# THE NUMBER EIGHT IN TOPOLOGY

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http://www.maths.ed.ac.uk/∼aar/eight.htm

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### Sociology and topology

▶ *It is a fact of sociology that topologists are interested in quadratic forms* (Serge Lang)

▶ The 8 in the title refers to the applications in topology of the mod 8 properties of the signatures of integral symmetric matrices, such as the celebrated $8 \times 8$ matrix $E_8$ with

$$\text{signature}(E_8) \; = \; 8 \in \mathbb{Z} \, .$$

▶ A compact oriented $4k$-manifold with boundary has an integral symmetric matrix of intersection numbers. The signature of the manifold is defined by

$$\text{signature(manifold)} \; = \; \text{signature(matrix)} \in \mathbb{Z} \, .$$

▶ Manifolds with intersection matrix $E_8$ have been used to distinguish the categories of differentiable, PL and topological manifolds, and so are of particular interest to topologists!

# Quadratic forms and manifolds

- ▶ The algebraic properties of quadratic forms were already studied in the 19th century: Sylvester, H.J.S. Smith, . . .
- ▶ Similarly, the study of the topological properties of manifolds reaches back to the 19th century: Riemann, Poincaré, . . .
- ▶ The combination of algebra and topology is very much a 20th century story. But in 1923 when Weyl first proposed the definition of the signature of a manifold, topology was so dangerous that he thought it wiser to write the paper in Spanish and publish it in Spain. And this is his signature :

## Symmetric matrices

- $R =$ commutative ring. Main examples today: $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Z}_4$, $\mathbb{Z}_2$.
- The **transpose** of an $m \times n$ matrix $\Phi = (\Phi_{ij})$ with $\Phi_{ij} \in R$ is the $n \times m$ matrix $\Phi^T$ with

$$(\Phi^T)_{ji} = \Phi_{ij} \ (1 \leqslant i \leqslant m, \ 1 \leqslant j \leqslant n) \ .$$

- Let $\mathrm{Sym}_n(R)$ be the set of $n \times n$ matrices $\Phi$ which are **symmetric** $\Phi^T = \Phi$.
- $\Phi, \Phi' \in \mathrm{Sym}_n(R)$ are **conjugate** if $\Phi' = A^T \Phi A$ for an invertible $n \times n$ matrix $A \in GL_n(R)$.
- Can also view $\Phi$ as a symmetric bilinear pairing on the $n$-dimensional f.g. free $R$-module $R^n$

$$\Phi \ : \ R^n \times R^n \to R \ ; \ ((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \mapsto \sum_{i=1}^{n} \sum_{j=1}^{n} \Phi_{ij} x_i y_j \ .$$

- $\Phi \in \mathrm{Sym}_n(R)$ is **unimodular** if it is invertible, or equivalently if $\det(\Phi) \in R$ is a unit.

## The signature

- The **signature** of $\Phi \in \mathrm{Sym}_n(\mathbb{R})$ is

$$\sigma(\Phi) = p_+ - p_- \in \mathbb{Z}$$

  with $p_+$ the number of eigenvalues $> 0$ and $p_-$ the number of eigenvalues $< 0$.

- **Law of Inertia** (Sylvester 1853)
  Symmetric matrices $\Phi, \Phi' \in \mathrm{Sym}_n(\mathbb{R})$ are conjugate if and only if

$$p_+ = p'_+ \, , \, p_- = p'_- \, .$$

- The signature of $\Phi \in \mathrm{Sym}_n(\mathbb{Z})$

$$\sigma(\Phi) = \sigma(\mathbb{R} \otimes_{\mathbb{Z}} \Phi) \in \mathbb{Z}.$$

  is an integral conjugacy invariant.

- The conjugacy classification of symmetric matrices is much harder for $\mathbb{Z}$ than $\mathbb{R}$. For example, can diagonalize over $\mathbb{R}$ but not over $\mathbb{Z}$.

## Type I and type II

- ▶ $\Phi \in \text{Sym}_n(\mathbb{Z})$ is of **type I** if at least one of the diagonal entries $\Phi_{ii} \in \mathbb{Z}$ is odd.
- ▶ $\Phi$ is of **type II** if each $\Phi_{ii} \in \mathbb{Z}$ is even.
- ▶ Type I cannot be conjugate to type II. So unimodular type II cannot be diagonalized, i.e. not conjugate to $\bigoplus_n \pm 1$.
- ▶ $\Phi$ is **positive definite** if $n = p_+$, or equivalently if $\sigma(\Phi) = n$. Choosing an orthonormal basis for $\mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{Z}^n, \Phi)$ defines an embedding as a lattice $(\mathbb{Z}^n, \Phi) \subset (\mathbb{R}^n, \text{dot product})$. Lattices (including $E_8$) much used in coding theory.
- ▶ **Examples**
  - (i) $\Phi = (1) \in \text{Sym}_1(\mathbb{Z})$ is unimodular, positive definite, type I, signature 1.
  - (ii) $\Phi = (2) \in \text{Sym}_1(\mathbb{Z})$ is positive definite, type II, signature 1.
  - (iii) $\Phi = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in \text{Sym}_2(\mathbb{Z})$ is unimodular, type I, signature 0.
  - (iv) $\Phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \text{Sym}_2(\mathbb{Z})$ is unimodular, type II, signature 0.

