ([0,1];\{0\},\{1\})$$

  and the dual Morse function

  $$1-f : (W;M',M) \to ([0,1];\{0\},\{1\}) \ .$$

- For $M = M' = \emptyset$ have $H^{n+1-*}(W) \cong H_*(W)$

The algebraic effect of a surgery

- Proposition If $(W; M, M')$ is the trace of a surgery on $S^i \times D^{n-i} \subset M$ there are homotopy equivalences

$$M \cup D^{i+1} \ \simeq \ W \ \simeq \ M' \cup D^{n-i} \ .$$

Thus M' is obtained from M by first attaching an $(i+1)$-cell and then detaching an $(n-i)$-cell, to restore Poincaré duality.

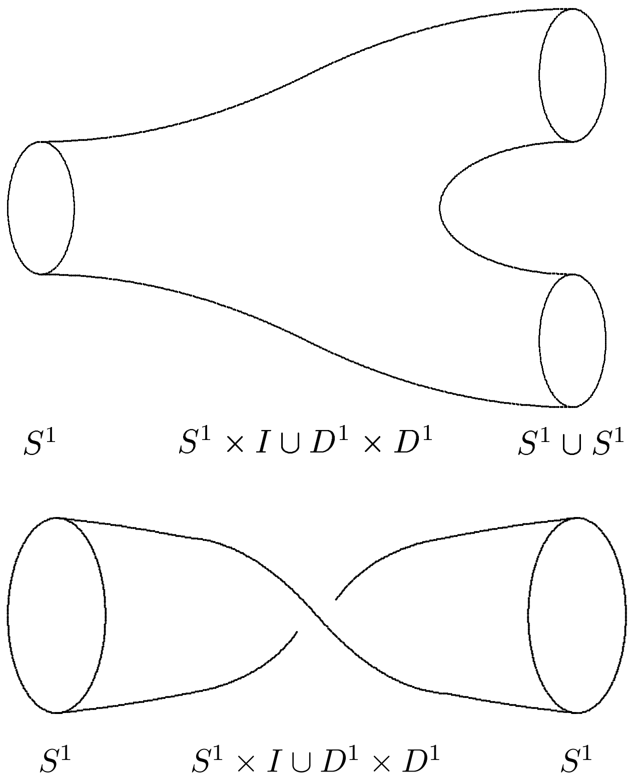
- Corollary The cellular chain complex $C(M')$ is such that $C(M')_r \ =$

$$\begin{cases} C(M)_r \oplus \mathbb{Z} & \text{for } r = i+1, n-i-1 \text{ distinct} \ , \\ C(M)_r \oplus \mathbb{Z} \oplus \mathbb{Z} & \text{for } r = i+1 = n-i-1 \ , \\ C(M)_r & \text{otherwise} \end{cases}$$

with differentials determined by the i-cycle $[S^i] \in C(M)_i = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, C(M)_i)$ and the Poincaré dual $(n-i)$-cocycle $[S^i]^* \in C(M)^{n-i} = \mathrm{Hom}_{\mathbb{Z}}(C(M)_{n-i}, \mathbb{Z})$.

Change of framing example

- There are two ways of extending $S^0 \subset S^1$ to an embedding $S^0 \times D^1 \subset S^1$, i.e. of trivializing the normal bundle $\nu_{S^0 \subset S^1}$, with correspondingly different surgeries.

- 

Vector bundles over spheres I.

- $O(k)$ = space of orthogonal $k \times k$ matrices. Every map $\omega : S^i \to O(k)$ is the clutching function of a k-plane bundle over S^{i+1}

$$\mathbb{R}^k \to E(\omega) \;=\; (D^{i+1} \times \mathbb{R}^k) \cup_\omega (D^{i+1} \times \mathbb{R}^k)$$
$$\to S^{i+1} \;=\; D^{i+1} \cup_{S^i} D^{i+1} \ .$$

  <u>Classification</u> The function

$$\pi_i(O(k)) \to \mathrm{Vect}_k(S^{i+1}); \omega \mapsto E(\omega)$$

  is a bijection. Oriented version

$$\pi_i(SO(k)) \;=\; \mathrm{SVect}_k(S^{i+1}) \ .$$

- For any k-plane bundle $\mathbb{R}^k \to E(\eta) \to X$ it may be assumed that the transition functions are orthogonal, so that there is defined a fibre bundle

$$(D^k, S^{k-1}) \to (D(\eta), S(\eta)) \to X \ .$$

Vector bundles over spheres II.

- For any $\omega : S^i \to O(n-i)$ surgery on

$$M^n = \partial(D^{i+1} \times D^{n-i}) = S^n$$

using the embedding

$$S^i \times D^{n-i} \hookrightarrow M = S^i \times D^{n-i} \cup D^{i+1} \times S^{n-i-1};$$
$$(x,y) \mapsto (x, \omega(x)(y)).$$

results in the $(n-i-1)$-sphere bundle

$$M' = S(\omega)^n = D^{i+1} \times S^{n-i-1} \cup_\omega D^{i+1} \times S^{n-i-1}$$

of the $(n-i)$-plane bundle

$$\mathbb{R}^{n-i} \to E(\omega) \to S^{i+1} = D^{i+1} \cup_{S^i} D^{i+1}$$

with clutching function ω.

- The trace of the surgery is

$$(W; M, M') = (\mathrm{cl.}(D(\omega) - D^{n+1}); S^n, S(\omega))$$

with $D(\omega)^{n+1}$ the $(n-i)$-disk bundle.

Classification of vector bundles

- Grassmann manifold $G_k(\mathbb{R}^n)$ of k-dimensional subspaces $V \subseteq \mathbb{R}^n$. Canonical k-plane bundle $\gamma_{k,n}$ over $G_k(\mathbb{R}^n)$ has total space
$$E(\gamma_{k,n}) \ = \ \{(V,x)\,|\,V \subseteq \mathbb{R}^n, x \in V\} \ .$$
Universal k-plane bundle $\gamma_k = \varinjlim_n \gamma_{k,n}$ over $BO(k) = \varinjlim_n G_k(\mathbb{R}^n)$.

- <u>Classification</u> Bijection
$$[X, BO(k)] \xrightarrow{\cong} \mathrm{Vect}_k(X); f \mapsto f^*\gamma_k \ .$$

- $H^*(BO(k);\mathbb{Q}) = \mathbb{Q}[p_1, p_2, \ldots, p_{\lfloor k/2 \rfloor}]$ with
$$p_i = (-)^i c_{2i}(\mathbb{C} \otimes \gamma_k) \in H^{4i}(BO(k))$$
the universal Pontrjagin classes.
Oriented case: $[X, BSO(k)] \cong \mathrm{SVect}_k(X)$.
Euler class $e \in H^{2k}(BSO(2k))$.

