DOUBLE POINTS

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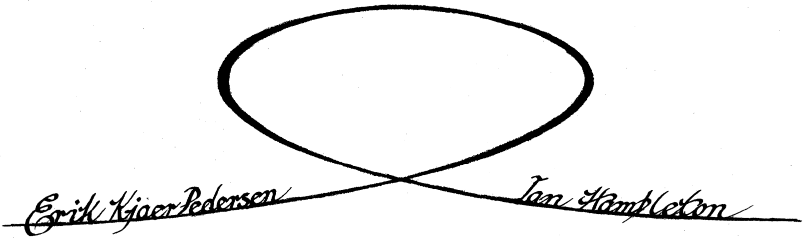

Homology classes

- Algebraic topologists have studied the homology classes represented by submanifolds $N^n \subset M^m$ and their intersections from the beginnings of the subject. So what is there to add?
- Theorem (Wall, 1966) The double points of an immersion $f : S^n \looparrowright M^{2n}$ are counted by an element

  $$\mu(f) \in H_0(\mathbb{Z}_2; \mathbb{Z}[\pi_1(M)], (-)^n) = \frac{\mathbb{Z}[\pi_1(M)]}{\{g - (-)^n g^{-1} \mid g \in \pi_1(M)\}}$$

  such that $\mu(f) = 0$ if (and for $n \geqslant 3$ only if) f is regular homotopic to an embedding.
- Traditional algebraic topology methods do not deal with μ. So surgery theory requires a better understanding of the algebraic topology of self-intersections.

The double point set $D(f_1, f_2)$

- The double point set of maps $f_1 : N_1 \to M$, $f_2 : N_2 \to M$ is

  $$D(f_1, f_2) \ = \ \{(x_1, x_2) \in N_1 \times N_2 \,|\, f_1(x_1) = f_2(x_2) \in M\} \ ,$$

  the pullback in the diagram

  $$\begin{array}{ccc} D(f_1, f_2) & \xrightarrow{h} & N_1 \times N_2 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f_1 \times f_2} \\ M & \xrightarrow{\Delta} & M \times M \end{array}$$

  with

  $$\begin{aligned} \Delta \ &: \ M \hookrightarrow M \times M \ ; \ x \mapsto (x, x) \ , \\ g \ &: \ D(f_1, f_2) \to M \ ; \ (x_1, x_2) \mapsto f_1(x_1) = f_2(x_2) \ , \\ h \ &: \ D(f_1, f_2) \hookrightarrow N_1 \times N_2 \ ; \ (x_1, x_2) \mapsto (x_1, x_2) \ . \end{aligned}$$

- $f_1(N_1), f_2(N_2) \subseteq M$ are disjoint if and only if $D(f_1, f_2) = \emptyset$.

The double point classes of immersions

- Manifolds are assumed to be oriented, unless specified otherwise!
- An m-dimensional manifold M has a fundamental class $[M] \in H_m(M)$, and Poincaré duality isomorphisms

$$[M] \cap - \ : \ H^{m-*}(M) \xrightarrow{\cong} H_*(M) \ ; \ x \mapsto [M] \cap x \ .$$

- A map of manifolds $f : N^n \to M^m$ represents a homology class $f[N] \in H_n(M)$, with Poincaré dual $f[N]^* \in H^{m-n}(M)$.
- If $f_1 : (N_1)^{n_1} \looparrowright M^m$, $f_2 : (N_2)^{n_2} \looparrowright M^m$ are transverse immersions the double point set $D(f_1, f_2)$ is an oriented $(n_1 + n_2 - m)$-dimensional submanifold of $N_1 \times N_2$.
- The immersion and embedding

$$g \ : \ D(f_1, f_2)^{n_1+n_2-m} \looparrowright M \ , \ h \ : \ D(f_1, f_2)^{n_1+n_2-m} \hookrightarrow N_1 \times N_2$$

represent the <u>double point classes</u>

$$g[D(f_1, f_2)] \in H_{n_1+n_2-m}(M) \ , \ h[D(f_1, f_2)] \in H_{n_1+n_2-m}(N_1 \times N_2) \ .$$