**Characteristic elements and the signature mod 8**

- An element $u \in R^n$ is **characteristic** for $\Phi \in \mathrm{Sym}_n(R)$ if

$$\Phi(x, u) - \Phi(x, x) \in 2R \subseteq R \text{ for all } x \in R^n .$$

- Every unimodular $\Phi$ admits characteristic elements $u \in R^n$ which constitute a coset of $2R^n \subseteq R^n$.

- **Theorem** (van der Blij, 1958) The mod 8 signature of a unimodular $\Phi \in \mathrm{Sym}_n(\mathbb{Z})$ is such that

$$\sigma(\Phi) \equiv \Phi(u, u) \bmod 8$$

for any characteristic element $u \in \mathbb{Z}^n$.

- **Corollary** A unimodular $\Phi \in \mathrm{Sym}_n(\mathbb{Z})$ is of type II if and only if $u = 0 \in \mathbb{Z}^n$ is characteristic, in which case

$$\sigma(\Phi) \equiv 0 \bmod 8 .$$

## The $E_8$-form I.

▶ **Theorem** (H.J.S. Smith 1867, Korkine and Zolotareff 1873)
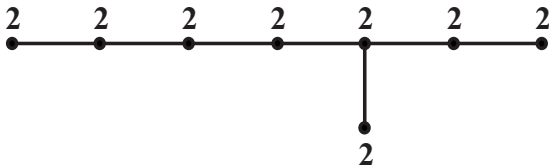There exists an 8-dimensional unimodular positive definite
type II symmetric matrix

$$E_8 \; = \; \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix} \in \mathsf{Sym}_8(\mathbb{Z}) \; .$$

▶ $E_8$ has signature

$$\sigma(E_8) \; = \; 8 \in \mathbb{Z} \; .$$

# The $E_8$-form II.

- $E_8 \in \mathrm{Sym}_8(\mathbb{Z})$ is determined by the Dynkin diagram of the simple Lie algebra $E_8$



weighted by $\chi(S^2) = 2$ at each vertex, with

$$\Phi_{ij} = \begin{cases} 1 & \text{if } i\text{th vertex is adjacent to } j\text{th vertex} \\ 2 & \text{if } i = j \\ 0 & \text{otherwise .} \end{cases}$$

- **Theorem** (Mordell, 1938) Any unimodular positive definite type II symmetric matrix $\Phi \in \mathrm{Sym}_8(\mathbb{Z})$ is conjugate to $E_8$.

**The intersection matrix of a** $4k$-**manifold**

▶ The **intersection matrix** of a $4k$-manifold with boundary
  $(M, \partial M)$ with respect to a basis $(b_1, b_2, \ldots, b_n)$ for
  $H_{2k}(M)/\text{torsion} \cong \mathbb{Z}^n$ is the symmetric matrix

$$\Phi(M) = (b_i \cap b_j)_{1 \leqslant i,j \leqslant n} \in \text{Sym}_n(\mathbb{Z})$$

  with $b_i \cap b_j \in \mathbb{Z}$ the homological intersection number.

▶ If $b_i, b_j$ are represented by disjoint closed $2k$-submanifolds
  $N_i, N_j \subset M$ which intersect transversely then $b_i \cap b_j \in \mathbb{Z}$ is
  the number of points in the actual intersection $N_i \cap N_j \subset M$,
  counted algebraically.



▶ A different basis gives a conjugate intersection matrix.

### $(2k-1)$-**connected** $4k$-**manifolds**

- A space $M$ is **0-connected** if it is connected.
- For $j \geqslant 1$ a space $M$ is $j$-**connected** if it is connected, and $\pi_i(M) = \{1\}$ for $1 \leqslant i \leqslant j$ or equivalently if $M$ is simply-connected ($\pi_1(M) = \{1\}$) and $H_i(M) = 0$ for $1 \leqslant i \leqslant j$.
- An $m$-manifold with boundary $(M, \partial M)$ is $j$-**connected** if $M$ is $j$-connected and $\partial M$ is $(j-1)$-connected.
- **Proposition** If $(M, \partial M)$ is a $(2k-1)$-connected $4k$-manifold with boundary then
  - $H_{2k}(M)$ is f.g. free,
  - there is an exact sequence

  $$0 \to H_{2k}(\partial M) \to H_{2k}(M) \xrightarrow{\Phi(M)} H_{2k}(M)^* \to H_{2k-1}(\partial M) \to 0$$

  with $H_{2k}(M)^* = \mathrm{Hom}_{\mathbb{Z}}(H_{2k}(M), \mathbb{Z})$.

## Homology spheres

▶ A **homology** $\ell$-**sphere** $\Sigma$ is a closed $\ell$-manifold such that

$$H_*(\Sigma) \ = \ H_*(S^\ell) \ .$$

▶ An $m$-manifold with boundary $(M, \partial M)$ is **almost closed** if

either $M$ is closed, i.e. $\partial M = \emptyset$,

or $\partial M$ is a homology $(m-1)$-sphere

$$H_*(\partial M) \ = \ H_*(S^{m-1}) \ .$$

▶ **Proposition** The intersection matrix $\Phi(M) \in \mathrm{Sym}_n(\mathbb{Z})$ of a $(2k-1)$-connected $4k$-dimensional manifold with boundary $(M, \partial M)$ with $H_{2k}(M) = \mathbb{Z}^n$ is unimodular if and only if $(M, \partial M)$ is almost closed.

