THE WORK OF C.T.C.WALL IN TOPOLOGY

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- 90+ papers, 2+ books

- Topics covered: cobordism groups, Steenrod algebra, homological algebra, manifolds of dimensions 3,4,$\geq$ 5, quadratic forms, finiteness obstruction, embeddings, bundles, Poincaré complexes, surgery obstruction theory, homology of groups, 2-dimensional complexes, topological space form problem, computations of K- and L-groups, ...

- MR 57Q12 Wall finiteness obstruction for CW-complexes

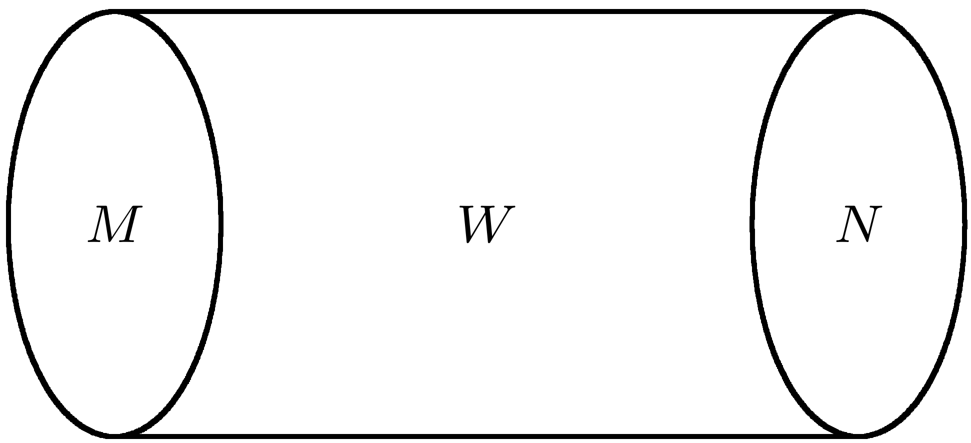
- MR 57R67 Surgery obstructions, Wall groups

Wall's manifold classifications

1. All manifolds at once
   - cobordism (1959-1961)
2. One manifold at a time
   - diffeomorphism (1962-1966)
3. Within a homotopy type
   - surgery (1967-1977)

Cobordism

- A <u>cobordism</u> between closed m-dimensional manifolds M, N is an $(m+1)$-dimensional manifold W with boundary $\partial W = M \cup N$



- $\Omega_m =$ abelian group of cobordism classes of oriented closed m-dimensional manifolds, addition by disjoint union

- $\Omega_* = \sum_{m=0}^{\infty} \Omega_m$ <u>oriented cobordism ring</u>, multiplication by cartesian product.

Computation of oriented cobordism

- Thom: expressed Ω_* as homotopy groups, computed $\Omega_* \otimes \mathbb{Q}$
  - no odd-primary torsion (Milnor).

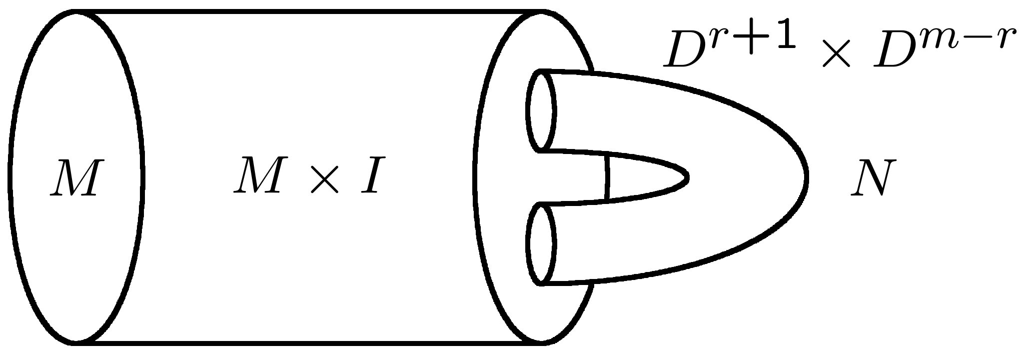
- Wall: **Determination of the cobordism ring** Annals of Mathematics 72, 292–311 (1960)

- Calculation of 2-primary torsion.