The Thom space

- An oriented k-plane bundle $\nu : X \to BSO(k)$ has a (D^k, S^{k-1})-bundle

$$(D^k, S^{k-1}) \to (B(\nu), S(\nu)) \to X \ .$$

- The Thom space of ν is the pointed space

$$T(\nu) \ = \ B(\nu)/S(\nu) \ .$$

  Cap product with the Thom class $U_\nu \in \dot{H}^k(T(\nu))$ defines a chain equivalence

$$U_\nu \cap - \ : \ \dot{C}(T(\nu)) \ \simeq \ C(X)_{*-k} \ .$$

- Example For the trivial k-plane bundle $\epsilon^k : X \to BSO(k)$

$$T(\epsilon^k) \ = \ (D^k \times X)/(S^{k-1} \times X) \ = \ \Sigma^k X^+$$

  with $X^+ = X \cup \{+\}$.

Normal bundles

- An immersion $f : N^n \looparrowright M^m$ has a normal bundle $\nu_f : N \to BSO(m-n)$ such that

$$f^*\tau_M \ = \ \tau_N \oplus \nu_f \ : \ N \to BSO(m) \ ,$$

with a codimension 0 immersion $B(\nu_f) \looparrowright M$ extending f.
- For transverse $f_1 : (N_1)^{n_1} \looparrowright M^m$, $f_2 : (N_2)^{n_2} \looparrowright M^m$ the normal bundle of $g : D(f_1, f_2)^{n_1+n_2-m} \looparrowright M$ is

$$\nu_g \ = \ h^*(\nu_{f_1} \times \nu_{f_2}) \ : \ D(f_1, f_2) \xrightarrow{h} N_1 \times N_2 \xrightarrow{\nu_{f_1} \times \nu_{f_2}} BSO(2m - n_1 - n_2) \ .$$

- The normal bundle of $h : D(f_1, f_2) \hookrightarrow N_1 \times N_2$ is

$$\nu_h \ = \ g^*\tau_M \ : \ D(f_1, f_2) \xrightarrow{g} M \xrightarrow{\tau_M} BSO(m) \ .$$

The Umkehr map I.

- The Umkehr of a map $f : N^n \to M^m$ is the chain map

$$f^! \ : \ C(M) \simeq C(M)^{m-*} \xrightarrow{f^*} C(N)^{m-*} \simeq C(N)_{*-m+n}$$

such that

$$(f^!)^*(1_N) \ = \ f[N]^* \in H^{m-n}(M) \ , \ f^![M] \ = \ [N] \in H_n(N) \ .$$

- Given an embedding $f : N^n \hookrightarrow M^m$ use the tubular neighbourhood $B(\nu_f) \hookrightarrow M$ and the Pontrjagin-Thom construction to define the geometric Umkehr map

$$F \ : \ M^+ \to M/(M - B(\nu_f)) \ = \ B(\nu_f)/S(\nu_f) \ = \ T(\nu_f)$$

inducing the Umkehr chain map

$$f^! \ : \ \dot{C}(M^+) \ = \ C(M) \to \dot{C}(T(\nu_f)) \ \simeq \ C(N)_{*-m+n} \ .$$

The Umkehr map II.

- Every immersion $f : N^n \looparrowright M^m$ can be approximated by an embedding

$$(e,f) \ : \ N \hookrightarrow D^k \times M \ ; \ x \mapsto (e(x), f(x))$$

for some $k \geqslant 2n - m + 1$, $e : N \to D^k$, with

$$\nu_{(e,f)} \ = \ \nu_f \oplus \epsilon^k \ : \ N \to BSO(m-n+k) \ .$$

The embedding (e,f) is regular homotopic to $(0,f)$.