**The $2k^{th}$ Wu class of an almost closed $(M^{4k}, \partial M)$**

► **Proposition** For an almost closed $(2k-1)$-connected $4k$-manifold with boundary $(M^{4k}, \partial M)$ and intersection matrix $\Phi(M) \in \mathrm{Sym}_n(\mathbb{Z})$ the Poincaré dual of the $2k^{th}$ Wu characteristic class of the normal bundle $\nu_M$

$$v_{2k}(\nu_M) \in H^{2k}(M; \mathbb{Z}_2) \cong H_{2k}(M; \mathbb{Z}_2)$$

is characteristic for $1 \otimes \Phi(M) \in \mathrm{Sym}_n(\mathbb{Z}_2)$. An element $u \in H_{2k}(M)$ is characteristic for $\Phi(M)$ if and only if

$$[u] = v_{2k}(\nu_M) \in H_{2k}(M)/2H_{2k}(M) = H_{2k}(M; \mathbb{Z}_2) .$$

.
► $\Phi(M)$ is of type II if and only if $v_{2k}(\nu_M) = 0$.
► By van der Blij's theorem, for any lift $u \in H_{2k}(M)$ of $v_{2k}(\nu_M)$.

$$\sigma(M) \equiv \Phi(u, u) \mod 8 .$$

► If $(M^{4k}, \partial M)$ is framed, i.e. $\nu_M$ is trivial, then

$$v_{2k}(\nu_M) = 0 , \ u = 0 \text{ and } \sigma(M) \equiv 0 \,(\mathrm{mod}\,8) .$$

## The Poincaré homology 3-sphere and $E_8$

▶ Poincaré (1904) constructed a differentiable homology 3-sphere

$$\Sigma^3 = \text{dodecahedron/opposite faces}$$

with $\pi_1(\Sigma^3) =$ binary icosahedral group of order $120 \neq \{1\}$. This disproved the naive **Poincaré conjecture** that every homology 3-sphere is homeomorphic to $S^3$.



Do you need a **plumber** now?

▶ Modern construction: $\Sigma^3 = \partial M$ is the boundary of the 1-connected framed differentiable 4–manifold with boundary $(M^4, \partial M)$ with intersection matrix $\Phi(M) = E_8$ obtained by the "geometric plumbing" of 8 copies of $\tau_{S^2}$ using the $E_8$ tree.

## Exotic spheres and $E_8$

- An **exotic $\ell$-sphere** $\Sigma^\ell$ is a differentiable $\ell$-manifold which is homeomorphic but not diffeomorphic to $S^\ell$.
- Milnor (1956) constructed the first exotic spheres, $\Sigma^7$, using the Hirzebruch signature theorem (1953) to detect non-standard differentiable structure.
- Kervaire and Milnor (1963) classified exotic $\ell$-spheres $\Sigma^\ell$ for all $\ell \geqslant 7$, involving the finite abelian groups $\Theta_\ell$ of differentiable structures on $S^\ell$.
- The subgroup $bP_{4k} \subseteq \Theta_{4k-1}$ consists of the exotic $(4k-1)$-spheres $\Sigma^{4k-1} = \partial M$ which are the boundary of a framed $(2k-1)$-connected $4k$-manifold $(M^{4k}, \partial M)$ obtained by geometric plumbing, with $\Phi(M) = \bigoplus E_8$.
- In particular, the Brieskorn (1965) exotic spheres arising in algebraic geometry are such boundaries, including the Poincaré homology 3-sphere $\Sigma^3$ as a special case.

$$bP_{4k}$$

▶ The subgroup $bP_{4k} \subseteq \Theta_{4k-1}$ of diffeomorphism classes of the bounding exotic spheres $\Sigma^{4k-1} = \partial M$ is a finite cyclic group $\mathbb{Z}_{bp_{4k}}$, with an isomorphism

$$bP_{4k} \xrightarrow{\cong} \mathbb{Z}_{bp_{4k}} \; ; \; \Sigma^{4k-1} = \partial M \mapsto \sigma(M)/8 \; .$$

▶ The order $|bP_{4k}| = bp_{4k}$ is related to the numerators of the Bernoulli numbers.

▶ The group

$$bP_8 = \Theta_7 = \mathbb{Z}_{28}$$
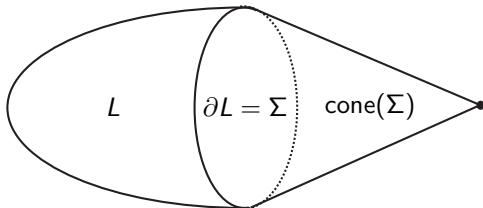
of 28 differentiable structures on $S^7$ is generated by $\Sigma^7 = \partial M$ with $\Phi(M) = E_8$.

**PL manifolds without differentiable structure I.**

▶ Cairns (1935) proved that a differentiable manifold has a canonical PL structure.

▶ If $(L^m, \partial L)$ is a differentiable $m$-manifold with boundary $\partial L = \Sigma^{m-1}$ an exotic $(m-1)$-sphere then

$$K^m = L^m \cup_\Sigma \operatorname{cone}(\Sigma)$$

is a closed PL $m$-manifold without a differentiable structure.