Transversality

- The Thom space of a k-plane bundle $\eta : \mathbb{R}^k \to E(\eta) \to X$ is
$$T(\eta) \;=\; D(\eta)/S(\eta)$$
(= the 1-point compactification of $E(\eta)$ for compact X). For oriented η have Thom isomorphism $\widetilde{H}_*(T(\eta)) \cong H_{*-k}(X)$.

- A map $g : L^{n+k} \to T(\eta)$ from an $(n+k)$-manifold is transverse at the zero-section $X \subset T(\eta)$ if the inverse image
$$M^n \;=\; g^{-1}(X) \subset L$$
is a codimension k submanifold with normal k-plane bundle $\nu_{M\subset L} = f^*\eta$ the pullback of η along $f = g| : L \to X$, with bundle map $b : \nu_{M\subset L} \to \eta$.

- Theorem (Sard-Thom) Every map $g : L^{n+k} \to T(\eta)$ is homotopic to one which is transverse at $X \subset T(\eta)$.

Cobordism I.

- Theorem (Pontrjagin-Thom) For any k-plane bundle $\mathbb{R}^k \to E(\eta) \to X$ the homotopy group $\pi_{n+k}(T(\eta))$ is isomorphic to the bordism group $\Omega_n(X,\eta)$ of n-dimensional submanifolds $M^n \subset S^{n+k}$ with a bundle map

$$(\ f : M \to X \ , \ b : \nu_{M \subset S^{n+k}} \to \eta \ ).$$

- Proof Define an isomorphism

$$\pi_{n+k}(T(\eta)) \xrightarrow{\cong} \Omega_n(X,\eta);$$

$$(g : S^{n+k} \to T(\eta)) \mapsto (g| : M^n = g^{-1}(X) \to X)$$

taking g to be transverse at $X \subset T(\eta)$.

- Let $MSO(k) = T(\gamma_k)$ be the Thom space of universal oriented k-plane bundle γ_k over $BSO(k)$. Oriented cobordism = stable homotopy of the Thom spectrum:

$$\Omega_n \ = \ \varinjlim_k \pi_{n+k}(MSO(k)) \ .$$

Cobordism II.

- Framed cobordism Ω_n^{fr} = cobordism ring of closed framed n-manifolds. Pontrjagin-Thom isomorphism
$$\Omega_n^{fr} \cong \varinjlim_k \pi_{n+k}(S^k) = \pi_n^S .$$

- Oriented cobordism Ω_n = cobordism ring of closed oriented n-manifolds. Pontrjagin-Thom isomorphism
$$\Omega_n \cong \varinjlim_k \pi_{n+k}(MSO(k))$$
with $MSO(k)$ the Thom space of the universal oriented k-plane bundle over $BSO(k)$.

- Low-dimensional computations
$\Omega_0 = \Omega_0^{fr} = \mathbb{Z}$,
$\Omega_1^{fr} = \Omega_2^{fr} = \mathbb{Z}_2$, $\Omega_3^{fr} = \mathbb{Z}_{24}$, $\Omega_4^{fr} = 0$
$\Omega_1 = \Omega_2 = \Omega_3 = 0$, $\Omega_4 = \mathbb{Z}$.

The intersection form

- The intersection form of an oriented $2i$-manifold M is $(-)^i$-symmetric bilinear form
$$\lambda : H^i(M,\partial M) \times H^i(M,\partial M) \to \mathbb{Z} ;$$
$$(x,y) \mapsto \langle x \cup y, [M] \rangle$$
with $[M] \in H_{2i}(M,\partial M)$ fundamental class.

- For homology classes $x,y \in H^i(M,\partial M) = H_i(M)$ represented by transverse immersions $x,y : S^i \to M$
$$\lambda(x,y) = \sum_{w \in x(S^i) \cap y(S^i)} \pm 1 \in \mathbb{Z}$$
is the geometric intersection number.

- If $\partial M = \emptyset$ or S^{2i-1} form is nonsingular:
$$H_i(M) \to H_i(M)^* = \mathrm{Hom}_{\mathbb{Z}}(H_i(M),\mathbb{Z});$$
$$x \mapsto (y \mapsto \lambda(x,y))$$
is an isomorphism, modulo torsion.

The signature

- The signature of oriented $4k$-manifold M is the signature $\sigma(M) \in \mathbb{Z}$ of $(H^{2k}(M), \lambda)$. Product formula $\sigma(M \times N) = \sigma(M)\sigma(N)$.

- Example The intersection pairing of $M = S^{2k} \times S^{2k}$ is the hyperbolic form

$$(H^{2k}(M), \lambda) = (\mathbb{Z} \oplus \mathbb{Z}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$$

  with signature $\sigma(M) = 0$.

- Example The complex projective space of lines in $\mathbb{C}^{2k+1}$

$$\mathbb{CP}^{2k} = (\mathbb{C}^{2k+1} - \{0\})/\{z \sim wz \,|\, w \neq 0 \in \mathbb{C}\}$$

  is an oriented $4k$-manifold with

$$(H^{2k}(\mathbb{CP}^{2k}), \lambda) = (\mathbb{Z}, 1) \ , \ \sigma(\mathbb{CP}^{2k}) = 1$$

Signature is a cobordism invariant

- Theorem (Thom) The signature defines a surjective ring morphism

$$\sigma : \Omega_{4k} \to \mathbb{Z} \ ; \ M \mapsto \sigma(M) \ .$$

Proof If $(W;M,M')$ is a $(4k+1)$-dimensional cobordism with c_i i-handles then $(H^{2k}(M'),\lambda')$ is obtained from $(H^{2k}(M),\lambda)$ by adding $c_{2k}-c_{2k+1}$ hyperbolic forms, each of which has signature 0.

- Computations (i) $\sigma : \Omega_4 \cong \mathbb{Z}$, generated by the complex projective plane $\mathbb{CP}^2$.

(ii) Ω_* finitely generated.

(iii) $\Omega_* \otimes \mathbb{Q} \ = \ \mathbb{Q}[\mathbb{CP}^{2k} \,|\, k \geqslant 1]$ as ring, with

$$\sigma(\mathbb{CP}^{2k_1} \times \mathbb{CP}^{2k_2} \times \cdots \times \mathbb{CP}^{2k_r}) \ = \ 1 \ .$$

The signature theorem

- Theorem (Hirzebruch, 1954) The signature of a closed oriented $4k$-manifold M is

  $$\sigma(M) = \langle \mathcal{L}_k(p_1(M), p_2(M), \dots, p_k(M)), [M] \rangle \in \mathbb{Z}$$

  a characteristic number of the tangent bundle τ_M, with $\mathcal{L}_k$ a polynomial with rational coefficients in the Pontrjagin classes

  $$p_i(M) = (-)^i c_{2i}(\tau_M \otimes \mathbb{C}) \in H^{4i}(M) .$$

  Proof The left and right hand sides are same morphism $\Omega_{4k} \to \mathbb{Z}$, since they agree on products $\mathbb{CP}^{2k_1} \times \mathbb{CP}^{2k_2} \times \cdots \times \mathbb{CP}^{2k_r}$.

