POLYNOMIALS, QUADRATIC FORMS AND THE TOPOLOGY OF MANIFOLDS

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In Edinburgh

“Signatures, braids and Seifert surfaces”

- A collection of old and new papers to appear later in 2016 in a volume edited by Étienne Ghys and myself of the Brazilian online journal Ensaios Matemáticos:
    - Étienne Ghys and Andrew Ranicki
      Signatures in algebra, topology and dynamics
    - Jean-Marc Gambaudo and Étienne Ghys
      Braids and signatures
    - Arjeh Cohen and Jack van Wijk
      Visualization of Seifert Surfaces
    - Julia Collins
      An algorithm for computing the Seifert matrix of a link from a braid representation
    - Maxime Bourrigan
      Quasimorphismes sur les groupes de tresses et forme de Blanchfield
    - Chris Palmer
      Seifert matrices of braids with applications to isotopy and signatures

In the beginning

- **Major problem from early 19th century**
  How many real roots does a degree n real polynomial $P(X) \in \mathbb{R}[X]$ have in an interval $[a, b] \subset \mathbb{R}$?
  That is, calculate

$$\begin{aligned} \#\mathbb{R}\text{-roots}(P(X); [a, b]) &= |\{x \in [a, b] \mid P(x) = 0\}| \\ &\in \{0, 1 \ldots, n\} \end{aligned}$$

- In 1829 Sturm solved the problem algorithmically, using the Euclidean algorithm in $\mathbb{R}[X]$ for the greatest common divisor of $P(X)$ and $P'(X)$ and counting sign changes.
- In 1853 Sylvester interpreted Sturm's theorem using the continued fraction expansion of $P(X)/P'(X)$ and the signatures of symmetric matrices. **This was the first ever application of the signature!**
- There have been very many applications of the signatures since then, particularly in the topology of manifolds.

Plan for today

1. The Sturm algorithm for $\#\mathbb{R}\text{-roots}(P(X);[a,b])$ for a degree n real polynomial $P(X) \in \mathbb{R}[X]$.
2. The Sylvester expression for $\#\mathbb{R}\text{-roots}(P(X);[a,b])$ as a difference of Witt classes

$$\big((\mathbb{R}^n, \mathrm{Tri}(b)) - (\mathbb{R}^n, \mathrm{Tri}(a))\big)/2 \in W(\mathbb{R}) = \mathbb{Z} \text{ (signature)}$$

   of **tridiagonal** symmetric matrices ($=$ forms) over $\mathbb{R}$.
3. The Ghys-R. expression for $\#\mathbb{R}\text{-roots}(P(X);[a,b])$ in terms of the Witt class

$$(\mathbb{R}(X), P(X)) \in W(\mathbb{R}(X)) = \bigoplus_{\infty} \mathbb{Z} \oplus \bigoplus_{\infty} \mathbb{Z}_2 \text{ (multisignature)}$$

   with $\mathbb{R}(X)$ the field of fractions of the polynomial ring $\mathbb{R}[X]$.
4. Tridiagonal symmetric matrices in the Milnor-Hirzebruch plumbing of sphere bundles, and the work of Barge-Lannes on the Maslov index and Bott periodicity.

Jacques Charles François Sturm (1803-1855)

The Sturm sequences

- Sturm's 1829 algorithmic formula for the number of real roots involved the **Sturm sequences** of $P(X) \in \mathbb{R}[X]$: the remainders $P_k(X)$ and quotients $Q_k(X)$ in the Euclidean algorithm (with sign change) in $\mathbb{R}[X]$ for finding the greatest common divisor of $P_0(X) = P(X)$ and $P_1(X) = P'(X)$

  $$P_*(X) = (P_0(X), \dots, P_n(X)) \ , \ Q_*(X) = (Q_1(X), \dots, Q_n(X))$$

  with $\deg(P_{k+1}(X)) < \deg(P_k(X)) \leqslant n - k$ and

  $$P_{k-1}(X) + P_{k+1}(X) \ = \ P_k(X)Q_k(X) \ (1 \leqslant k \leqslant n) \ .$$

- **Simplifying assumption** $P(X)$ is generic: the roots of $P_0(X)$, $P_1(X), \dots, P_n(X)$ are distinct, so that $\deg(P_k(X)) = n - k$, $P_n(X)$ is a non-zero constant, and $\deg(Q_k(X)) = 1$.

Variation

- The **variation** $\mathrm{var}(p)$ of $p = (p_0, p_1, \ldots, p_n) \in (\mathbb{R} \backslash \{0\})^{n+1}$ is the number of sign changes $p_0 \to p_1 \to \cdots \to p_n$.
- The variation is expressed in terms of the sign changes $p_{k-1} \to p_k$ by

$$\mathrm{var}(p) \;=\; (n - \sum_{k=1}^{n} \mathrm{sign}(p_k/p_{k-1}))/2 \in \{0, 1, \ldots, n\} \; .$$

- Sturm's root-counting formula involved the variations of the Sturm remainders $P_k(X)$ evaluated at 'regular' $x \in \mathbb{R}$.
- Call $x \in \mathbb{R}$ **regular** if $P_k(x) \neq 0$ $(0 \leqslant k \leqslant n-1)$, so that the variation in the values of the Sturm remainders

$$\mathrm{var}(P_*(x)) \;=\; \mathrm{var}(P_0(x), P_1(x), \ldots, P_n(x)) \in \{0, 1, \ldots, n\}$$

is defined.