- The <u>geometric Umkehr</u> of f is the geometric Umkehr of (e,f)

$$F \ : \ (D^k \times M)/(S^{k-1} \times M) = \Sigma^k M^+ \to T(\nu_{(e,f)}) = \Sigma^k T(\nu_f) \ ,$$

a stable map inducing the Umkehr chain map

$$F = f^! : \dot{C}(\Sigma^k M^+) \simeq C(M)_{*-k} \to \dot{C}(\Sigma^k T(\nu_f)) \simeq C(N)_{*-m+n-k} \ .$$

Capturing $[D(f_1, f_2)]$ by homology I.

- Proposition (Modern version of Lefschetz, 1930)
  The double point classes of transverse immersions $f_i : (N_i)^{n_i} \looparrowright M^m$ $(i = 1, 2)$ are given by

$$g[D(f_1, f_2)] \;=\; (f_1[N_1]^* \cup f_2[N_2]^*) \cap [M] \in H_{n_1+n_2-m}(M) \;,$$
$$h[D(f_1, f_2)] \;=\; (f_1 \times f_2)^! \Delta[M] \in H_{n_1+n_2-m}(N_1 \times N_2) \;.$$

- Proof Approximate $f_i : N_i \looparrowright M$ by an embedding $(e_i, f_i) : N_i \hookrightarrow D^{k_i} \times M$ with geometric Umkehr map

$$F_i \;:\; \Sigma^{k_i} M^+ \to \Sigma^{k_i} T(\nu_{f_i}) \;.$$

  The immersion $g : D(f_1, f_2) \looparrowright M$ is approximated by the embedding

$$(e_1, e_2, g) \;:\; D(f_1, f_2) \hookrightarrow D^{k_1} \times D^{k_2} \times M \;=\; D^{k_1+k_2} \times M$$

  with a geometric Umkehr map

$$G \;:\; \Sigma^{k_1+k_2} M^+ \to \Sigma^{k_1+k_2} T(\nu_g) \;.$$

Capturing $[D(f_1, f_2)]$ by homology II.

- The formulae for the double point classes follow from the commutative diagrams

$$\begin{array}{ccc} \Sigma^{k_1+k_2} T(\nu_f) & \xrightarrow{T(h)} & \Sigma^{k_1} T(\nu_{f_1}) \wedge \Sigma^{k_2} T(\nu_{f_2}) \\ {\scriptstyle G}\uparrow & & \uparrow{\scriptstyle F_1 \wedge F_2} \\ \Sigma^{k_1+k_2} M^+ & \xrightarrow{\Delta_M} & \Sigma^{k_1} M^+ \wedge \Sigma^{k_2} M^+ \end{array}$$

$$\begin{array}{ccc} C(D(f_1, f_2))_{*-2m+n_1+n_2} & \xrightarrow{h} & C(N_1 \times N_2)_{*-2m+n_1+n_2} \\ {\scriptstyle g^!}\uparrow & & \uparrow{\scriptstyle (f_1 \times f_2)^!} \\ C(M) & \xrightarrow{\Delta_M} & C(M \times M) \end{array}$$

- $g[D(f_1, f_2)]^* = (hg^!)^*(1_{N_1 \times N_2}) = ((f_1 \times f_2)^! \Delta_M)^*(1_{N_1 \times N_2})$.
- $h[D(f_1, f_2)] = hg^![M] = (f_1 \times f_2)^! \Delta_M[M]$. $\square$

The double point sets $\overline{D}(f)$, $D(f)$

- For any map $f : N \to M$ there is defined a $\mathbb{Z}_2$-equivariant map

$$f \times f \ : \ N \times N \to M \times M \ ; \ (x, y) \mapsto (f(x), f(y))$$

with the generator $T \in \mathbb{Z}_2$ acting by $T(x, y) = (y, x)$.
$D(f, f)$ is $\mathbb{Z}_2$-invariant, with fixed points $D(f, f)^{\mathbb{Z}_2} = \Delta_N$.