- Example (i) $\mathcal{L}_1 = p_1/3$, $p_1(\mathbb{CP}^2) = 3$.

  (ii) $\mathcal{L}_2 = (7p_2 - (p_1)^2)/45$, $p_1(\mathbb{CP}^4) = 5$, $p_2(\mathbb{CP}^4) = 10$.

- Corollary If M is framed (= tangent bundle τ_M stably trivial) then $\sigma(M) = 0$.

Some $(i-1)$-connected $2i$-manifolds

- For $i \geqslant 3$ an $(i-1)$-connected $2i$-manifold has a handle decomposition

$$M^{2i} \;=\; h^0 \cup \bigcup_g h^i$$

with g i-handles $h^i = D^i \times D^i$ attached to the 0-handle $h^0 = D^{2i}$ at embeddings $S^{i-1} \times D^i \subset S^{2i-1}$. In oriented case have intersection form on

$$H^i(M, \partial M) \;=\; H_i(M) \;=\; \mathbb{Z}^g \ .$$

If $\partial M \;=\; S^{2i-1}$ can close by $h^{2i} \;=\; D^{2i}$.

- The disk bundle of $\omega \in \pi_{i-1}(O(i)) \;=\; \mathrm{Vect}_i(S^i)$ is an $(i-1)$-connected $2i$-manifold $D(\omega)$ with boundary $\partial D(\omega) \;=\; S(\omega)$

$$(D^i, S^{i-1}) \to (D(\omega), S(\omega)) \to S^i$$

with intersection form $(\mathbb{Z}, e(\omega))$ given by the Euler number. Important special case: $\omega \;=\; \tau_{S^i}$, with $e(\omega) \;=\; \chi(S^i) \;=\; 1+(-1)^i$.

Plumbing

- Let Γ be a finite tree with vertices v_r $(1 \leqslant r \leqslant g)$, and $\omega_r \in \pi_{i-1}(SO(i)) = \mathrm{SVect}_i(S^i)$. Define $(-)^i$-symmetric form $(\mathbb{Z}^g, \lambda)$ by

$$\lambda(e_r, e_s) = \begin{cases} e(\omega_r) & \text{if } r = s \\ 1 & \text{if } r < s \text{ and } v_r, v_s \text{ incident} \\ 0 & \text{otherwise} \end{cases}$$

  with $e_r = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^g$.

- <u>Theorem</u> For $i \geqslant 3$ can realize $(\mathbb{Z}^g, \lambda)$ as the intersection form of an $(i-1)$-connected $2i$-manifold M with $H^i(M) = H_i(M) = \mathbb{Z}^g$,

$$H_i(\partial M) = \ker(\lambda : \mathbb{Z}^g \to \mathbb{Z}^g) ,$$

$$H_{i-1}(\partial M) = \mathrm{coker}(\lambda : \mathbb{Z}^g \to \mathbb{Z}^g) .$$

  <u>Proof</u> Splice together the disc bundles $D(\omega_r)$ $(1 \leqslant r \leqslant g)$ according to Γ.

- $(\mathbb{Z}^g, \lambda)$ unimodular iff $\partial M \simeq S^{2i-1}$.

An example of plumbing

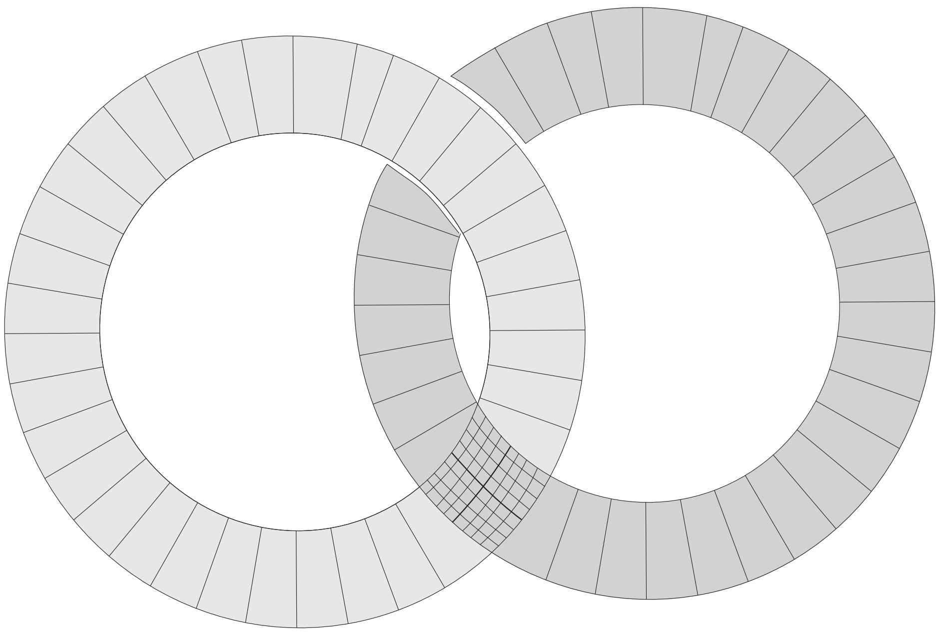
- The plumbing of two trivial 1-plane bundles $\omega_1 = \omega_2 \in \mathrm{Vect}_1(S^1)$ along the tree

$$\Gamma : v_1 \text{---} v_2$$

is $M = T^2 - D^2 =$ punctured torus, with boundary $\partial M = S^1$ and intersection form

$$(H^1(M), \lambda) \ = \ (\mathbb{Z} \oplus \mathbb{Z}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$$

- $D(\omega_1) = D(\omega_2) = S^1 \times D^1$



The Milnor E_8-plumbing I.

- Plumb 8 copies of

  $$\tau_{S^{2k}} \in \pi_{2k}(BSO(2k)) = \mathrm{SVect}_{2k}(S^{2k})$$

  along the Dynkin diagram of Lie group E_8

$$\begin{array}{ccccccccccccc} & & & & v_1 & & & & & & & & \\ & & & & | & & & & & & & & \\ v_2 & — & v_3 & — & v_4 & — & v_5 & — & v_6 & — & v_7 & — & v_8 \end{array}$$

- The symmetric integral matrix

$$E_8 = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

  is unimodular, with signature 8. For $k \geqslant 2$ intersection form of $(2k-1)$-connected $4k$-manifold M with $\partial M = \Sigma^{4k-1} \simeq S^{4k-1}$.

An exotic 7-sphere

- Theorem (Milnor, 1956) There exists a 7-manifold Σ^7 which is homeomorphic but not diffeomorphic to S^7.