Sturm's Theorem I.

- **Theorem** (1829) The number of real roots of a generic $P(X) \in \mathbb{R}[X]$ in $[a,b] \subset \mathbb{R}$ for regular $a < b$ is

$$|\{x \in [a,b] \mid P(x) = 0 \in \mathbb{R}\}| \ = \ \text{var}(P_*(a)) - \text{var}(P_*(b)) \ .$$

- **Idea of proof** The function

$$f \ : \ [a,b] \to \{0,1,\dots,n\} \ ; \ x \mapsto \text{var}(P_*(a)) - \text{var}(P_*(x))$$

jumps by $\begin{cases} 1 \\ 0 \end{cases}$ at root x of $P_k(X)$ if $k = \begin{cases} 0 \\ 1,2,\dots,n. \end{cases}$

- For $k = 0$ the jump in f at a root x of $P_0(x)$ is 1, since for y close to x

$$P_0(y)P_1(y) = d/dy(P(y)^2)/2 = \begin{cases} < 0 & \text{if } y < x \\ > 0 & \text{if } y > x \ , \end{cases}$$

$$\text{var}(P_0(y), P_1(y)) = \begin{cases} \text{var}(+,-) = \text{var}(-,+) = 1 & \text{if } y < x \\ \text{var}(+,+) = \text{var}(-,-) = 0 & \text{if } y > x \ . \end{cases}$$

Sturm's Theorem II.

- For $k = 1, 2, \ldots, n$ the jump in f at a root x of $P_k(x)$ is 0.
- $k = n$ trivial, since $P_n(X)$ is non-zero constant.
- For $k = 1, 2, \ldots, n-1$ the numbers $P_{k-1}(x)$, $P_{k+1}(x) \neq 0 \in \mathbb{R}$ have opposite signs since

$$P_{k-1}(x) + P_{k+1}(x) \;=\; P_k(x)Q_k(x) \;=\; 0\ .$$

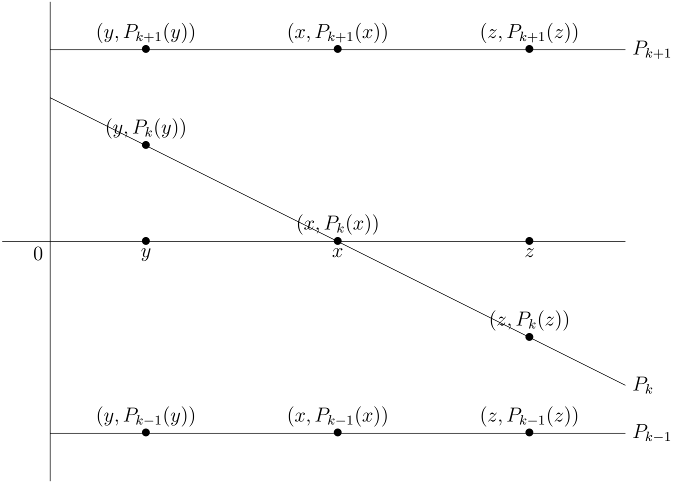
- For y, z close to x with $y < x < z$

$$\begin{aligned}\mathrm{sign}(P_{k-1}(y)) &= -\mathrm{sign}(P_{k+1}(y)) \\ &= \mathrm{sign}(P_{k-1}(z)) = -\mathrm{sign}(P_{k+1}(z))\ , \\ \mathrm{var}(P_{k-1}(y), P_k(y), P_{k+1}(y)) & \\ &= \mathrm{var}(P_{k-1}(z), P_k(z), P_{k+1}(z)) = 1\ ,\end{aligned}$$

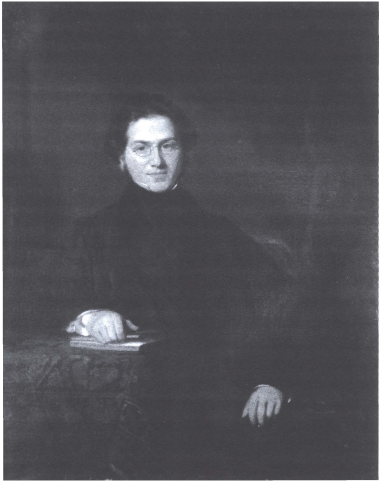
  that is

  $$\mathrm{var}(+,+,-) = \mathrm{var}(+,-,-) = \mathrm{var}(-,+,+) = \mathrm{var}(-,-,+) = 1.$$

Sturm's theorem III.