- The <u>ordered double point set</u> of f is the free $\mathbb{Z}_2$-set

$$\begin{aligned} \overline{D}(f) \ &= \ D(f, f) - D(f, f)^{\mathbb{Z}_2} \\ &= \ \{(x, y) \in N \times N \,|\, x \neq y \in N,\ f(x) = f(y) \in M\} \ . \end{aligned}$$

The $\mathbb{Z}_2$-set $D(f, f) \subseteq N \times N$ is the union

$$\begin{aligned} D(f, f) \ &= \ D(f, f)^{\mathbb{Z}_2} \cup (D(f, f) - D(f, f)^{\mathbb{Z}_2}) \\ &= \ \mathbb{Z}_2\text{-fixed points} \cup \text{free } \mathbb{Z}_2\text{-set} \ = \ \Delta_N \cup \overline{D}(f) \ . \end{aligned}$$

- The <u>unordered double point set</u> of $f : N \to M$ is

$$D(f) \ = \ \overline{D}(f)/\mathbb{Z}_2 \ .$$

- $f : N \to M$ is an embedding if and only if $D(f) = \emptyset$.

The ordered double point classes of an immersion

- The double point set of a self-transverse immersion $f : N^n \looparrowright M^m$ with $n < m$ is a stratified set

$$D(f,f) \ = \ \Delta_N \cup \overline{D}(f) \cup (\leqslant 3n-2m)\text{-dimensional strata}$$

  with Δ_N n-dimensional and $\overline{D}(f)$ $(2n-m)$-dimensional.
- $\overline{D}(f)$ is oriented, with a fundamental class

$$[\overline{D}(f)] \in H_{2n-m}(\overline{D}(f)) \ .$$

- The <u>ordered double point classes</u> are the images

$$g[\overline{D}(f)] \in H_{2n-m}(M) \ , \ \ h[\overline{D}(f)] \in H_{2n-m}(N \times N)$$

  with $g : \overline{D}(f) \looparrowright M$, $h : \overline{D}(f) \hookrightarrow N \times N$ as before.
- The covering translation $T : \overline{D}(f) \to \overline{D}(f)$ is orientation-preserving if and only if $m-n$ is even. Thus $D(f)$ only has a twisted fundamental class $[D(f)] \in H_{2n-m}(D(f);\mathbb{Z}^{(-)^{m-n}})$.

Capturing $[\overline{D}(f)]$ by homology I.

- Proposition (Modern version of Whitney, 1940)
  The ordered double point classes of $f : N^n \looparrowright M^m$ are
  $$\begin{aligned} g[\overline{D}(f)] &= (f[N]^* \cup f[N]^* - f^* e(\nu_f)) \cap [M] \in H_{2n-m}(M) , \\ h[\overline{D}(f)] &= ((f \times f)^! \Delta_M - \Delta_{T(\nu_f)} f^!)[M] \in H_{2n-m}(N \times N) \end{aligned}$$
  with $e(\nu_f) \in H^{m-n}(N)$ the Euler class.
- Proof The immersion
  $$g \ : \ \overline{D}(f) \looparrowright M \ ; \ (x,y) \mapsto f(x) = f(y)$$
  has normal bundle
  $$\nu_g \ = \ h^*(\nu_f \times \nu_f) \ : \ \overline{D}(f) \to BSO(2(m-n))$$
  with $h : \overline{D}(f) \hookrightarrow N \times N$ the inclusion. If f is approximated by an embedding $(e,f) : N \hookrightarrow D^k \times M$ then f and g have geometric Umkehr maps
  $$F : \Sigma^k M^+ \to \Sigma^k T(\nu_f) \ , \ G : \Sigma^{2k} M^+ \to \Sigma^{2k} T(\nu_g) \ .$$

Capturing $[\overline{D}(f)]$ by homology II.