  Proof The E_8-plumbing of 8 copies of τ_{S^4} is a framed 3-connected 8-manifold M with intersection form $(\mathbb{Z}^8, E_8)$. As $\partial M = \Sigma^7$ admits a Morse function with two critical points, it is homeomorphic to S^7. If Σ^7 were diffeomorphic to S^7 then
  $$N = M \cup_{\Sigma^7} D^8$$
  would be a framed-except-at-a-point closed 8-manifold. By cobordism theory $\sigma(N)$ must be a multiple of 224. But
  $$\sigma(N) = \sigma(M) = 8 \not\equiv 0 \pmod{224}$$
  so Σ^7 is not diffeomorphic to S^7.

- Failure of Hirzebruch signature theorem for manifolds with boundary, such as (M, Σ^7)
  $\sigma(M) \neq \langle \mathcal{L}_2(p_1(M), p_2(M)), [M] \rangle = 0 \in \mathbb{Z}$.

Poincaré complexes: definition

- An <u>n-dimensional Poincaré complex</u> X is a finite CW complex with a homology class $[X] \in H_n(X)$ such that there are Poincaré duality isomorphisms
$$[X] \cap - : H^{n-*}(X) \cong H_*(X)$$
with arbitrary coefficients.

- Similarly for an <u>n-dimensional Poincaré pair</u> $(X, \partial X)$, with $[X] \in H_n(X, \partial X)$ and
$$[X] \cap - : H^{n-*}(X) \cong H_*(X, \partial X).$$

- If X is simply-connected, i.e. $\pi_1(X) = \{1\}$, it is enough to just use $\mathbb{Z}$-coefficients.

- For non-oriented X need twisted coefficients.

Poincaré complexes: examples

- A closed n-manifold is an n-dimensional Poincaré complex.

- A finite CW complex homotopy equivalent to an n-dimensional Poincaré complex is an n-dimensional Poincaré complex.

- If M_1, M_2 are n-manifolds with boundary and $h : \partial M_1 \simeq \partial M_2$ is a homotopy equivalence then $X = M_1 \cup_h M_2$ is an n-dimensional Poincaré complex.

  If h is homotopic to a diffeomorphism then X is homotopy equivalent to an n-manifold.

  Conversely, if X is not homotopy equivalent to an n-manifold then h is not homotopic to a diffeomorphism.

Poincaré complexes vs. manifolds

- Theorem Let $n = 0, 1$ or 2.

  (i) Every n-dimensional Poincaré complex X is homotopy equivalent to an n-manifold. (Non-trivial for $n = 2$). (ii) Every homotopy equivalence $M \to M'$ of n-manifolds is homotopic to a diffeomorphism.

- Theorem is false for $n \geqslant 3$.

- (Reidemeister, 1930) Homotopy equivalences $L \simeq L'$ of 3-dimensional lens spaces $L = S^3/\mathbb{Z}_p$ which are not homotopic to diffeomorphisms. (Lens spaces classified by Whitehead torsion).

Homotopy types of manifolds

- The manifold structure set $\mathcal{S}(X)$ of an n-dimensional Poincaré complex X is the set of equivalence classes of pairs (M,h) with M an n-manifold and $h : M \to X$ a homotopy equivalence, subject to
  $(M,h) \sim (M',h')$ if $h^{-1}h' : M' \to M$
  is homotopic to a diffeomorphism.

- Existence Problem Is $\mathcal{S}(X)$ non-empty?

- Uniqueness Problem If $\mathcal{S}(X)$ is non-empty, compute it by algebraic topology.

- Example If $\pi_1(X) = \{1\}$ and $H_*(X) = H_*(S^n)$ then X is homotopy equivalent to S^n and

$$\mathcal{S}(X) = \mathcal{S}(S^n) \neq \emptyset \ .$$

The h-cobordism theorem

- Theorem (Smale, 1962) Let $(W; M, M')$ be an $(n+1)$-dimensional h-cobordism, so that the inclusions $i : M \subset W$, $i' : M' \subset W$ are homotopy equivalences. If $n \geqslant 5$ and W is simply-connected then $(W; M, M')$ is diffeomorphic to $M \times ([0,1]; \{0\}, \{1\})$ with the identity on M. In particular, the homotopy equivalence $h = i^{-1}i' : M' \to M$ is homotopic to diffeomorphism, and

$$(M', h) = (M, 1) \in \mathcal{S}(M).$$

- Need $n \geqslant 5$ for 'Whitney trick' realizing algebraic moves by handle cancellations.

- The non-simply-connected version is called the s-cobordism theorem (Barden, Mazur and Stallings, 1964), and requires the Whitehead torsion condition

$$\tau(i) = \tau(i') = 0 \in Wh(\pi_1(M)) .$$

A converse of the Hirzebruch signature theorem

- Theorem (Browder, 1962) For $k \geqslant 2$ a simply-connected $4k$-dimensional Poincaré complex X is homotopy equivalent to a manifold if and only if there exists a vector bundle $E \in \mathrm{Vect}_j(X)$ with a map $\rho : S^{j+4k} \to T(E)$ with Hurewicz image
$$[\rho] = [X] \in \widetilde{H}_{j+4k}(T(E)) = H_{4k}(X)$$
such that
$$\sigma(X) = \langle \mathcal{L}_k(p_1(-E), \ldots, p_k(-E)), [X] \rangle \in \mathbb{Z}$$
with $\sigma(X)$ the signature of the intersection form $(H^{2k}(X), \lambda)$ and $-E$ any vector bundle over X such that $E \oplus -E$ is trivial.

- For any n-manifold M the normal bundle ν_M of embedding $M \subset S^{j+n}$ (j large) have $\rho : S^{j+n} \to S^{j+n}/(S^{j+n} - D(\nu_M)) = T(\nu_M)$ (Pontrjagin-Thom map) such that
$$[\rho] = [M] \in \widetilde{H}_{j+n}(T(\nu_M)) = H_n(M).$$

The J-homomorphism

- $J : \pi_m(O(k)) \to \pi_{m+k}(S^k)$ defined by

$$\begin{aligned} J(\omega) : S^{m+k} &= S^m \times D^k \cup D^{m+1} \times S^{k-1} \\ \to S^k &= D^k/S^{k-1} \ ; \end{aligned}$$

$$(x,y) \mapsto \omega(x)(y) \ (x \in S^m, y \in D^k)$$

- Stable spherical fibrations $S^{k-1} \to E \to X$ are classified by homotopy classes of maps $X \to BG$ to a space BG constructed using self-homotopy equivalences $S^{k-1} \to S^{k-1}$, with

$$\pi_{m+1}(BG) = \pi_m^S = \varinjlim_k \pi_{m+k}(S^k)$$

the stable homotopy groups of spheres. BO and BG are related by a fibration sequence

$$G/O \longrightarrow BO \xrightarrow{J} BG \xrightarrow{t} B(G/O)$$

with J inducing the stable J-homomorphism

$$J : \pi_{m+1}(BO) = \varinjlim_k \pi_m(O(k)) \to \pi_{m+1}(BG) = \pi_m^S.$$

Normal maps

- Let X be an n-dimensional Poincaré complex. A normal map $(f,b) : M \to X$ is a map $f : M \to X$ from an n-manifold M which is degree 1

$$f_*[M] \;=\; [X] \in H_n(X)$$

together with an embedding $M \subset S^{j+n}$, a vector bundle $\eta \in \mathrm{Vect}_j(X)$ and a bundle map $b : \nu_{M\subset S^{j+n}} \to \eta$ over f.