**PL manifolds without differentiable structure II.**

▶ The first PL manifold without a differentiable structure was the closed 4-connected PL 10-manifold constructed by Kervaire (1960)
$$K^{10} \; = \; L^{10} \cup_{\partial L} c\partial L$$

using a framed differentiable 4-connected 10-manifold $(L^{10}, \partial L)$ with boundary an exotic 9-sphere $\partial L$, obtained by plumbing two $\tau_{S^5}$'s. The corresponding $\mathbb{Z}_2$-valued quadratic form on $H^5(K; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ has Arf invariant $1 \in \mathbb{Z}_2$.

▶ The $E_8$-plumbing $(M^8, \partial M)$ gives a closed 3-connected PL 8-manifold $M^8 \cup_{\partial M} c\partial M$ without a differentiable structure.

▶ In fact, there is a close connection between the $\mathbb{Z}_8$-valued signature mod 8 and the $\mathbb{Z}_2$-valued Arf invariant, which is best understood using symmetric matrices in $\mathbb{Z}_4$.

### The classification of 1-connected 4-manifolds

▶ Milnor (1958) proved that $M^4 \mapsto \Phi(M)$ defines a bijection

{homotopy equivalence classes of closed

1-connected differentiable 4-manifolds $M^4$} $\xrightarrow{\ \cong\ }$

{conjugacy classes of unimodular integral symmetric matrices $\Phi$} .

▶ **Diagonalisation Theorem** (Donaldson 1982) If $M^4$ is a closed 1-connected differentiable 4-manifold and $\Phi(M)$ is positive definite then $\Phi(M)$ is diagonalizable over $\mathbb{Z}$.

▶ **Non-diagonalisation Theorem** (Freedman 1982) Every unimodular matrix $\Phi \in \mathsf{Sym}_n(\mathbb{Z})$ is realized as $\Phi = \Phi(M)$ for a closed 1-connected topological 4-manifold $M^4$. If $\Phi$ is of type II and $M$ has a PL structure then $\sigma(M) \equiv 0 (\mathrm{mod}\,16)$ (Rochlin 1952).

▶ **Nontriangulable manifolds** Casson (1990) : $M^4$ with $\Phi(M) = E_8$ is nontriangulable. Manolescu (2013) : there are nontriangulable topological $m$-manifolds $M^m$ for all $m \geqslant 4$.

**Which integral symmetric matrices are realized as intersection matrices of manifolds? I.**

- Adams (1962) proved that there exists a map $S^{4k-1} \to S^{2k}$ of Hopf invariant 1 if and only if $k = 1, 2, 4$. It followed that there exists a closed differentiable $(2k-1)$-connected $4k$-manifold $M^{4k}$ with intersection matrix $\Phi(M)$ of type I if and only if $k = 1, 2, 4$.

- The standard examples of $(2k-1)$-connected $M^{4k}$ with

$$(H_{2k}(M), \Phi(M)) = (\mathbb{Z}, 1)$$

of type I :

(i) $k = 1$ : the complex projective plane $\mathbb{CP}^2$,

(ii) $k = 2$ : the quaternionic projective plane $\mathbb{HP}$ (Hamilton),

(iii) $k = 4$ : the octonionic projective plane $\mathbb{OP}$ (Cayley).

**Which integral symmetric matrices are realized as intersection matrices of manifolds? II.**

▶ **Theorem** (Milnor, Hirzebruch 1962) For every symmetric matrix $\Phi \in \mathsf{Sym}_n(\mathbb{Z})$ of type II and every $k \geqslant 1$ there exists a differentiable $(2k-1)$-connected $4k$-manifold $(M, \partial M)$ with intersection matrix $\Phi(M) = \Phi$.

▶ $(M, \partial M)$ is constructed by the "geometric plumbing" of a sequence of $n$ oriented $2k$-plane bundles over $S^{2k}$

$$\mathbb{R}^{2k} \to E(w_i) \to S^{2k} \ (1 \leqslant i \leqslant n)$$

classified by $w_i \in \pi_{2k}(BSO(2k))$, with Euler numbers $\chi(w_i) = \Phi_{ii} \in 2\mathbb{Z} \subset \mathbb{Z}$.

▶ The geometry reflects the way in which $\Phi$ is built up from $0$ by the "algebraic plumbing" of its $n$ principal minors

$$(\Phi_{11}) \ , \ \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \ , \ \begin{pmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} \end{pmatrix} \ , \ \ldots \ , \ \Phi$$

## Algebraic plumbing

▶ **Definition** The **algebraic plumbing** of a symmetric $n \times n$ matrix $\Phi \in \mathsf{Sym}_n(\mathbb{Z})$ with respect to $v \in \mathbb{Z}^n$, $w \in \mathbb{Z}$ is the symmetric $(n+1) \times (n+1)$ matrix

$$\Phi' \; = \; \begin{pmatrix} \Phi & v^T \\ v & w \end{pmatrix} \in \mathsf{Sym}_{n+1}(\mathbb{Z}) \; .$$

▶ Let $\Phi = \Phi(M) \in \mathsf{Sym}_n(\mathbb{Z})$ is the intersection matrix of a $(2k-1)$-connected $4k$-manifold with boundary $(M, \partial M)$, taken to be $(D^{4k}, S^{4k-1})$ if $n = 0$. It is frequently possible to realize the algebraic plumbing $\Phi \mapsto \Phi'$ by a geometric plumbing
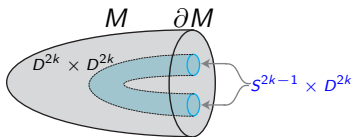
$$(M, \partial M) \mapsto (M', \partial M') \; , \; \Phi(M') \; = \; \Phi' \in \mathsf{Sym}_{n+1}(\mathbb{Z})$$

and $(M', \partial M')$ also $(2k-1)$-connected.