- The formulae for the ordered double point classes follow from the commutative diagrams

$$\begin{array}{ccc}
\Sigma^{2k}T(\nu_f)\vee\Sigma^{2k}T(\nu_g) & \xrightarrow{\Delta_{T(\nu_f)}\vee T(h)} & \Sigma^kT(\nu_f)\wedge\Sigma^kT(\nu_f) \\
\uparrow{\scriptstyle F\vee G} & & \uparrow{\scriptstyle F\wedge F} \\
\Sigma^{2k}M^+ & \xrightarrow{\Delta_M} & \Sigma^kM^+\wedge\Sigma^kM^+
\end{array}$$

$$\begin{array}{ccc}
C(N)_{*-m+n}\oplus C(\overline{D}(f))_{*-2m+2n} & \xrightarrow{\Delta_Ne(\nu_f)\oplus h} & C(N\times N)_{*-2m+2n} \\
\uparrow{\scriptstyle f^!\oplus g^!} & & \uparrow{\scriptstyle (f\times f)^!} \\
C(M) & \xrightarrow{\Delta_M} & C(M\times M)
\end{array}$$

□

The quadratic construction

- In order to capture $D(f)$ by homology need to take account of the $\mathbb{Z}_2$-action on $\overline{D}(f)$.
- The <u>quadratic construction</u> on a space X is

$$Q(X) \;=\; S^\infty \times_{\mathbb{Z}_2} (X \times X)$$

with $T(x,y) = (y,x)$ on $X \times X$ and

$$T \;:\; S^\infty \;=\; \varinjlim_k S^k \to S^\infty \;;\; s \mapsto -s \;.$$

The projection $Q(X) \to \mathbb{R}\mathbb{P}^\infty$ classifies the double cover

$$\overline{Q(X)} \;=\; S^\infty \times (X \times X) \to Q(X) \;.$$

- The <u>reduced quadratic construction</u> on a pointed space Y is

$$\dot{Q}(Y) \;=\; (S^\infty)^+ \wedge_{\mathbb{Z}_2} (Y \wedge Y) \;.$$

In particular, for an unpointed space X

$$\dot{Q}(X^+) \;=\; Q(X)^+ \;.$$

The unordered double point class of an immersion I.

- Approximate the immersion $f : N^n \looparrowright M^m$ by an embedding $(e, f) : N \hookrightarrow D^k \times M$. The $\mathbb{Z}_2$-equivariant map

$$\overline{d} \ : \ \overline{D}(f) \to S^{k-1} \times (N \times N) \ ; \ (x, y) \mapsto \left(\frac{e(x) - e(y)}{\|e(x) - e(y)\|}, x, y \right)$$

induces a map

$$d \ : \ D(f) \to S^{k-1} \times_{\mathbb{Z}_2} (N \times N) \subset Q(N) \ .$$

- The unordered double point class of f is

$$[D(f)] \ \equiv \ d[D(f)] \in H_{2n-m}(Q(N); \mathbb{Z}^{(-)^{m-n}}) \ .$$

- The composite $D(f) \to \mathbb{R}\mathbb{P}^{k-1} \subset \mathbb{R}\mathbb{P}^{\infty}$ classifies the double cover $p : \overline{D}(f) \to D(f)$. The transfer of p sends $[D(f)]$ to the ordered double point class

$$p^![D(f)] \ = \ h[\overline{D}(f)] \in H_{2n-m}(\overline{Q(N)}) \ = \ H_{2n-m}(N \times N) \ .$$

The unordered double point class of an immersion II.