- A normal bordism of normal maps $(f,b) : M \to X$, $(f',b') : M' \to X$ is a normal map of cobordisms

$$(F,B) : (W; M, M') \to X \times ([0,1]; \{0\}, \{1\}) .$$

Let $\mathcal{N}(X)$ be the bordism group of normal maps $(f,b) : M \to X$.

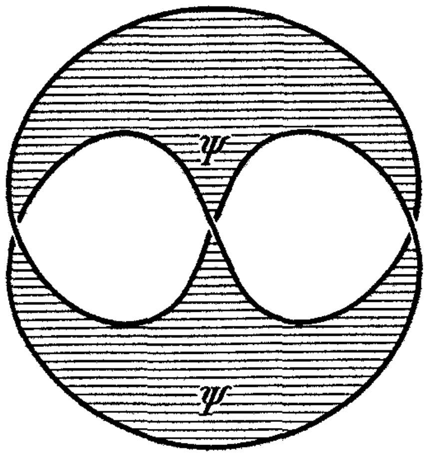
Knot theory examples of normal maps

- Every knot $k : \Sigma^n \subset S^{n+2}$ has a <u>Seifert surface</u>, a codimension 1 framed submanifold $M^{n+1} \subset S^{n+2}$ with $\partial M = k(\Sigma^n) \subset S^{n+2}$. The inclusion is a normal map

$$(f, b) : (M, \partial M) \subset (D^{n+3}, k(S^n))$$

with $(D^{n+3}, k(S^n))$ an $(n+1)$-dimensional Poincaré pair, and $\partial f = 1 : \partial M \to k(S^n)$.

- <u>Trefoil knot</u> $S^1 \subset S^3$, $M^2 = T^2 - D^2$



High-dimensional knot theory (Springer, 1998)

The Spivak normal fibration

- Proposition An n-dimensional Poincaré complex X has a canonical map $\nu_X : X \to BG$, classifying the 'Spivak normal fibration'

$$(D^k, S^{k-1}) \to (W, \partial W) \to X \quad (k \text{ large})$$

with a map $\rho : S^{n+k} \to W/\partial W$ representing the fundamental class

$$\rho[S^{n+k}] = [X] \in \widetilde{H}_{n+k}(W/\partial W) = H_n(X) .$$

- Proof For large j there exists an embedding $X \subset S^{n+j}$ with regular neighbourhood a codimension 0 submanifold $W \subset S^{n+j}$ such that $(W, \partial W)$ is a (D^k, S^{k-1})-fibration over X. Define

$$\rho : S^{n+j} \to S^{n+j}/(S^{n+j} - W) = W/\partial W .$$

The transversality construction of normal maps

- Proposition (Browder, Novikov, 1962)
  (i) An n-dimensional geometric Poincaré complex X admits a normal map iff ν_X lifts to $\widetilde{\nu}_X : X \to BO$, iff $t\nu_X = 0 \in [X, B(G/O)]$. (Key point: a purely homotopy theoretic condition).
  (ii) For an n-manifold M bijection
  $$\mathcal{N}(M) \cong [M, G/O].$$

- Consider the 'Pontrjagin-Thom' map
  $$\rho \ : \ S^{n+j} \to S^{n+j}/(S^{n+j} - W) \ = \ W/\partial W$$
  for the Spivak normal fibration. If $(W, \partial W) = (D(\eta), S(\eta))$ is the (disk,sphere)-fibration of a vector bundle $\eta : X \to BO(j)$ then $W/\partial W = T(\eta)$ is the Thom space, and ρ can be made transverse at the zero section $X \subset T(\eta)$, with $(f, b) = \rho| : M = \rho^{-1}(X) \to X$ a normal map.

The two obstructions

- When is a n-dimensional Poincaré complex homotopy equivalent to an n-manifold?

- Proposition For any $n \geqslant 0$, X is homotopy equivalent to an n-manifold if and only if

  (i) there is a normal map $(f, b) : M \to X$,

  (ii) there is an (f, b) which is normal bordant to a normal map $(f', b') : M' \to X$ with $f' : M' \to X$ is a homotopy equivalence.

- The obstruction to (i) is homotopy-theoretic, the homotopy class of $t\nu_X : X \to B(G/O)$, i.e. topological K-theory, with the bordism classes of (f, b)'s classified by $X \to G/O$.

- For $n \geqslant 5$ and any particular choice of (f, b) the obstruction to (ii) is algebraic.

The algebraic L-theory obstruction

- Theorem (Wall, 1970)
  (i) For any ring with involution A there are defined 4-periodic Grothendieck-Witt type surgery obstruction groups $L_n(A) = L_{n+4}(A)$ of quadratic forms on f.g. free A-modules and their automorphisms.
  (Key point: purely algebraic)

  (ii) An n-dimensional normal map $(f, b) : M \to X$ with $f| : \partial M \to \partial X$ a homotopy equivalence determines a surgery obstruction
  $$\sigma_*(f, b) \in L_n(\mathbb{Z}[\pi_1(X)]) \ .$$
  If $n \geqslant 5$ (f, b) is normal bordant to a homotopy equivalence if and only if $\sigma_*(f, b) = 0$. Same dimension condition $n \geqslant 5$ as in h-cobordism theorem.

- The extensive computations of $L_*(\mathbb{Z}[\pi])$ are by algebraic number theory for finite π, and by geometric topology for infinite π.

The surgery exact sequence

- Main Theorem (Browder-Novikov-Sullivan-Wall, 1962-1970). Let $n \geqslant 5$.

  (i) An n-dimensional Poincaré complex X is homotopy equivalent to an n-manifold (i.e. $\mathcal{S}(X)$ is non-empty) if and only if there exists a normal map $(f, b) : M \to X$ with $\sigma_*(f, b) = 0 \in L_n(\mathbb{Z}[\pi_1(X)])$. (Topological K-theory + algebraic L-theory).

  (ii) The structure set $\mathcal{S}(M)$ of an n-manifold M fits into exact sequence of pointed sets

$$[\Sigma M, G/O] \to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \xrightarrow{r} \mathcal{S}(M)$$
$$\to [M, G/O] \xrightarrow{\sigma_*} L_n(\mathbb{Z}[\pi_1(M)])$$

  with $\sigma_* : (f, b) \mapsto \sigma_*(f, b)$. The map r is a non-simply-connected generalization of plumbing.

Realization of groups and forms

- The fundamental group $\pi_1(M)$ of a closed manifold M is finitely presented.