James Joseph Sylvester (1814-1897)



Sylvester's 4 papers related to Sturm's theorem

- *On the relation of Sturm's auxiliary functions to the roots of an algebraic equation.* (1841)
- *A demonstration of the theorem that every homogeneous quadratic polynomial is reducible by real orthogonal substitutions to the form of a sum of positive and negative squares.* (1852)
- *On a remarkable modification of Sturm's Theorem* (1853)
- *On a theory of the syzygetic relations of two rational integral functions, comprising an application to the theory of Sturm's functions, and that of the greatest algebraical common measure.* (1853)

The signature

- The **transpose** of an $n \times n$ matrix $A = (a_{ij})$ is $A^* = (a_{ji})$.
- **Spectral Theorem** (Cauchy, 1829) For any symmetric $n \times n$ matrix $S = S^*$ in $\mathbb{R}$ there exists an orthogonal $A = A^{*-1}$ with

$$A^*SA \;=\; \mathrm{diag}(\lambda_1, \dots, \lambda_n) \;=\; \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

- The **signature** of a symmetric $n \times n$ matrix S is

$$\tau(S) \;=\; \tau(A^*SA) \;=\; \sum_{i=1}^{n} \mathrm{sign}(\lambda_i)$$

- If S is invertible $\tau(S) = n - 2\,\mathrm{var}(1, \lambda_1, \dots, \lambda_n) \equiv n \bmod 2$.
- **Law of Inertia** (Sylvester, 1853) For any invertible $n \times n$ matrix A in $\mathbb{R}$

$$\tau(S) \;=\; \tau(A^*SA)\ .$$

Tridiagonal symmetric matrices (Jacobi)

- **Definition** The **tridiagonal symmetric matrix** of $q = (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n$ is

$$\mathrm{Tri}(q) \;=\; \begin{pmatrix} q_1 & 1 & 0 & \ldots & 0 \\ 1 & q_2 & 1 & \ldots & 0 \\ 0 & 1 & q_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & q_n \end{pmatrix}$$

- The principal minors of $\mathrm{Tri}(q)$

$$\mu_k \;=\; \det(\mathrm{Tri}(q_1, q_2, \ldots, q_k)) \; (1 \leqslant k \leqslant n)$$

satisfy the recurrence of the Euclidean algorithm

$$\mu_k \;=\; q_k \mu_{k-1} - \mu_{k-2} \; (\mu_0 = 1, \mu_{-1} = 0) \; .$$

The signature of a tridiagonal matrix

- **Theorem** (Sylvester, 1853) Assume the principal minors

$$\mu_k \;=\; \mu_k(\mathrm{Tri}(q)) \;=\; \det(\mathrm{Tri}(q_1, q_2, \ldots, q_k))\ (1 \leqslant k \leqslant n)$$

are non-zero. The invertible $n \times n$ matrix

$$A \;=\; \begin{pmatrix} 1 & -\mu_0/\mu_1 & \mu_0/\mu_2 & \ldots & (-1)^{n-1}\mu_0/\mu_{n-1} \\ 0 & 1 & -\mu_1/\mu_2 & \ldots & (-1)^{n-2}\mu_1/\mu_{n-1} \\ 0 & 0 & 1 & \ldots & (-1)^{n-3}\mu_2/\mu_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

is such that

$$A^*\mathrm{Tri}(q)A \;=\; \mathrm{diag}(\mu_1/\mu_0, \mu_2/\mu_1, \ldots, \mu_n/\mu_{n-1})$$

so that

$$\tau(\mathrm{Tri}(q)) \;=\; \sum_{k=1}^{n} \mathrm{sign}(\mu_k/\mu_{k-1}) \;=\; n - 2\,\mathrm{var}(\mu)\ .$$

Continued fractions and the Sturm sequences

- The **improper continued fraction** of $(q_1, q_2, \ldots, q_n)$ is

$$[q_1, q_2, \ldots, q_n] \;=\; q_1 - \cfrac{1}{q_2 - \ddots \; - \cfrac{1}{q_n}}$$

  assuming there are no divisions by 0.
- The continued fraction expansion of $P(X)/P'(X)$ is

$$P(X)/P'(X) \;=\; [Q_1(X), Q_2(X), \ldots, Q_n(X)] \in \mathbb{R}(X)$$

  with $Q_1(X), Q_2(X), \ldots, Q_n(X)$ the Sturm quotients.
- The Sturm remainders $(P_0(X), P_1(X), \ldots, P_n(X))$ are the numerators in the **reverse convergents** $(0 \leqslant k \leqslant n)$

$$[Q_{k+1}(X), Q_{k+2}(X), \ldots, Q_n(X)] = P_k(X)/P_{k+1}(X) \in \mathbb{R}(X) \; .$$

- $P_k(X)/P_n(X) = \det(\mathrm{Tri}(Q_{k+1}(X), Q_{k+2}(X), \ldots, Q_n(X)))$

Convergents

- The **convergents** of $[Q_1(X), Q_2(X), \ldots, Q_n(X)] \in \mathbb{R}(X)$ are

$$\begin{aligned} & [Q_1(X), Q_2(X), \ldots, Q_k(X)] \\ & \qquad = \frac{P_k^*(X)}{\det(\mathrm{Tri}(Q_2(X), Q_3(X), \ldots, Q_k(X)))} \end{aligned}$$

with numerators

$$\begin{aligned} P_k^*(X) &= \mu_k(\mathrm{Tri}(Q_1(X), Q_2(X), \ldots, Q_n(X))) \\ &= \det(\mathrm{Tri}(Q_1(X), Q_2(X), \ldots, Q_k(X))) \in \mathbb{R}[X] \end{aligned}$$

the principal minors of $\mathrm{Tri}(Q_1(X), Q_2(X), \ldots, Q_n(X))$.