▶ Need $k = 1, 2, 4$ for type I. All $k \geqslant 1$ possible for type II. For $k = 1$ have to distinguish differentiable and topological categories.
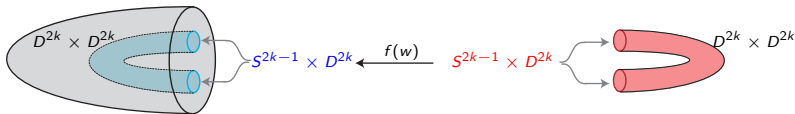
## Geometric plumbing I.

▶ **Input** (i) A $4k$-manifold with boundary $(M, \partial M)$,
(ii) an embedding $v : (D^{2k} \times D^{2k}, S^{2k-1} \times D^{2k}) \subseteq (M, \partial M)$



(iii) a map $w : S^{2k-1} \to SO(2k)$, the clutching map of the oriented $2k$-plane bundle over $S^{2k} = D^{2k} \cup_{S^{2k-1}} D^{2k}$ classified by $w \in \pi_{2k-1}(SO(2k)) = \pi_{2k}(BSO(2k))$

$$\mathbb{R}^{2k} \to E(w) = D^{2k} \times \mathbb{R}^{2k} \cup_{f(w)} D^{2k} \times \mathbb{R}^{2k} \to S^{2k}$$

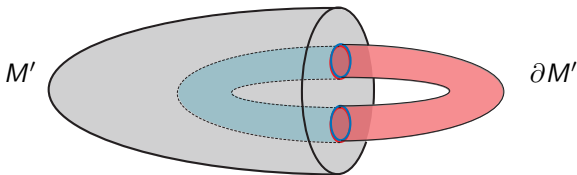$$f(w) : S^{2k-1} \times \mathbb{R}^{2k} \to S^{2k-1} \times \mathbb{R}^{2k} ; (x, y) \mapsto (x, w(x)(y)) .$$

# Geometric plumbing II.

▶ **Output** The **plumbed** $4k$-manifold with boundary

$(M', \partial M')$

$= (M \cup_{f(w)} D^{2k} \times D^{2k}, \mathrm{cl}.(\partial M \backslash S^{2k-1} \times D^{2k}) \cup D^{2k} \times S^{2k-1})$ .



$M'$                                                $\partial M'$

▶ $M'$ is obtained from $M$ by attaching a $2k$-handle $D^{2k} \times D^{2k}$ at $S^{2k-1} \times D^{2k} \subset \partial M$.

▶ $\partial M'$ is obtained from $\partial M$ by surgery on $S^{2k-1} \times D^{2k} \subset \partial M$.

**The algebraic effect of geometric plumbing**

▶ **Proposition** If $(M^{4k}, \partial M)$ has symmetric intersection matrix $\Phi(M) \in \mathrm{Sym}_n(\mathbb{Z})$ the geometric plumbing $(M', \partial M')$ has the symmetric intersection matrix given by algebraic plumbing

$$\Phi(M') \ = \ \begin{pmatrix} \Phi(M) & v^T \\ v & \chi(w) \end{pmatrix} \in \mathrm{Sym}_{n+1}(\mathbb{Z})$$

with

$$v \ = \ v[D^{2k} \times D^{2k}] \in H_{2k}(M, \partial M) \ = \ H_{2k}(M)^* = \ \mathbb{Z}^n \ ,$$
$$\chi(w) \ = \ \mathrm{degree}(S^{2k-1} \to^w SO(2k) \to S^{2k-1}) \in \mathbb{Z} \ ,$$
$$SO(2k) \to S^{2k-1} \ ; \ A \mapsto A(0, \ldots, 0, 1) \ .$$

## Graph manifolds

- A **graph manifold** is a differentiable $4k$-manifold with boundary constructed from $(D^{4k}, S^{4k-1})$ by the geometric plumbing of $n$ oriented $2k$-plane bundles $w_i \in \pi_{2k}(BSO(2k))$ over $S^{2k}$, using a graph with vertices $i = 1, 2, \ldots, n$ and weights $\chi_i = \chi(w_i) \in \mathbb{Z}$.

- **Theorem** (Milnor 1959, Hirzebruch 1961) Let $\Phi \in \text{Sym}_n(\mathbb{Z})$. If $\Phi$ is of type I assume $k = 1, 2$ or $4$. If $\Phi$ is of type II take any $k \geqslant 1$. Then $\Phi$ is the intersection matrix of a graph $4k$-manifold with boundary $(M, \partial M)$ such that

$$(H_{2k}(M), \Phi(M)) = (\mathbb{Z}^n, \Phi) .$$

- If the graph is a tree then $(M, \partial M)$ is $(2k-1)$-connected, and if $\Phi$ is unimodular then $(M, \partial M)$ is almost closed.

## The $A_2$ graph manifold

▶ The Dynkin diagram of the simple Lie algebra $A_2$ is the tree

$$\overset{\mathbf{2}}{\bullet} \rule{3cm}{0.4pt} \overset{\mathbf{2}}{\bullet}$$
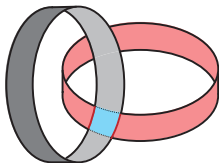
which is here weighted by $\chi(S^2) = 2$ at each vertex.