- Let $n \geqslant 4$. Every finitely presented group π is the fundamental group $\pi = \pi_1(M)$ of a closed n-manifold M. For every $x \in L_{n+1}(\mathbb{Z}[\pi])$ there exists a normal bordism
  $$(f,b) : (W; M, M') \to M \times ([0,1]; \{0\}, \{1\})$$
  with $f| = 1 : M \to M$, $h = f| : M' \to M$ a homotopy equivalence, and surgery obstruction
  $$\sigma_*(f,b) \;=\; x \in L_{n+1}(\mathbb{Z}[\pi]) \ .$$
  Define $r : L_{n+1}(\mathbb{Z}[\pi]) \to \mathcal{S}(M); x \mapsto (M', h)$.

- The homotopy equivalence $h : M' \to M$ is h-cobordant to a diffeomorphism if and only if $x \in \mathrm{im}([\Sigma M, G/O] \to L_{n+1}(\mathbb{Z}[\pi]))$. (Modulo Whitehead torsion can replace 'h-cobordant' by 'homotopic').

The simply-connected surgery obstruction groups

- Computation (Kervaire and Milnor, 1963)

$$L_n(\mathbb{Z}) = \begin{cases} \mathbb{Z}\ (\sigma/8) \\ 0 \\ \mathbb{Z}_2\ (\mathrm{Arf}) \\ 0 \end{cases} \quad \text{if } n \equiv \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \pmod 4$$

- $L_{2i+1}(\mathbb{Z}) = 0$: for $i \geqslant 2$ every $(f,b) : M^{2i+1} \to X$ with $\pi_1(X) = \{1\}$ is normal bordant to a homotopy equivalence. A $(2i+1)$-dimensional Poincaré complex X with $\pi_1(X) = \{1\}$ is homotopy equivalent to a manifold if and only if $t\nu_X = 0 \in [X, B(G/O)]$.

The kernel modules

- The <u>kernel</u> $\mathbb{Z}[\pi_1(X)]$-modules of an n-dimensional normal map $(f,b) : M \to X$

$$K_i(M) \;=\; \text{ker}(f_* : H_i(\widetilde{M}) \to H_i(\widetilde{X}))$$

fit into direct sum system

$$H_i(\widetilde{M}) \;=\; K_i(M) \oplus H_i(\widetilde{X})$$

with $\widetilde{X} =$ universal cover of X, $\widetilde{M} \;=\; f^*\widetilde{X}$.

- <u>Proposition</u> (i) For $i > 1$ $(f,b) : M \to X$ is i-connected iff $f_* : \pi_1(M) \to \pi_1(X)$ is an isomorphism and $K_j(M) = 0$ for $j < i$, in which case $K_i(M) = \pi_{i+1}(f)$.

  (ii) If $n = 2i$ or $2i+1$ (f,b) is a homotopy equivalence iff $(i+1)$-connected.

  <u>Proof</u> The theorems of Hurewicz and Whitehead, the universal coefficient theorem and Poincaré duality.

Surgery on a normal map

- Suppose given an n-dimensional normal map $(f,b) : M \to X$, and an element $x \in K_i(M)$ which is represented by an embedding $S^i \times D^{n-i} \subset M$ and a null-homotopy in X. Can extend (f,b) to a normal map on the trace

  $$(F,B) : (W;M,M') \to X \times ([0,1];\{0\},\{1\})$$

  with restriction $(F,B)| = (f',b') : M' \to X$.

- The effect on the kernel modules is to <u>kill</u> $x \in K_i(M)$

  $$K_r(W) = \begin{cases} K_i(M)/\langle x\rangle & \text{if } r = i \\ K_r(M) & \text{if } r \neq i \end{cases}$$

  with $\langle x\rangle \subseteq K_i(M)$ the $\mathbb{Z}[\pi_1(X)]$-submodule generated by x.

Surgery below the middle dimension

- Proposition Let $(f, b) : M \to X$ be an n-dimensional normal map, $n \geqslant 5$.
(i) If (f, b) is i-connected and $2i < n$ then every element $x \in K_i(M)$ can be killed by surgery on (f, b).
(ii) If $n = 2i$ or $2i+1$ there exists a normal bordism of (f, b) to an i-connected normal map $(f', b') : M' \to X$, with

$$K_j(M') \;=\; \pi_{j+1}(f') \;=\; 0 \text{ for } j < i \;.$$

- $L_{2i+1}(\mathbb{Z}) = 0$, so if $n = 2i + 1 \geqslant 5$ and $\pi_1(X) = \{1\}$ can go one connectivity further, and (f, b) is normal bordant to a homotopy equivalence. But in general there is a middle-dimensional obstruction.

Quadratic forms I.

- An <u>involution</u> on a ring A is a function $A \to A; a \mapsto \overline{a}$ such that :

  $$\overline{a+b} = \overline{a} + \overline{b} \ , \ \overline{ab} = \overline{b}.\overline{a} \in A \quad (a, b \in A) \ .$$

  <u>Examples</u>: (i) A commutative, $\overline{a} = a$.
  (ii) $A = \mathbb{Z}[\pi]$ group ring, $\overline{g} = g^{-1}$ $(g \in \pi)$.

- A <u>$(-)^i$-quadratic form over A</u> (K, λ, μ) is a f.g. free A-module K together with a sesquilinear $(-)^i$-symmetric pairing $\lambda : K \times K \to A$ and a $(-)^i$-quadratic function $\mu : K \to Q_{(-)^i}(A) = A/\{a - (-)^i\overline{a} \,|\, a \in A\}$ such that for all $x, y \in K$, $b \in A$

  $$\begin{aligned}
  \lambda(x,y) &= (-)^i\overline{\lambda(y,x)}, \ \lambda(x,by) = b\lambda(x,y) \in A, \\
  \lambda(x,x) &= \mu(x) + (-)^i\overline{\mu(x)} \in A, \\
  \lambda(x,y) &= \mu(x+y) - \mu(x) - \mu(y) \in Q_{(-)^i}(A), \\
  \mu(bx) &= b\mu(x)\overline{b} \in Q_{(-)^i}(A) \ .
  \end{aligned}$$

  <u>Example</u>: For $A = \mathbb{Z}$

  $$Q_+(\mathbb{Z}) = \mathbb{Z} \ , \ Q_-(\mathbb{Z}) = \mathbb{Z}_2 \ .$$

Quadratic forms II.

- A form (K, λ, μ) over A is nonsingular if the A-module morphism

  $K \to K^* = \text{Hom}_A(K, A); x \mapsto (y \mapsto \lambda(x, y))$

  is an isomorphism.