Sylvester's reformulation of Sturm's Theorem

- **Duality Theorem** Let $x \in \mathbb{R}$ be regular for a degree n $P(X) \in \mathbb{R}[X]$. The variations of the sequences of the numerators of the convergents and reverse convergents are equal

$$\begin{aligned}&\mathrm{var}(P_0(x), P_1(x), \ldots, P_n(x))\\ &\qquad = \mathrm{var}(P_0^*(x), P_1^*(x), \ldots, P_n^*(x)) .\end{aligned}$$

- **Roots and signatures** The number of real roots of $P(X) \in \mathbb{R}[X]$ in an interval $[a, b] \subset \mathbb{R}$ is

$$\begin{aligned}&\#\mathbb{R}\text{-roots}(P(X); [a, b])\\ &= \mathrm{var}(P_0(a), P_1(a), \ldots, P_n(a)) - \mathrm{var}(P_0(b), P_1(b), \ldots, P_n(b))\\ &= \mathrm{var}(P_0^*(a), P_1^*(a), \ldots, P_n^*(a)) - \mathrm{var}(P_0^*(b), P_1^*(b), \ldots, P_n^*(b))\\ &= \big(\tau(\mathrm{Tri}(Q_*(b))) - \tau(\mathrm{Tri}(Q_*(a)))\big)/2 \in \{0, 1, 2, \ldots, n\} .\end{aligned}$$

Sylvester's musical inspiration for the Duality Theorem

As an artist delights in recalling the particular time and atmospheric effects under which he has composed a favourite sketch, so I hope to be excused putting upon record that it was in listening to one of the magnificent choruses in the 'Israel in Egypt' that, unsought and unsolicited, like a ray of light, silently stole into my mind the idea (simple, but previously unperceived) of the equivalence of the Sturmian residues to the denominator series formed by the reverse convergents. The idea was just what was wanting,—the key-note to the due and perfect evolution of the theory.

The Witt group $W(R)$

- Let R be a commutative ring. For simplicity assume $1/2 \in R$.
- A **symmetric form** (F, ϕ) over R is a f.g. free R-module F with a symmetric pairing

$$\phi \ = \ \phi^* \ : \ F \times F \to R \ ; \ (x, y) \mapsto \phi(x, y) = \phi(y, x).$$

- The form is **nonsingular** if the adjoint R-module morphism

$$\phi \ : \ F \to F^* \ = \ \mathrm{Hom}_R(F, R) \ ; \ x \mapsto (y \mapsto \phi(x, y))$$

  is an isomorphism, or equivalently $\det(\phi) \in R^\bullet$.
- A **lagrangian** of (F, ϕ) is a direct summand $L \subset F$ such that $L^\perp = L$, with $L^\perp := \ker(\phi| : F \to L^*)$. The **hyperbolic form** $H(L) = (L \oplus L^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$ is nonsingular, with L a lagrangian.
- The **Witt group** $W(R)$ is the abelian group of equivalence classes of nonsingular symmetric forms (F, ϕ) over R, with $(F, \phi) \sim (F', \phi')$ if there exists an isomorphism

$$(F, \phi) \oplus H(L) \cong (F', \phi') \oplus H(L') \ .$$

The linking Witt group $W(R,S)$

- Let R be a commutative ring, and $S \subset R$ a multiplicative subset of non-zero divisors with $1 \in S$. The **localization of** R **inverting** S is the ring of fractions
$$S^{-1}R \;=\; \{r/s \,|\, r \in R, s \in S\} \;.$$
- A **symmetric linking form** (T,λ) over (R,S) is an h.d. 1 R-module $T = \mathrm{coker}(d : R^n \to R^n)$ with $\det(d) \in S$, and with a symmetric pairing
$$\lambda \;=\; \lambda\hat{}\;:\; T \times T \to S^{-1}R/R \;;\; (x,y) \mapsto \lambda(x,y) = \lambda(y,x)$$
- The linking form is **nonsingular** if the adjoint R-module morphism
$$\lambda \;:\; T \to T\hat{} \;=\; \mathrm{Hom}_R(T, S^{-1}R/R) \;;\; x \mapsto (y \mapsto \lambda(x,y))$$
is an isomorphism.
- The **linking Witt group** $W(R,S)$ is defined by analogy with $W(R)$, but using exact sequences rather than direct sums.

The localization exact sequence of Witt groups

- A symmetric form (F, ϕ) over R is S-**nonsingular** if $S^{-1}(F, \phi)$ is a nonsingular symmetric form over K. Equivalently $\det(\phi) \in R^{\bullet}S$, and $\phi : F \to F^*$ is injective.
- The **boundary** of (F, ϕ) is the nonsingular symmetric linking form over (R, S)

$$\begin{aligned} \partial(F, \phi) &= (\mathrm{coker}(\phi : F \to F^*), (f, g) \mapsto f(\phi^{-1}(g))) \\ &= (F^{\#}/F, (v/s, w/t) \mapsto \phi(v, w)/st) \end{aligned}$$

  with $F^{\#} = \{v/s \in S^{-1}F \,|\, \phi(v) \in sF^* \subset F^*\}$.
- **Theorem** (Milnor, Karoubi, Pardon, R. 1970's) The Witt groups of R and $S^{-1}R$ are related by an exact sequence

$$\ldots \longrightarrow W(R) \longrightarrow W(S^{-1}R) \xrightarrow{\partial} W(R, S) \longrightarrow \ldots$$

  with $\partial : S^{-1}(F, \phi) \mapsto \partial(F, \phi)$. If R is a principal ideal domain this is a split short exact sequence.