▶ The corresponding symmetric matrix of type II

$$A_2 \;=\; \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \in \mathsf{Sym}_2(\mathbb{Z})$$

is the intersection matrix $\Phi(M)$ of the graph 1-connected 4-manifold with boundary $(M, \partial M)$ obtained by plumbing two copies of $\tau_{S^2}$, with $\partial M = S^3/\mathbb{Z}_3 = L(3,2)$ a lens space.
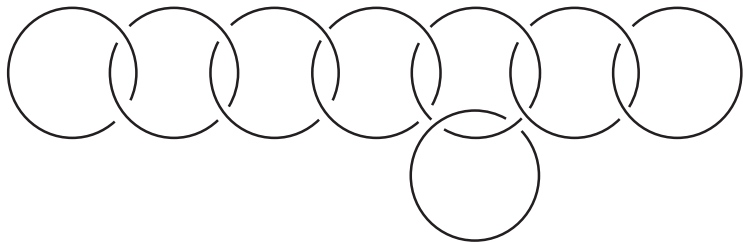
# The $E_8$ graph manifold

► Geometric plumbing using $\Phi = E_8 \in \mathrm{Sym}_8(\mathbb{Z})$ and the Dynkin diagram of $E_8$ gives for each $k \geqslant 1$ a $(2k-1)$-connected graph $4k$-manifold $(M, \partial M)$ with

$$(H_{2k}(M), \Phi(M)) \ = \ (\mathbb{Z}^8, E_8) \ .$$

► The boundary $\partial M = \Sigma^{4k-1}$ is one of the interesting homology $(4k-1)$-spheres discussed already!

**A doughnut of genus 2**

**The multiplicativity mod 8 signature of fibre bundles**

▶ Convention: $\sigma(M) = 0 \in \mathbb{Z}$ for a $(4j+2)$-manifold $M$.

▶ What is the relationship between the signatures $\sigma(E), \sigma(B), \sigma(F) \in \mathbb{Z}$ of the manifolds in a fibre bundle

$$F^{2m} \to E^{4k} \to B^{2n} ?$$

▶ **Theorem** (Chern, Hirzebruch, Serre 1956)
If $\pi_1(B)$ acts trivially on $H_*(F; \mathbb{R})$ then

$$\sigma(E) = \sigma(B)\sigma(F) \in \mathbb{Z} .$$

▶ Kodaira, Atiyah and Hirzebruch (1970) constructed examples with $\sigma(E) \neq \sigma(B)\sigma(F) \in \mathbb{Z}$.

▶ **Theorem** (Meyer 1972 for $k = 1$ using the **first Chern class**, Hambleton, Korzeniewski, Ranicki 2004 for all $k \geqslant 1$)

$$\sigma(E) \equiv \sigma(B)\sigma(F) \bmod 4 .$$

▶ What about mod 8? What is $(\sigma(E) - \sigma(B)\sigma(F))/4 \bmod 2$ ?

## Symmetric forms over $\mathbb{Z}_2$

▶ A **symmetric form over** $\mathbb{Z}_2$ $(V, \lambda)$ is a finite-dimensional vector space $V$ over $\mathbb{Z}_2$ together with bilinear pairing

$$\lambda \; : \; V \times V \to \mathbb{Z}_2 \; ; \; (x, y) \mapsto \lambda(x, y) \; .$$

▶ The form is **nonsingular** if the adjoint $\mathbb{Z}_2$-linear map

$$\lambda \; : \; V \to V^* \; = \; \mathrm{Hom}_{\mathbb{Z}_2}(V, \mathbb{Z}_2)$$

is an isomorphism.

▶ A nonsingular $(V, \lambda)$ has a unique **characteristic element** $v \in V$ such that

$$\lambda(x, x) \; = \; \lambda(x, v) \in \mathbb{Z}_2 \; (x \in V) \; .$$

▶ $(V, \lambda)$ is **isotropic** if $v = 0$, and **anisotropic** if $v \neq 0$.

## $\mathbb{Z}_4$-quadratic enhancements

- Let $(V, \lambda)$ be a nonsingular symmetric form over $\mathbb{Z}_2$.
- A $\mathbb{Z}_4$-**quadratic enhancement** of $(V, \lambda)$ is a function $q : V \to \mathbb{Z}_4$ such that for all $x, y \in V$

$$q(x + y) - q(x) - q(y) = 2\lambda(x, y) \in \mathbb{Z}_4 \ ,$$
$$[q(x)] = \lambda(x, x) \in \mathbb{Z}_2 \ .$$

- Every $(V, \lambda)$ admits $\mathbb{Z}_4$-quadratic enhancements $q$.
- **Example** $(V, \lambda) = (\mathbb{Z}_2, 1)$ has two $\mathbb{Z}_4$-quadratic enhancements

$$q_+(1) = 1 \in \mathbb{Z}_4 \ \text{ and } \ q_-(1) = -1 \in \mathbb{Z}_4 \ .$$

## The Brown-Kervaire invariant

▶ The **Brown-Kervaire invariant** (1972) of a nonsingular symmetric form $(V, \lambda)$ over $\mathbb{Z}_2$ with a $\mathbb{Z}_4$-quadratic enhancement $q$ is the Gauss sum

$$BK(V, \lambda, q) = \frac{1}{\sqrt{|V|}} \sum_{x \in V} e^{\pi i q(x)/2}$$

$$\in \mathbb{Z}_8 = \{\text{eighth roots of unity}\} \subset \mathbb{C} .$$

▶ The Brown-Kervaire invariant has mod 4 reduction

$$[BK(V, \lambda, q)] = q(v) \in \mathbb{Z}_4$$

where $v \in V$ is the characteristic element for $(V, \lambda)$.