- The hyperbolic nonsingular $(-)^i$-quadratic form over A

  $$H_{(-)^i}(F) = (F \oplus F^*, \lambda, \mu)$$

  is defined for any f.g. free A-module F, with

  $\lambda : F \oplus F^* \times F \oplus F^* \to A;$
  $((x, f), (y, g)) \mapsto f(y) + (-)^i \overline{g(x)}\ ,$
  $\mu : F \oplus F^* \to Q_{(-)^i}(A); (x, f) \mapsto f(x)\ .$

The even-dimensional L-group $L_{2i}(A)$

- A stable isomorphism of nonsingular $(-)^i$-quadratic forms (K,λ,μ) (K',λ',μ') over A is an isomorphism

  $$(K,\lambda,\mu)\oplus H_{(-)^i}(F) \cong (K',\lambda',\mu)\oplus H_{(-)^i}(F')$$

  for some f.g. free A-modules F, F'.

- Definition $L_{2i}(A)$ = the abelian group of stable isomorphism classes of nonsingular $(-)^i$-quadratic forms over A, with addition by

  $$(K,\lambda,\mu)+(K',\lambda',\mu') = (K\oplus K',\lambda\oplus\lambda',\mu\oplus\mu')$$

  and inverses by

  $$-(K,\lambda,\mu) = (K,-\lambda,-\mu) .$$

The even-dimensional surgery obstruction

- The kernel form of an i-connected $2i$-dimensional normal map $(f,b) : M \to X$ is the nonsingular $(-)^i$-quadratic form $(K_i(M), \lambda, \mu)$ over $\mathbb{Z}[\pi_1(X)]$ defined by geometric intersection and self-intersection numbers of immersions $S^i \to M$. For $i \geqslant 3$ can kill $x \in K_i(M) = \pi_{i+1}(f)$ by surgery on $S^i \times D^i \subset M$ if and only if $\mu(x) = 0 \in Q_{(-)^i}(\mathbb{Z}[\pi_1(X)])$.

- For $j = i-1$ (resp. i) the effect on the kernel form of a surgery on $S^j \times D^{2i-j} \subset M$ is to add (resp. subtract) $H_{(-)^i}(\mathbb{Z}[\pi_1(X)])$.

- The surgery obstruction of i-connected $2i$-dimensional normal map $(f,b) : M \to X$ is

  $$\sigma_*(f,b) \ = \ (K_i(M), \lambda, \mu) \in L_{2i}(\mathbb{Z}[\pi_1(X)]) \ .$$

  The surgery obstruction of any $2i$-dimensional normal map is the surgery obstruction of any bordant i-connected normal map.

The odd-dimensional surgery obstruction

- (Heegaard, 1898) Every closed connected 3-manifold M^3 has a handle decomposition of the type $M = h^0 \cup \bigcup_r h^1 \cup \bigcup_r h^2 \cup h^3$, so $M = N \cup_\alpha N$ with $N^3 = \#_r(S^1 \times D^2)$ a solid torus and $\alpha : \partial N \to \partial N$ a self-diffeomorphism of $\partial N = \#_r T^2$ inducing an automorphism of the intersection form $H^1(\partial N) = H_-(\mathbb{Z}^r)$.

- <u>Definition</u> (Wall, 1970)
  (i) $L_{2i+1}(A) = \varinjlim_r \mathrm{Aut}(H_{(-)^i}(A^r))^{ab}/\{\begin{pmatrix} 0 & 1 \\ (-)^i & 0 \end{pmatrix}\}$
  the abelian group of automorphisms of $(-)^i$-hyperbolic forms $H_{(-)^i}(A^r)$.
  (ii) The <u>surgery obstruction</u> of an i-connected $(2i+1)$-dimensional normal map $(f, b) : M \to X$ is the class
  $$\sigma_*(f, b) = \alpha \in L_{2i+1}(\mathbb{Z}[\pi_1(X)])$$
  of the automorphism α of $H_{(-)^i}(\mathbb{Z}[\pi_1(X)]^r)$ in a Heegaard-type decomposition of (f, b).

The signature/8

- A quadratic form (K, λ, μ) over $\mathbb{Z}$ is essentially the same as a symmetric form (K, λ) with each $\lambda(x, x) \in \mathbb{Z}$ $(x \in K)$ even.

- The signature of a nonsingular quadratic form (K, λ, μ) is divisible by 8, with isomorphism

$$\sigma/8 : L_{4k}(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}; (K\,\lambda, \mu) \mapsto \sigma(K, \lambda)/8$$

- The surgery obstruction of $(f, b) : M^{4k} \to X$ with $\pi_1(X) = \{1\}$ is

$$\sigma_*(f, b) = (\sigma(M) - \sigma(X))/8 \in L_{4k}(\mathbb{Z}) = \mathbb{Z}\,.$$

The Arf invariant

- The <u>Arf invariant</u> of a nonsingular (-1)-quadratic form (K,λ,μ) over $\mathbb{Z}$ is
$$A(K,\lambda,\mu) \;=\; \sum_{j=1}^{2g} \mu(x_j) \in \mathbb{Z}_2$$
for any basis $x_1, x_2, \ldots, x_{2g} \in K$ such that
$$\lambda(x_i,x_j) \;=\; \begin{cases} \pm 1 & \text{if } j-i \;=\; \pm\, g \\ 0 & \text{otherwise} \end{cases}$$

- The Arf invariant defines an isomorphism
$$A : L_{4k+2}(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}_2 ; (K,\lambda,\mu) \mapsto A(K,\lambda,\mu)$$
The surgery obstruction of $(2k+1)$-connected $(f,b) : M^{4k+2} \to X$ with $\pi_1(X) = \{1\}$ is $\sigma_*(f,b) \;=\; A(K_{2k+1}(M),\lambda,\mu) \in \mathbb{Z}_2$.

- <u>Example</u> $\Omega_2^{fr} = \pi_2^S = \mathbb{Z}_2$, generated by T^2 with exotic framing of Arf invariant 1.

Framed plumbing

- Let Γ be a finite tree with vertices v_r $(1 \leqslant r \leqslant g)$, and stably trivialized i-plane bundles over S^i

$$\mu_r \in \pi_{i+1}(BSO, BSO(i)) = Q_{(-)^i}(\mathbb{Z})$$

(generated by τ_{S^i}). Define $(-)^i$-quadratic form $(\mathbb{Z}^g, \lambda, \mu)$ by

$$\lambda(e_r, e_s) = \begin{cases} e(\omega_r) & \text{if } r = s \\ 1 & \text{if } r < s \text{ and } v_r, v_s \text{ incident} \\ 0 & \text{otherwise} \end{cases}$$

and $\mu(e_r) = \mu_r$. Assume nonsingular.

- Theorem Framed plumbing gives an i-connected $2i$-dimensional normal map

$$(f, b) : (M, \partial M) \to (D^{2i}, S^{2i-1})$$

with $\partial f : \partial M \to S^{2i-1}$ a homotopy equivalence, and surgery obstruction

$$\sigma_*(f, b) = (\mathbb{Z}^g, \lambda, \mu) \in L_{2i}(\mathbb{Z}) \ .$$

Homotopy spheres

- Generalized Poincaré conjecture (Smale, 1962): for $n \geqslant 5$ an n-manifold Σ^n is homeomorphic to S^n if and only if Σ^n is homotopy equivalent to S^n. (Proved by Morse theory and h-cobordism theorem).