- **Theorem** (Wall) Two oriented manifolds are cobordant if and only if they have the same Stiefel and Pontrjagin numbers
  - ultimate achievement of pioneering phase of cobordism theory.

Handles and surgery

- Given m-manifold M and $S^r \times D^{m-r} \subset M$ define elementary cobordism $(W; M, N)$ by attaching an $(r+1)$-handle to $M \times I$



$$W \ = \ M \times I \cup D^{r+1} \times D^{m-r}$$

- $N = (M \backslash S^r \times D^{m-r}) \cup D^{r+1} \times S^{m-r-1}$ manifold obtained from M by surgery on $S^r \times D^{m-r} \subset M$

- Handles are the building blocks of manifolds

  - need surgeries to attach handles

Structure of manifolds

- Every cobordism $(W; M, N)$ is a union of elementary cobordisms.

- <u>h-cobordism</u> = cobordism $(W; M, N)$ with $M \subset W$, $N \subset W$ homotopy equivalences

- <u>h-cobordism theorem</u> (Smale): every simply-connected h-cobordism with $\dim(W) \geq 6$ is diffeomorphic to $M \times (I; \{0\}, \{1\})$
  - needs Whitney trick for removing double points in dimensions > 4
  - <u>s-cobordism theorem</u> is non-simply-connected version $\pi_1(W) \neq \{1\}$
  - possible rearrangements of handles governed by <u>algebraic K-theory</u> (Whitehead torsion)

Intersection form

- M = oriented $2n$-dimensional manifold.

- Intersection form: $(-)^n$-symmetric pairing

$$H_n(M) \times H_n(M) \to \mathbb{Z}$$

- Isomorphism class of form is an oriented homotopy invariant.

- Signature defined for even n, an oriented cobordism invariant.

- The boundary of an $(n-1)$-connected $2n$-dimensional manifold M with unimodular intersection form is a homotopy sphere $\partial M = \Sigma^{2n-1}$, with a potentially exotic differential structure for $n \geq 4$ (Milnor).

Classification of highly-connected manifolds

- Wall: **Classification of $(n-1)$-connected $2n$-manifolds** Annals of Mathematics 75, 163–189 (1962)

- **Theorem** (Wall) For $n \geq 3$ the diffeomorphism classes of differentiable $(n-1)$-connected $2n$-manifolds with boundary an exotic sphere = the isomorphism classes of $\mathbb{Z}$-valued $(-)^n$-symmetric forms with a quadratic refinement in $\pi_n(BSO(n))$

- Classification of handlebodies by homotopy theory, subsequently generalized to other cases:

  - Wall: **Classification problems in differential topology I–VI** Topology, Inventiones Math. (1963–1967)

4-manifolds

- Simply-connected 4-manifolds are homotopy equivalent if and only if intersection forms are isomorphic (Milnor).

- Wall: **On simply-connected 4-manifolds** Journal LMS 39, 141–149 (1964)

- **Theorem** (Wall) Simply-connected 4-manifolds are h-cobordant if and only if intersection forms are isomorphic.

- **Theorem** (Wall) h-cobordant simply-connected 4-manifolds M, N are stably diffeomorphic

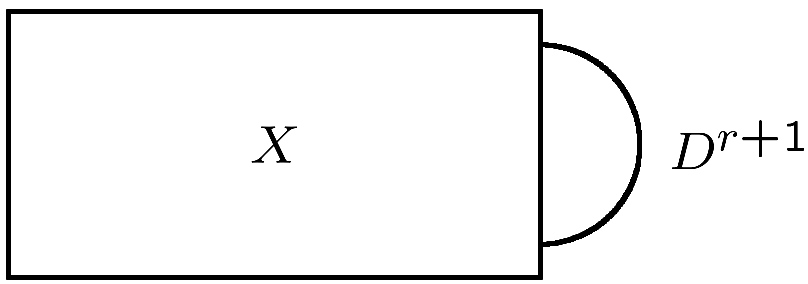
$$M \# \underset{k}{\#} S^2 \times S^2 \;\cong\; N \# \underset{k}{\#} S^2 \times S^2$$

  for some $k \geq 0$. $\#$ = connected sum

CW **complexes**

- X space, $f : S^r \to X$ map

- $X \cup_f D^{r+1}$ = space obtained from X by attaching an $(r+1)$-cell



- CW complex = space obtained from $\emptyset$ by attaching cells

- When is a space homotopy equivalent to a finite CW complex?