▶ The exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \overset{4}{\longrightarrow} \mathbb{Z}_8 \longrightarrow \mathbb{Z}_4 \longrightarrow 0$$

identifies a Brown-Kervaire invariant which has mod 4 reduction $0 \in \mathbb{Z}_4$ with a $\mathbb{Z}_2$-valued Arf invariant.

**The Brown-Kervaire invariant of a symmetric matrix over $\mathbb{Z}$**

- A unimodular symmetric matrix $\Phi \in \mathrm{Sym}_n(\mathbb{Z})$ determines

$$(V, \lambda, q) = ((\mathbb{Z}_2)^n, [\Phi], [x] \mapsto [\Phi(x, x)]) .$$

- Any lift of the characteristic element $v \in (\mathbb{Z}_2)^n$ for $[\Phi] \in \mathrm{Sym}_n(\mathbb{Z}_2)$ is a characteristic element $u \in \mathbb{Z}^n$ for $\Phi$.
- The Brown-Kervaire invariant is the mod 8 reduction of the signature

$$\mathrm{BK}(V, \lambda, q) = [\sigma(\Phi)] = [\Phi(u, u)] \in \mathbb{Z}_8 .$$

- **Example** The unimodular symmetric matrix $\Phi = 1 \in \mathrm{Sym}_1(\mathbb{Z})$ determines

$$(V, \lambda, q) = (\mathbb{Z}_2, 1, 1) , \ u = 1 \in \mathbb{Z} ,$$
$$\mathrm{BK}(V, \lambda, q) = 1 \in \mathbb{Z}_8 .$$

**The Brown-Kervaire invariant of a symmetric matrix over $\mathbb{Z}_4$**

- A unimodular symmetric matrix $\Phi \in \mathrm{Sym}_n(\mathbb{Z}_4)$ with mod 2 reduction $[\Phi] \in \mathrm{Sym}_n(\mathbb{Z}_2)$ determines

$$(V, \lambda, q) = ((\mathbb{Z}_2)^n, [\Phi], [x] \mapsto \Phi(x, x)) .$$

- Any lift of the characteristic element $v \in V$ for $[\Phi] \in \mathrm{Sym}_n(\mathbb{Z}_2)$ is a characteristic element $u \in (\mathbb{Z}_4)^n$ for $\Phi$.

- The mod 4 reduction of the Brown-Kervaire invariant is

$$[BK(V, \lambda, q)] = q(v) = \Phi(u, u) \in \mathbb{Z}_4$$

for any characteristic element $u \in (\mathbb{Z}_4)^n$ for $\Phi$.

- **Example** The unimodular symmetric matrix $\Phi = 1 \in \mathrm{Sym}_1(\mathbb{Z}_4)$ has

$$(V, \lambda, q) = (\mathbb{Z}_2, 1, 1) , \ u = 1 ,$$
$$BK(V, \lambda, q) = 1 \in \mathbb{Z}_8 .$$

## The Brown-Kervaire invariant of $A_2$

▶ The unimodular symmetric matrix over $\mathbb{Z}_4$

$$A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \in \mathsf{Sym}_2(\mathbb{Z}_4)$$

has characteristic element $u = 0 \in (\mathbb{Z}_4)^2$ .

▶ $A_2$ determines

$$
\begin{aligned}
&(V, \lambda, q) \\
&= (\mathbb{Z}_2 \oplus \mathbb{Z}_2, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, (x, y) \mapsto 2(x^2 + xy + y^2)) , \\
v &= 0 \in V , \\
\mathsf{BK}&(V, \lambda, q) = 4 \in \mathsf{im}(4 : \mathbb{Z}_2 \to \mathbb{Z}_8) = \mathsf{ker}(\mathbb{Z}_8 \to \mathbb{Z}_4) .
\end{aligned}
$$

### Brown-Kervaire = signature mod 8

▶ **Theorem** (Morita 1974) A closed oriented $4k$-manifold $M$ determines a nonsingular symmetric form $(H^{2k}(M; \mathbb{Z}_2), \lambda_M)$ over $\mathbb{Z}_2$, with

$$\lambda_M(x, y) = \langle x \cup y, [M] \rangle \in \mathbb{Z}_2$$

and characteristic element $v = v_{2k}(\nu_M) \in H^{2k}(M; \mathbb{Z}_2)$. The Pontrjagin square is a $\mathbb{Z}_4$-quadratic refinement

$$q_M = \mathcal{P}_{2k} : H^{2k}(M; \mathbb{Z}_2) \to H^{4k}(M; \mathbb{Z}_4) = \mathbb{Z}_4$$

with Brown-Kervaire invariant = the mod 8 reduction of the signature

$$\mathrm{BK}(H^{2k}(M; \mathbb{Z}_2), \lambda_M, q_M) = [\sigma(M)] \in \mathbb{Z}_8$$