- Exact sequence of abelian groups

$$\cdots \to \pi_{n+1}(G/O) \xrightarrow{\sigma_*} L_{n+1}(\mathbb{Z}) \xrightarrow{r} \mathcal{S}(S^n)$$
$$\to \pi_n(G/O) \to L_n(\mathbb{Z}) \to \ldots$$

with $r : L_{2i}(\mathbb{Z}) \to \mathcal{S}(S^{2i-1})$ the framed plumbing realization map.

- Example

$$\pi_8(G/O) = \mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{(28\ 0)} L_8(\mathbb{Z}) = \mathbb{Z} \xrightarrow{r} \mathcal{S}(S^7)$$
$$\to \pi_7(G/O) = 0 \to L_7(\mathbb{Z}) = 0$$

and $\mathcal{S}(S^7) \xrightarrow{\cong} \mathbb{Z}_{28}; \Sigma^7 \mapsto \sigma(N)/8$ with N^8 any framed 8-manifold such that $\partial N = \Sigma^7$.

The Milnor E_8-plumbing II.

- The plumbing of 8 copies of $\tau_{S^{2k}} \in \mathrm{Vect}_{2k}(S^{2k})$ $(k \geqslant 2)$ along the E_8-tree

$$\begin{tikzcd}[column sep=small]
 & & v_1 \arrow[d, no head] & & & & \\
v_2 \arrow[r, no head] & v_3 \arrow[r, no head] & v_4 \arrow[r, no head] & v_5 \arrow[r, no head] & v_6 \arrow[r, no head] & v_7 \arrow[r, no head] & v_8
\end{tikzcd}$$

  with $\mu_r = 1 \in \pi_{2k+1}(BSO, BSO(2k)) = \mathbb{Z}$ is a $2k$-connected $4k$-dimensional normal map

$$(f, b) : (M, \Sigma^{4k-1}) \to (D^{4k}, S^{4k-1})$$

  with surgery obstruction given by the signature/8

$$\sigma_*(f, b) = (\mathbb{Z}^8, E_8) = 1 \in L_{4k}(\mathbb{Z}) = \mathbb{Z}.$$

  The boundary $\partial M = \Sigma^{4k-1}$ is an exotic sphere, which is homeomorphic to S^{4k-1}.

- Can embed $M \subset S^{4k+1}$, get knot

$$\partial M = \Sigma^{4k-1} \subset S^{4k+1} .$$

The Kervaire-Arf plumbing

- The plumbing of two copies of $\tau_{S^{2k+1}} \in \text{Vect}_{2k+1}(S^{2k+1})$ along tree $\Gamma : v_1 \text{---} v_2$ with

  $\mu_1 = \mu_2 = 1 \in \pi_{2k+2}(BSO, BSO(2k+1)) = \mathbb{Z}_2$

  is a $(2k+1)$-connected $(4k+2)$-dimensional normal map

  $$(f, b) : (M, \Sigma^{4k+1}) \to (D^{4k+2}, S^{4k+1})$$

  with surgery obstruction given by the Arf invariant

  $\sigma_*(f, b) = (\mathbb{Z} \oplus \mathbb{Z}, \lambda, \mu) = 1 \in L_{4k+2}(\mathbb{Z}) = \mathbb{Z}_2.$

  For $k \geqslant 2$ boundary $\partial M = \Sigma^{4k+1}$ is an exotic sphere, homeomorphic to S^{4k+1}. Embed $M \subset S^{4k+3}$, knot $\Sigma^{4k+1} \subset S^{4k+3}$. (For $k = 0$ trefoil knot $\Sigma^1 = S^1 \subset S^3$).

- Theorem (Kervaire, 1960) The closed topological 10-manifold $N^{10} = M^{10} \cup_{\Sigma^9} D^{10}$ ($k = 2$ here) does not have a differentiable structure.

Homotopy types of topological manifolds

- The topological manifold structure set $\mathcal{S}^{TOP}(X)$ of an n-dimensional Poincaré complex X is the set of equivalence classes of pairs (M, h) with M a topological n-manifold and $h : M \to X$ a homotopy equivalence, subject to $(M, h) \sim (M', h')$ if $h^{-1}h' : M' \to M$ is homotopic to a homeomorphism.

- Example $\mathcal{S}^{TOP}(S^n) = \{*\}$ for $n \geqslant 5$.

- Theorem (Ranicki, using Kirby-Siebenmann) The structure set $\mathcal{S}^{TOP}(M)$ of a topological n-manifold M for $n \geqslant 5$ fits into exact sequence of abelian groups

$$[\Sigma M, G/TOP] \to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to \mathcal{S}^{TOP}(M) \to [M, G/TOP] \to L_n(\mathbb{Z}[\pi_1(M)])$$

with $\pi_*(G/TOP) = L_*(\mathbb{Z})$. (See *Algebraic L-theory and topological manifolds*, Cambridge Tract, 1992)

Aspherical manifolds

- An n-dimensional Poincaré duality group G is a group with the Eilenberg-MacLane space $K(G,1)$ an n-dimensional Poincaré complex. (G is infinite and torsion-free.) E.g. if G acts freely on $\mathbb{R}^n$ with $\mathbb{R}^n/G$ compact.

- Generalized Borel conjecture For any such G there exists an n-manifold M with $\pi_1(M) = G$, $\pi_i(M) = 0$ ($i \geqslant 2$), so that $M \simeq K(G,1)$. Moreover, for any homotopy equivalences $h : M \to K(G,1)$, $h' : M' \to K(G,1)$ the composite $h^{-1}h' : M' \to M$ is homotopic to a homeomorphism, $\mathcal{S}^{TOP}(K(G,1)) = \{*\}$. Many verifications, no counterexamples.

- Example The free abelian group $\mathbb{Z}^n$ is an n-dimensional Poincaré duality group, with
$$K(\mathbb{Z}^n,1) = T^n = S^1 \times S^1 \times \cdots \times S^1$$
and $\mathcal{S}^{TOP}(T^n) = \{*\}$ for $n \geqslant 5$.

Further reading

- *Novikov Conjectures, Index theorems and Rigidity*, Oberwolfach 1993, LMS Lecture Notes 226, 227, Cambridge (1995)

- *Surveys on surgery theory*, C.T.C. Wall 60th birthday Festschrift, Ann. of Maths. Studies 145, 149, Princeton (2000)

- *Topology of high-dimensional manifolds*, ICTP Trieste 2001, World Scientific (2003)
  Lecture notes from
  www.ictp.trieste.it/~pub_off/lectures/vol9.html

- *The Novikov Conjecture*, Oberwolfach 2004, Kreck & Lück, Birkhäuser (2004)

- www.maths.ed.ac.uk/~aar/surgery

"Nurse, get on the internet, go to SURGERY.COM, scroll down and click on the 'Are you totally lost?' icon."