- For $\pi_1(M) = \{1\}$ Wall's self-intersection invariant for $f : N^n \looparrowright M^{2n}$ is the unordered double point class

$$\mu(f) \;=\; [D(f)] \in H_0(Q(N); \mathbb{Z}^{(-)^n}) \;=\; H_0(\mathbb{Z}_2; \mathbb{Z}, (-)^n)$$

- The algebraic theory of surgery (R., 1980) identified

$$[D(f : N^n \looparrowright M^m)] \in H_{2n-m}(Q(N); \mathbb{Z}^{(-)^{m-n}})$$

  for any f with a chain level desuspension obstruction for a geometric Umkehr $F : \Sigma^k M^+ \to \Sigma^k T(\nu_f)$, including a $\pi_1(M)$-equivariant version.
- Joint project with Michael Crabb: apply $\mathbb{Z}_2$-equivariant stable homotopy theory and the 'geometric Hopf invariant' to provide a homotopy theoretic treatment of $[D(f)]$.

The geometric Hopf invariant $h(F)$ I.

- When is a k-stable map $F : \Sigma^k X \to \Sigma^k Y$ homotopic to the k-fold suspension $\Sigma^k F_0$ of an unstable map $F_0 : X \to Y$?
- The geometric Hopf invariant of F is the stable $\mathbb{Z}_2$-equivariant map

$$h(F) = (F \wedge F)\Delta_X - \Delta_Y F \;:\; X \to Y \wedge Y \;.$$

- If $F \simeq \Sigma^k F_0$ for an unstable map $F_0 : X \to Y$ then $h(F) \simeq *$.
- *The stable $\mathbb{Z}_2$-equivariant homotopy class of $h(F)$ depends only on the homotopy class of F, and is the primary obstruction to the k-fold desuspension of F.*
- *For the geometric Umkehr map $F : \Sigma^k M^+ \to \Sigma^k T(\nu_f)$ of an immersion $f : N^n \looparrowright M^m$ the stable $\mathbb{Z}_2$-equivariant homotopy class of $h(F)$ factors through the ordered double point $\mathbb{Z}_2$-set $\overline{D}(f)$.*

The stable $\mathbb{Z}_2$-equivariant homotopy groups

- Given pointed $\mathbb{Z}_2$-spaces X, Y let $[X, Y]_{\mathbb{Z}_2}$ be the set of $\mathbb{Z}_2$-equivariant homotopy classes of $\mathbb{Z}_2$-equivariant maps $X \to Y$.
- The stable $\mathbb{Z}_2$-equivariant homotopy group is

$$\{X; Y\}_{\mathbb{Z}_2} = \varinjlim_{k} [\Sigma^{k,k} X, \Sigma^{k,k} Y]_{\mathbb{Z}_2}$$

where

$$T : \Sigma^{k,k} X = S^k \wedge S^k \wedge X \to \Sigma^{k,k} X ; (s,t,x) \mapsto (t,s,T(x)) .$$

- Example The $\mathbb{Z}_2$-equivariant Pontrjagin-Thom isomorphism identifies $\{S^0; S^0\}_{\mathbb{Z}_2}$ with the cobordism group of 0-dimensional framed $\mathbb{Z}_2$-manifolds (= finite $\mathbb{Z}_2$-sets). The decomposition of finite $\mathbb{Z}_2$-sets as fixed $\cup$ $\mathbb{Z}_2$-free determines an isomorphism

$$\{S^0; S^0\}_{\mathbb{Z}_2} \cong \mathbb{Z} \oplus \mathbb{Z} ; A = A^{\mathbb{Z}_2} \cup (A - A^{\mathbb{Z}_2}) \mapsto \left(|A^{\mathbb{Z}_2}|, \frac{|A| - |A^{\mathbb{Z}_2}|}{2} \right) .$$

$\mathbb{Z}_2$-equivariant stable homotopy theory = fixed-points + fixed-point-free

- Theorem (Crabb, 1980) For any pointed spaces X, Y there is a split exact sequence of abelian groups

$$0 \to \{X; \dot{Q}(Y)\} \longrightarrow \{X; Y \wedge Y\}_{\mathbb{Z}_2} \xrightarrow{\rho} \{X; Y\} \to 0$$

with the injection induced by the projection $S^\infty \to \{*\}$

$$\{X; \dot{Q}(Y)\} = \{X; (S^\infty)^+ \wedge (Y \wedge Y)\}_{\mathbb{Z}_2} \to \{X; Y \wedge Y\}_{\mathbb{Z}_2} .$$

- ρ is given by the $\mathbb{Z}_2$-fixed points, split by

$$\sigma : \{X; Y\} \to \{X; Y \wedge Y\}_{\mathbb{Z}_2} ; \ F \mapsto \Delta_Y F .$$

The geometric Hopf invariant $h(F)$ II.