and mod 4 reduction

$$q_M(v) = [[\sigma(M)]] \in \mathbb{Z}_4 .$$

### The Arf invariant I.

- Let $(V, \lambda)$ be a nonsingular symmetric form over $\mathbb{Z}_2$.
- A $\mathbb{Z}_2$-**quadratic enhancement** of $(V, \lambda)$ is a function $h : V \to \mathbb{Z}_2$ such that

$$h(x + y) - h(x) - h(y) = \lambda(x, y) \in \mathbb{Z}_2 \ (x, y \in V) \ .$$

- $(V, \lambda)$ admits an $h$ if and only if $\lambda$ is isotropic, in which case there exists a basis $(b_1, b_2, \ldots, b_n)$ for $V$ with $n$ even, such that

$$\lambda(b_i, b_j) = \begin{cases} 1 & \text{if } (i, j) = (1, 2) \text{ or } (2, 1) \text{ or } (3, 4) \text{ or } (4, 3) \ \ldots \\ 0 & \text{otherwise.} \end{cases}$$

- The **Arf invariant** of $(V, \lambda, h)$ is defined using any such basis

$$\text{Arf}(V, \lambda, h) = \sum_{i=1}^{n/2} h(b_{2i-1}) h(b_{2i}) \in \mathbb{Z}_2 \ .$$

### The Arf invariant II.

- Let $(V, \lambda)$ be a nonsingular symmetric form over $\mathbb{Z}_2$.
- A $\mathbb{Z}_2$-quadratic enhancement $h : V \to \mathbb{Z}_2$ determines a $\mathbb{Z}_4$-quadratic enhancement

$$q = 2h : V \to \mathbb{Z}_4 ; x \mapsto q(x) = 2h(x)$$

  such that

$$BK(V, \lambda, q) = 4 \operatorname{Arf}(V, \lambda, h) \in 4\mathbb{Z}_2 \subset \mathbb{Z}_8 .$$

- A $\mathbb{Z}_4$-quadratic enhancement $q : V \to \mathbb{Z}_4$ is such that $q(V) \subseteq 2\mathbb{Z}_2 \subset \mathbb{Z}_4$ if and only if $(V, \lambda)$ is isotropic, and

$$h = q/2 : V \to \mathbb{Z}_2 ; x \mapsto h(x) = q(x)/2$$

  is a $\mathbb{Z}_2$-quadratic enhancement as above.

- **Example** For the symmetric form $A_2 \in \operatorname{Sym}_2(\mathbb{Z}_4)$

$$(V, \lambda, q) = (\mathbb{Z}_2 \oplus \mathbb{Z}_2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, q(x, y) = 2(x^2 + xy + y^2))$$

$$BK(V, \lambda, q) = 4 \in \mathbb{Z}_8 , \operatorname{Arf}(V, \lambda, h) = 1 \in \mathbb{Z}_2 .$$

**Carmen Rovi's Edinburgh thesis I.**

▶ **Theorem** (CR 2015)
(i) Let $(V, \lambda)$ be a nonsingular symmetric form over $\mathbb{Z}_2$ with a
$\mathbb{Z}_4$-quadratic enhancement $q : V \to \mathbb{Z}_4$, and characteristic
element $v \in V$.
The Brown-Kervaire invariant $BK(V, \lambda, q) \in \mathbb{Z}_8$ has mod 4
reduction $[BK(V, \lambda, q)] = 0 \in \mathbb{Z}_4$ if and only if $q(v) = 0 \in \mathbb{Z}_4$.
In this case $\lambda(v, v) = 0 \in \mathbb{Z}_2$ and the maximal isotropic
nonsingular subquotient of $(V, \lambda, q)$

$$(V', \lambda', q') = (\{x \in V \mid \lambda(x, v) = 0 \in \mathbb{Z}_2\}/\{v\}, [\lambda], [q])$$

has $\mathbb{Z}_2$-quadratic enhancement $h' = q'/2 : V' \to \mathbb{Z}_2$ such that

$$\begin{aligned}
BK(V, \lambda, q) &= BK(V', \lambda', q') = 4\,\mathrm{Arf}(V', \lambda', h') \\
&\in \mathrm{im}(4 : \mathbb{Z}_2 \to \mathbb{Z}_8) = \ker(\mathbb{Z}_8 \to \mathbb{Z}_4) .
\end{aligned}$$

**Carmen Rovi's Edinburgh thesis II.**

▶ (ii) For any fibre bundle $F^{2m} \to E^{4k} \to B^{2n}$

$$(\sigma(E) - \sigma(B)\sigma(F))/4 \ = \ \mathrm{Arf}(V', \lambda', h') \in \mathbb{Z}_2$$

with

$$(V, \lambda, q)$$
$$= \ (H^{2k}(E; \mathbb{Z}_2), \lambda_E, q_E) \oplus (H^{2k}(B \times F; \mathbb{Z}_2), -\lambda_{B \times F}, -q_{B \times F}) \ .$$

▶ (iii) If the action of $\pi_1(B)$ on $(H_m(F; \mathbb{Z})/\text{torsion}) \otimes \mathbb{Z}_4$ is trivial then the Arf invariant in (ii) is 0 and

$$\sigma(E) \ \equiv \ \sigma(B)\sigma(F) \ \mathrm{mod} \ 8 \ .$$