The Witt group localization exact sequence for $R = \mathbb{R}[X]$

- **Theorem** (Milnor, 1970) The localization exact sequence for the principal ideal domain $R = \mathbb{R}[X]$ with fraction field $S^{-1}\mathbb{R}[X] = \mathbb{R}(X)$ is

$$0 \longrightarrow W(\mathbb{R}[X]) = W(\mathbb{R}) = \mathbb{Z} \longrightarrow W(\mathbb{R}(X)) = \mathbb{Z} \oplus \mathbb{Z}[\mathbb{R}] \oplus \mathbb{Z}_2[\mathfrak{H}]$$
$$\xrightarrow{\partial} W(\mathbb{R}[X], S) = \bigoplus_{\mathfrak{P} \triangleleft \mathbb{R}[X] \text{ prime}} W(\mathbb{R}[X]/\mathfrak{P}) = \mathbb{Z}[\mathbb{R}] \oplus \mathbb{Z}_2[\mathfrak{H}] \longrightarrow 0$$

  with $\mathfrak{H} = \{u + iv \in \mathbb{C} \,|\, v > 0\}$ the complex upper half plane.
- Isomorphism

$$\begin{aligned} \mathbb{Z} \oplus \mathbb{Z}[\mathbb{R}] \oplus \mathbb{Z}_2[\mathfrak{H}] \xrightarrow{\cong} & \ W(\mathbb{R}(X)) \ ; \\ (1, 0, 0) \mapsto & \ (\mathbb{R}(X), 1), \\ (0, x, 0) \mapsto & \ (\mathbb{R}(X), X - x), \\ (0, 0, u + iv) \mapsto & \ (\mathbb{R}(X), (X - u)^2 + v^2) \ . \end{aligned}$$

Signature differences

- Let $P(X) \in \mathbb{R}[X]$ be generic of degree n, with Sturm sequences $P_*(X)$, $Q_*(X)$. For regular $a \in \mathbb{R}$ the composite

$$\epsilon(a) : W(\mathbb{R}(X)) \xrightarrow{\partial} W(\mathbb{R}[X], S) \xrightarrow{\text{proj.}} \mathbb{Z}[\mathbb{R}] \xrightarrow{\text{eval. at } a} \mathbb{Z}$$

sends $(\mathbb{R}(X)^n, \mathrm{Tri}(Q_*(X))) \cong \bigoplus_{k=1}^{n} (\mathbb{R}(X), P_{k-1}(X)/P_k(X))$ to

$$\sum_{k=1}^{n} (\tau(\mathbb{R}, P_{k-1}(a)/P_k(a))$$
$$- \lim_{x\to\infty} \big(\tau(\mathbb{R}, P_{k-1}(x)/P_k(x)) + \tau(\mathbb{R}, P_{k-1}(-x)/P_k(-x))\big)/2) \in \mathbb{Z}$$

- For any regular $a < b \in \mathbb{R}$ the morphism

$$\epsilon(b) - \epsilon(a) : W(\mathbb{R}(X)) \to \mathbb{Z}$$

sends $(\mathbb{R}(X)^n, \mathrm{Tri}(Q_*(X)))$ to

$$\sum_{k=1}^{n} (\tau(\mathbb{R}, P_{k-1}(b)/P_k(b)) - \tau(\mathbb{R}, P_{k-1}(a)/P_k(a))) \in \mathbb{Z} \ .$$

The Sturm-Sylvester Theorem via the Witt group

- A polynomial $P(X) \in \mathbb{R}[X]$ is a unit in $\mathbb{R}(X)$, so Witt class $(\mathbb{R}[X], P(X)) \in W(\mathbb{R}(X))$ defined.
- Assume $P(X)$ is monic of degree $n = 2r + s$ with r distinct real roots and $2s$ distinct complex roots

$$\begin{aligned} P(X) \;&=\; (X - x_1)(X - x_2)\dots(X - x_r) \\ &\qquad ((X - u_1)^2 + v_1^2)\dots((X - u_s)^2 + v_s^2) \in \mathbb{R}[X] \end{aligned}$$

  with Sturm sequences $P_*(X)$, $Q_*(X)$.
- The Ghys-R. paper gives detailed proofs that

$$\begin{aligned} (\mathbb{R}(X)^n, \mathrm{Tri}(Q_*(X))) &= (-s, \sum_{j=1}^{r} 1.x_j, \sum_{k=1}^{s} 1.(u_k + iv_k)) \\ &\in W(\mathbb{R}(X)) = \mathbb{Z} \oplus \mathbb{Z}[\mathbb{R}] \oplus \mathbb{Z}_2[\mathfrak{H}] \end{aligned}$$

- For regular $a < b \in \mathbb{R}$ $\epsilon(b) - \epsilon(a) : W(\mathbb{R}(X)) \to \mathbb{Z}$ has image

$$\begin{aligned} &(\epsilon(b) - \epsilon(a))(\mathbb{R}(X)^n, \mathrm{Tri}(Q_*(X))) = \tau(\mathrm{Tri}(Q_*(b))) - \tau(\mathrm{Tri}(Q_*(a))) \\ &= 2\,\#\mathbb{R}\text{-roots}(P(X); [a, b]) = 2\,|\{j \mid a < x_j < b\}| \in \{0, 1, \dots, r\}\ . \end{aligned}$$