Finite domination

- A space X is finitely dominated if it is a homotopy retract of a finite CW complex K, i.e. if there exist maps $f : X \to K$, $g : K \to X$ and a homotopy $gf \simeq 1 : X \to X$.

- Is a finitely dominated space homotopy equivalent to a finite CW complex?

- Every compact ANR, e.g. a topological manifold, is finitely dominated (Borsuk).

- A finite group π with cohomology of period q acts freely on an infinite CW complex Y homotopy equivalent to S^{q-1}, with Y/π finitely dominated (Swan).

Finiteness obstruction

- Wall: **Finiteness conditions for CW-complexes** Annals of Mathematics 81, 56–89 (1965)

- Wall finiteness obstruction $[X] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ of finitely dominated space X
  - fundamental algebraic invariant of non-compact topology.

- **Theorem** (Wall) X is homotopy equivalent to finite CW complex if and only if $[X] = 0$

- Many applications to topology of manifolds
  - Siebenmann end obstruction for closing tame ends of open manifolds
  - Topologically stratified sets

The surgery method

- Standard method for classifying manifolds within a homotopy type.

- An m-dimensional manifold M has Poincaré duality $H^{m-*}(M) \cong H_*(M)$.

- Is a space X with m-dimensional Poincaré duality $H^{m-*}(X) \cong H_*(X)$ homotopy equivalent to an m-dimensional manifold?

- Is a homotopy equivalence of manifolds homotopic to a diffeomorphism?

  – relative version of previous question

- Formulation by Browder, Novikov, Sullivan in terms of <u>normal maps</u> $(f, b) : M \to X$ from manifolds to Poincaré duality spaces, with f degree 1 and b a bundle map.

Wall surgery theory

- Wall: **Surgery on compact manifolds**
  LMS Monograph 1, Academic Press (1970)
  - the surgeon's bible
  - algebraic L-groups $L_*(\mathbb{Z}[\pi])$ of group ring $\mathbb{Z}[\pi]$ = quadratic algebraic K-groups
  - surgery obstruction of normal map $(f,b) : M \to X$

$$\sigma_*(f,b) \in L_m(\mathbb{Z}[\pi_1(X)])$$

- **Theorem** (Wall) For $m \geq 5$ an m-dimensional Poincaré duality space X is homotopy equivalent to an m-dimensional manifold if and only if there exists a normal map $(f,b) : M \to X$ with $\sigma_*(f,b) = 0$.

Properties of Wall groups $L_m(\mathbb{Z}[\pi])$

- Quadratic forms over $\mathbb{Z}[\pi]$ for m even
- Automorphisms of forms for m odd
- Govern existence and effects of surgeries on m-dimensional manifolds with fundamental group π
- Computations for finite π using algebra
  - Wall: **Classification of Hermitian Forms I–VI**, (Compositio Math., Inventiones Math., Annals of Maths. 1970–1976)
- Computations for infinite π using topology
- Many, many applications to both algebra and topology

The topological space form problem

- Wall: **The topological space-form problem**, pp 319-351 in Topology of manifolds, Markham, 1970

- Wall: **Free actions of finite groups on spheres**, pp 115-124 in Proc Symp in Pure Math 32, AMS 1978

- +3 further papers (with Madsen and Thomas)

- Complete classification of finite groups π which have a free topological action on S^m for $m \geq 5$, using:

  – group cohomology

  – homotopy theory

  – algebraic K- and L-theory of $\mathbb{Z}[\pi]$.

PL **structures on tori**

- Wall: **On homotopy tori and the annulus theorem** Bulletin LMS 1, 95–97 (1969)

- Uses geometric computation of $L_*(\mathbb{Z}[\mathbb{Z}^m])$ to classify PL manifolds homotopy equivalent to m-torus T^m for $m \geq 5$

- Applied by Kirby to prove the annulus theorem for $m \geq 5$: if $D^m \subset \text{int}(D^m)$ is an embedding then $D^m \backslash \text{int}(D^m)$ is homeomorphic to $S^{m-1} \times I$

- Crucial ingredient of Kirby-Siebenmann handlebody theory of topological manifolds of dimension ≥ 5

- Now know as much about topological manifolds as about differentiable manifolds.