- The geometric Hopf invariant of $F : \Sigma^k X \to \Sigma^k Y$

$$\begin{aligned} h(F) &= (F \wedge F)\Delta_X - \Delta_Y F \\ &\in \ker(\rho : \{X; Y \wedge Y\}_{\mathbb{Z}_2} \to \{X; Y\}) \\ &= \operatorname{im}(\{X; \dot{Q}(Y)\} \hookrightarrow \{X; Y \wedge Y\}_{\mathbb{Z}_2}) \end{aligned}$$

  has the following properties:

  (i) The function

$$h \ : \ \{X; Y\} \to \{X; \dot{Q}(Y)\} \ ; \ F \mapsto h(F)$$

  is nonadditive, being quadratic in nature:

$$h(F + G) \ = \ h(F) + h(G) + (F \wedge G)\Delta_X \ .$$

  (ii) If $F \in \operatorname{im}([X, Y] \to \{X; Y\})$ then $h(F) = 0$.

The $\mathbb{Z}_2$-equivariant Umkehr map

- An immersion $f : N^n \looparrowright M^m$ determines a commutative square of $\mathbb{Z}_2$-equivariant immersions and embeddings

$$\begin{array}{ccc} N \cup \overline{D}(f) & \xrightarrow{\Delta_N \cup h} & N \times N \\ {\scriptstyle f \cup g}\big\downarrow & & \big\downarrow{\scriptstyle f \times f} \\ M & \xrightarrow{\Delta_M} & M \times M \end{array}$$

$$g(x,y) = f(x) = f(y)\ ,\ \ h(x,y) = (x,y)\ ,\ \ \nu_g = h^*(\nu_f \times \nu_f)\ .$$

- An approximating embedding $(e,f) : N \hookrightarrow D^k \times M$ determines $\mathbb{Z}_2$-equivariant embeddings

$$(e \times e, f \times f)\ :\ N \times N \hookrightarrow D^k \times D^k \times M \times M\ ,$$

$$(e \times e|, g)\ :\ \overline{D}(f) \hookrightarrow D^k \times D^k \times M\ .$$

- The Umkehr of $(e \times e|, g)$ is a $\mathbb{Z}_2$-equivariant Umkehr map

$$G\ :\ \Sigma^{k,k} M^+ \to \Sigma^{k,k} T(\nu_g)\ .$$

Capturing $[D(f)]$ by homology I.

- Proposition (Crabb+R.) If $f : N^n \looparrowright M^m$ is an immersion with Umkehr map $F : \Sigma^k M^+ \to \Sigma^k T(\nu_f)$ the geometric Hopf invariant $h(F)$ factors through $T(\nu_g)$

$$\begin{aligned} h(F) &= T(h)G \\ &\in \ker\big(\rho : \{M^+; T(\nu_f) \wedge T(\nu_f)\}_{\mathbb{Z}_2} \to \{M^+; T(\nu_f)\}\big) \\ &= \operatorname{im}\big(\{M^+; \dot{Q}(T(\nu_f))\} \hookrightarrow \{M^+; T(\nu_f) \wedge T(\nu_f)\}_{\mathbb{Z}_2}\big) \end{aligned}$$

with $h : \overline{D}(f) \hookrightarrow N \times N$ the inclusion, i.e.

$$h(F) \;:\; M^+ \xrightarrow{G} T(\nu_g) \xrightarrow{T(h)} T(\nu_f \times \nu_f) = T(\nu_f) \wedge T(\nu_f)\ .$$

Capturing $[D(f)]$ by homology II.