Manifolds, intersections and linking

- An oriented 4-dimensional manifold with boundary $(M, \partial M)$ has an **intersection** symmetric form $(F_2(M), \phi)$ over $\mathbb{Z}$, with $F_2(M) = H_2(M)/\text{torsion}$ and
$$\phi(N_1^2 \subset M, N_2^2 \subset M) \ = \ N_1 \cap N_2 \in \mathbb{Z} \ .$$
Nonsingular if $H_*(\partial M; \mathbb{Q}) = H_*(S^3; \mathbb{Q})$.
- An oriented closed 3-dimensional manifold L has a symmetric **linking** form $(T_1(L), \lambda)$ over $(\mathbb{Z}, \mathbb{Z}\backslash\{0\})$, with $T_1(L) = \text{torsion}(H_1(L))$ and
$$\lambda(K_1^1 \subset L, K_2^1 \subset L) = (\delta K_1 \cap K_2)/s \in \mathbb{Q}/\mathbb{Z}$$
if $\delta K_1^2 \subset L$ extends $\partial \delta K_1 = \bigcup\limits_s K_1 \subset L$ for some $s \geqslant 1$.
- **Linking (geometric ∂) = algebraic ∂ (intersection)**
If $L = \partial M$ then $(T_1(L), \lambda) = \partial(F_2(M), \phi)$ corresponding to the exact sequence
$$0 \longrightarrow F_2(M) \xrightarrow{\phi} F_2(M)^* \longrightarrow T_1(L) \longrightarrow 0$$

Why is $\partial : W(S^{-1}R) \to W(R,S)$ onto for a principal ideal domain R?

- Every nonsingular symmetric linking form over (R,S) is a direct sum of $(R/(p_1), p_0/p_1)$'s, with $p_0, p_1 \in R$ coprime.
- The Euclidean algorithm in R gives Sturm sequences $p = (p_0, p_1, \ldots, p_n) \in S^{n+1}$, $q = (q_1, q_2, \ldots, q_n) \in R^n$

$$p_k q_k \ = \ p_{k-1} + p_{k+1} \ (1 \leqslant k \leqslant n)$$

  with $p_n = \text{g.c.d.}(p_0, p_1) \in R^\bullet$, $p_{n+1} = 0$.
- **Proposition** (Wall 1964 for $R = \mathbb{Z}$, Ghys-R. 2016) The Sturm sequences lift $(R/(p_1), p_0/p_1)$ to $S^{-1}(R^n, \mathrm{Tri}(q))$, with

$$\partial S^{-1}(R^n, \mathrm{Tri}(q)) = \partial(S^{-1}R, p_0/p_1) = (R/(p_1), p_0/p_1) \in W(R,S)$$

- Illustrated by the Hirzebruch-Milnor plumbing construction of a 4-dimensional manifold M with boundary $\partial M = L(c,a)$ a lens space in the case $R = \mathbb{Z}$, $S^{-1}R = \mathbb{Q}$ – a topological proof of the Sylvester Duality Theorem for integral symmetric forms.

The lens spaces

- For any coprime $a, c \in \mathbb{Z}$ define the **lens space**

$$L(c,a) \;=\; S^1 \times D^2 \cup_A S^1 \times D^2$$

using any $b, d \in \mathbb{Z}$ such that $ad - bc = 1$. Heegaard decomposition, with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ realized by

$$A \;:\; S^1 \times S^1 \to S^1 \times S^1 \;;\; (z,w) \mapsto (z^a w^b, z^c w^d) \;.$$

- $L(c,a)$ is a closed oriented 3-dimensional manifold with symmetric linking form $(H_1(L(c,a)), \lambda) = (\mathbb{Z}_c, a/c)$.
- Surgery on $S^1 \times D^2 \subset L(c,a)$ results in an oriented cobordism $(M(c,a); L(c,a), L(a,c))$ with

$$M(c,a) \;=\; L(c,a) \times I \cup D^2 \times D^2 \;,$$
$$-L(a,c) \;=\; (L(c,a) \backslash S^1 \times D^2) \cup D^2 \times S^1 \;.$$

Symmetric intersection form $(H_2(M(c,a)), \phi) = (\mathbb{Z}, ac)$.

Topological proof of the Sylvester Duality Theorem I.

- (Hirzebruch, 1962) For coprime $c > a > 0$ the Euclidean algorithm for g.c.d.$(a, c) = 1$

$$p_0 = c \ , \ p_1 = a \ , \ \dots \ , \ p_n = 1 \ , \ p_{n+1} = 0 \ ,$$
$$p_k q_k = p_{k-1} + p_{k+1} \ (1 \leqslant k \leqslant n) \ .$$