- The formula for the unordered double point classes follows from the commutative diagrams of $\mathbb{Z}_2$-equivariant maps

$$\begin{array}{ccc}
\Sigma^{k,k}T(\nu_f) \vee \Sigma^{k,k}T(\nu_g) & \xrightarrow{\Delta_{T(\nu_f)} \vee T(h)} & \Sigma^k T(\nu_f) \wedge \Sigma^k T(\nu_f) \\
{\scriptstyle F \vee G}\uparrow & & \uparrow{\scriptstyle F \wedge F} \\
\Sigma^{k,k}M^+ & \xrightarrow{\quad\Delta_M\quad} & \Sigma^k M^+ \wedge \Sigma^k M^+
\end{array}$$

□

- Corollary The unordered double point class of $f : N^n \looparrowright M^m$ is

$$[D(f)] \;=\; h(F)[M] \in \dot{H}_m(\dot{Q}(T(\nu_f))) \;=\; H_{2n-m}(Q(N); \mathbb{Z}^{(-)^{m-n}}) \,,$$

regarding $h(F)$ as a stable map $M^+ \to \dot{Q}(T(\nu_f))$.

The π-equivariant geometric Hopf

- Let π be a group, and let X be a pointed π-space. The diagonal map $\Delta : X \to X \wedge X$ is π-equivariant, so induces

$$\Delta/\pi \ : \ X/\pi \to X \wedge_\pi X \ ; \ [x] \mapsto [x,x] \ .$$

- Let X, Y be pointed π-spaces. The geometric Hopf invariant of a π-equivariant stable map $F : \Sigma^k X \to \Sigma^k Y$ is the stable $\mathbb{Z}_2$-equivariant map

$$h_\pi(F) \ = \ ((F \wedge F)\Delta_X - \Delta_Y F)/\pi \ : \ X/\pi \to Y \wedge_\pi Y$$

which can be regarded as a stable map

$$h_\pi(F) \ : \ X/\pi \to \dot{Q}_\pi(Y) \ = \ (S^\infty)^+ \wedge_{\mathbb{Z}_2} (Y \wedge_\pi Y) \ .$$

The π-equivariant unordered double point class

- An immersion $f : N^n \looparrowright M^m$ lifts to a π-equivariant immersion $\widetilde{f} : \widetilde{N} \looparrowright \widetilde{M}$, with $\pi = \pi_1(M)$, $\widetilde{M}$ the universal cover of M and $\widetilde{N} = f^*\widetilde{M}$.
- Proposition (C.+R.) The π-equivariant unordered double point class of $\widetilde{f}$ is the evaluation on $[M] \in H_m(M)$ of the π-equivariant geometric Hopf invariant of a π-equivariant geometric Umkehr $F : \Sigma^k \widetilde{M}^+ \to \Sigma^k T(\nu_{\widetilde{f}})$ for $\widetilde{f}$, that is

$$\begin{aligned}[D(\widetilde{f})/\pi] \ &= \ h_\pi(F)[M] \\ &\in \dot{H}_m(\dot{Q}_\pi(T(\nu_{\widetilde{f}}))) \ = \ H_{2n-m}(Q_\pi(\widetilde{N}); \mathbb{Z}^{(-)^{m-n}}) \ .\end{aligned}$$

- For $f : S^n \looparrowright M^{2n}$ this is Wall's self-intersection invariant

$$\begin{aligned}\mu(f) \ &= \ [D(\widetilde{f})/\pi] \ = \ h_\pi(F)[M] \\ &\in \dot{H}_{2n}(\dot{Q}_\pi(T(\nu_{\widetilde{f}}))) \ = \ H_0(Q_\pi(\widetilde{N}); \mathbb{Z}^{(-)^n}) \ = \ H_0(\mathbb{Z}_2; \mathbb{Z}[\pi], (-)^n) \ .\end{aligned}$$