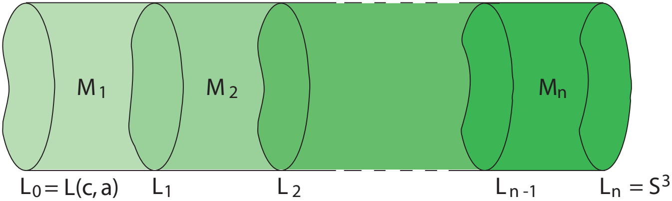
  determines an expression of the lens space $L(c, a) = \partial M$ as the boundary of an oriented 4-dimensional manifold M with intersection form $(H_2(M), \phi) = (\mathbb{Z}^n, \mathrm{Tri}(q))$.
- The continued fraction $a/c = [q_1, q_2, \dots, q_n]$ is realized topologically by a sequence of oriented cobordisms

$$(M, \partial M) = (M_1; L_0, L_1) \cup (M_2; L_1, L_2) \cup \cdots \cup (M_n \cup D^4; L_{n-1}, \emptyset)$$

  with $L_0 = L(p_0, p_1) = L(c, a),\ L_k = L(p_k, p_{k+1}) = -L(p_k, p_{k-1}),$

$$L_n = L(p_n, p_{n+1}) = L(1, 0) = S^3 \ ,$$
$$M_k = \text{trace of surgery on } S^1 \times D^2 \subset L_{k-1} \ (1 \leqslant k \leqslant n) \ .$$

Topological proof of the Sylvester Duality Theorem II.

- 

- M is obtained by glueing together the cobordisms $(M_k; L_{k-1}, L_k)$ for $k = 1, 2, \ldots, n$ (A_n-plumbing) with

$$L_{k-1} = L(p_{k-1}, p_k)\ ,\ \ M_k = M(p_{k-1}, p_k)$$

$$(M, \partial M) = (M_1; L_0, L_1) \cup (M_2; L_1, L_2) \cup \cdots \cup (M_n \cup D^4; L_{n-1}, \emptyset)\ .$$

- Algebraic plumbing: construction of a tridiagonal symmetric form $(\bigoplus_n F, \mathrm{Tri}(q))$ over a ring with involution R, using any sequence $\{(F, q_k) \mid 1 \leqslant k \leqslant n\}$ of symmetric forms over R.

Topological proof of the Sylvester Duality Theorem III.

- The union $U_k = \bigcup\limits_{j=1}^{k} M_j$ has
  $$(H_2(U_k;\mathbb{Q}),\phi_{U_k}) = \bigoplus_{j=1}^{k}(\mathbb{Q},p_{j-1}p_j)\ ,\ \tau(U_k) = \sum_{j=1}^{k}\text{sign}(p_j/p_{j-1})$$
  with $p_j = \det(\text{Tri}(q_{j+1},\dots,q_n))$.
- The union $F_k = \bigcup\limits_{j=n-k+1}^{n} M_j$ has
  $$(H_2(F_k),\phi_{F_k}) = (\mathbb{Z}^k,\text{Tri}(q_1,q_2,\dots,q_k))\ ,$$
  $$\tau(F_k) = \sum_{j=1}^{k}\text{sign}(p_j^*/p_{j-1}^*) \text{ with } p_j^* = \det(\text{Tri}(q_1,q_2,\dots,q_j))\ .$$
- It now follows from $M = U_n = F_n$ that
  $$\begin{aligned}\tau(M) &= \tau(\text{Tri}(q_1,q_2,\dots,q_n)) \\ &= \sum_{j=1}^{n}\text{sign}(p_j/p_{j-1}) = \sum_{j=1}^{n}\text{sign}(p_j^*/p_{j-1}^*)\ .\end{aligned}$$

Generalized tridiagonal symmetric matrices I.

- Following book by J.Barge and J.Lannes "Suites de Sturm, indice de Maslov et périodicité de Bott" (Birkhäuser, 2008)
- For a commutative ring R and $k \geqslant 1$ let $\mathrm{Lag}_k(R)$ be the set of f.g. free lagrangians $L \subset R^k \oplus R^k$ of the symplectic form
$$(H_k(R), J_k) \;=\; (R^k \oplus R^k, \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}) \;.$$
- The symplectic group
$$\mathrm{Sp}_{2k}(R) = \mathrm{Aut}(H_k(R), J_k) = \{\alpha \in \mathrm{GL}_{2k}(R) \,|\, \alpha^* J_k \alpha = J_k\}$$
acts transitively on the lagrangians by
$$\mathrm{Sp}_{2k}(R) \times \mathrm{Lag}_k(R) \to \mathrm{Lag}_k(R) \;;\; (\alpha, L) \mapsto \alpha(L) \;.$$
- An **algebraic path** in $\mathrm{Lag}_k(R)$ is an $\alpha \in \mathrm{Sp}_{2k}(R[X])$, **starting** at $\alpha(0)(R^k \oplus 0)$ and **ending** at $\alpha(1)(R^k \oplus 0) \in \mathrm{Lag}_k(R)$.
- $\Omega\mathrm{Lag}_k(R) \subset \mathrm{Sp}_{2k}(R[X])$ is the set of **loops**, the paths α with
$$\alpha(0)(R^k \oplus 0) \;=\; \alpha(1)(R^k \oplus 0) \in \mathrm{Lag}_k(R) \;.$$

Generalized tridiagonal symmetric matrices II.

- A sequence $q_1, q_2, \ldots, q_n$ of symmetric $k \times k$ matrices in $R[X]$ determines an algebraic path in $\mathrm{Lag}_k(R)$

$$\alpha = E(q_1)E(q_2)\ldots, E(q_n) \in \mathrm{Sp}_{2k}(R[X])$$

  with each

$$E(q_j) \;=\; \begin{pmatrix} q_j & -I_k \\ I_k & 0 \end{pmatrix}$$

  an elementary symplectic matrix.
- The symmetric form $(R[X]^{nk}, \mathrm{Tri}(q))$ over $R[X]$ is defined by the generalized tridiagonal symmetric matrix with

$$\mathrm{Tri}(q) \;=\; \begin{pmatrix} q_1 & I_k & \ldots & 0 \\ I_k & q_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & q_n \end{pmatrix} \;.$$

The Maslov index and Bott periodicity

- For any $\ell \geqslant 1$ let $\mathrm{Sym}_\ell(R)$ be the pointed set of nonsingular symmetric forms (R^ℓ, ϕ) over R, based at (R^ℓ, I_ℓ).
- **Theorem** (Barge-Lannes, 2008) For a noetherian commutative ring R with $1/2 \in R$ every algebraic loop $\alpha \in \Omega\mathrm{Lag}_k(R) \subset \mathrm{Sp}_{2k}(R[X])$ is

$$\alpha \ = \ E(q_1)E(q_2)\ldots E(q_n) \in \mathrm{Sp}_{2k}(R[X]) \ (n \text{ large})$$

with $(R[X]^{nk}, \mathrm{Tri}(q))$ a symmetric form over $R[X]$ such that the symmetric forms $(R^{nk}, \mathrm{Tri}(q)(0))$, $(R^{nk}, \mathrm{Tri}(q)(1))$ over R are nonsingular. The **Maslov index** map

$$\Omega\mathrm{Lag}_k(R) \to \mathrm{Sym}_{2nk}(R) \ ;$$
$$\alpha \mapsto \mathrm{Maslov}(\alpha) = (R^{nk}, \mathrm{Tri}(q)(1)) \oplus (R^{nk}, -\mathrm{Tri}(q)(0))$$

induces the **algebraic Bott periodicity** isomorphism

$$\varinjlim_k \pi_1(\mathrm{Lag}_k(R)) \ \cong \ \varinjlim_\ell \pi_0(\mathrm{Sym}_\ell(R)) \ .$$

The 1-dimensional case I.

- Every 1-dimensional subspace $L \subset \mathbb{R} \oplus \mathbb{R}$ is a lagrangian in
$$H_-(\mathbb{R}) \;=\; (\mathbb{R} \oplus \mathbb{R}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \;.$$
- The function
$$S^1 \to \mathrm{Lag}_1(\mathbb{R}) \;=\; P(\mathbb{R}^2) \;=\; \mathbb{R}\,\mathbb{P}^1 \;;\; e^{2\pi i x} \mapsto \{(\cos \pi x, \sin \pi x)\}$$
is a diffeomorphism, such that the image of $S^1 \backslash \{1\} \cong \mathbb{R}$ is the contractible subspace
$$\mathrm{Lag}_1(\mathbb{R})_0 \;=\; \mathrm{Lag}_1(\mathbb{R}) \backslash \{\mathbb{R} \oplus 0\} \subset \mathrm{Lag}_1(\mathbb{R}) \;.$$
- For generic $P(X) \in \mathbb{R}[X]$ with $0, 1 \in \mathbb{R}$ regular the algebraic path $\alpha = E(Q_1(X))E(Q_2(X)) \dots E(Q_n(X)) \in \mathrm{Sp}_2(\mathbb{R}[X])$ given by the Sturm sequence corresponds to the actual path
$$\alpha \;:\; [0,1] \to \mathrm{Lag}_1(\mathbb{R}) \;;\; x \mapsto \{(P(x), P'(x))\}$$
with $\alpha(0), \alpha(1) \in \mathrm{Lag}_1(\mathbb{R})_0$,

The 1-dimensional case II.

- For $R = \mathbb{R}$ signature gives a canonical surjection

$$\varinjlim_{\ell} \pi_0(\mathrm{Sym}_\ell(\mathbb{R})) \to W(\mathbb{R}) = \mathbb{Z} \ ; \ S \mapsto (\tau(S) - \ell)/2 \ .$$

- **Theorem** (Barge-Lannes, 2008) The degree of the topological loop associated to $P(X) \in \mathbb{R}[X]$

$$[\alpha] \ : \ S^1 \ = \ [0,1]/\{0,1\} \to \mathrm{Lag}_1(\mathbb{R})/\mathrm{Lag}_1(\mathbb{R})_0 \ \simeq \ S^1 \ ;$$
$$e^{2\pi i x} \mapsto \{(P(x), P'(x))\} \ (0 \leqslant x \leqslant 1)$$

  is the Maslov index of α.
- **Proof**

$$\begin{aligned} \mathrm{degree}([\alpha]) \ &= \ |[\alpha]^{-1}\{(0,1)\}| \ = \ \#\mathbb{R}\text{-roots}(P(X);[0,1]) \\ &= \ \mathrm{var}(P_0(0), P_1(0), \dots, P_n(0)) - \mathrm{var}(P_0(1), P_1(1), \dots, P_n(1)) \\ &= \ \big(\tau(\mathrm{Tri}(Q)(1)) - \tau(\mathrm{Tri}(Q)(0))\big)/2 \ = \ \mathrm{Maslov}(\alpha) \in W(\mathbb{R}) = \mathbb{Z} \end{aligned}$$

(by Sturm and Sylvester).