THE POINCARÉ DUALITY THEOREM AND ITS CONVERSE II.

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The total surgery obstruction
I. Existence of manifold structures

- The $\mathbb{S}$-**groups** $\mathbb{S}_*(X)$ are $\mathbb{Z}$-graded abelian groups defined for any space X. A map $f : X \to Y$ induces $f_* : \mathbb{S}_*(X) \to \mathbb{S}_*(Y)$. If f is a homotopy equivalence, then f_* is an isomorphism
- The **total surgery obstruction** $s(X) \in \mathbb{S}_n(X)$ of an n-dimensional Poincaré duality space X with the following properties.
- If $f : X \to Y$ is a homotopy equivalence of n-dimensional Poincaré duality spaces then $f_* s(X) = s(Y) \in \mathbb{S}_n(Y)$.
- If X is an n-dimensional homology manifold then $s(X) = 0 \in \mathbb{S}_n(X)$.
- **Main Theorem** If $n \geqslant 5$ and $s(X) = 0 \in \mathbb{S}_n(X)$ then X is homotopy equivalent to an n-dimensional topological manifold.
- Global input $\Longrightarrow$ local output.
- Proof by Browder-Novikov-Sullivan-Wall theory.

The total surgery obstruction
II. Uniqueness of manifold structures

- The **total surgery obstruction** of a homotopy equivalence $h : N \to M$ of n-dimensional topological manifolds is an element $s(h) \in \mathbb{S}_{n+1}(M)$ with the following properties.
- If the point inverses $h^{-1}(x) \subset N$ $(x \in M)$ are acyclic

$$h| \ : \ H_*(h^{-1}(x)) \ \cong \ H_*(\{x\})$$

then $s(h) = 0 \in \mathbb{S}_{n+1}(M)$.
- If $n \geqslant 5$ and $s(h) = 0 \in \mathbb{S}_{n+1}(M)$ then h is homotopic to a homeomorphism. (Need also Whitehead torsion $\tau(h) = 0$). Every $s \in \mathbb{S}_{n+1}(M)$ is $s = s(h)$ for some h.
- Global input $\Longrightarrow$ local output.
- (A.R.) **The total surgery obstruction** (Proc. 1978 Aarhus Topology Conference, Springer Lecture Notes)

The Wall surgery obstruction

- In 1969 C.T.C. Wall constructed the **surgery obstruction groups** $L_n(A)$ of a ring with involution A, using quadratic structures on f.g. free A-modules.
- 4-periodic: $L_n(A) = L_{n+4}(A)$
- $L_0(A) =$ Witt group of quadratic forms over A.
- $L_1(A) =$ stable automorphism group of quadratic forms over A.
- $L_2(A) =$ Witt group of symplectic-quadratic forms over A.
- $L_3(A) =$ stable automorphism group of symplectic-quadratic forms over A.
- A normal map $(f, b) : M \to X$ from an n-dimensional manifold M to an n-dimensional Poincaré duality space X has a surgery obstruction $\sigma(f, b) \in L_n(\mathbb{Z}[\pi_1(X)])$ such that $\sigma(f, b) = 0$ if (and for $n \geqslant 5$ only if) (f, b) is normal bordant to a homotopy equivalence.

Local $\Longrightarrow$ global in surgery theory

- The algebraic L-groups $L_*(\mathbb{Z}[\pi_1(X)])$ depend only on the fundamental group $\pi_1(X)$ of a space X, so are global.
- The Witt groups of sheaves of quadratic forms over X define the generalized homology groups $H_*(X;\mathbf{L}(\mathbb{Z}))$, which are local. Here $\mathbf{L}(\mathbb{Z})$ is a spectrum with

$$\pi_*(\mathbf{L}(\mathbf{Z})) \;=\; L_*(\mathbb{Z}) \;=\; \mathbb{Z}, 0, \mathbb{Z}_2, 0, \dots \text{ (4-periodic)}$$

  the simply-connected surgery obstruction groups.
- For any space X there is an exact sequence

$$\begin{aligned} \dots \to H_n(X;\mathbf{L}(\mathbb{Z})) &\xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \\ &\to \mathbb{S}_n(X) \to H_{n-1}(X;\mathbf{L}(\mathbb{Z})) \to \dots \end{aligned}$$

- A is the local $\Longrightarrow$ global **assembly** map in L-theory. Originally defined geometrically by Quinn.
- The $\mathbb{S}$-groups $\mathbb{S}_*(X)$ measure the failure of A to be an isomorphism.

The failure of local Poincaré duality

- Let X be an n-dimensional Poincaré duality space. The failure of local Poincaré duality at $x \in X$ is measured by the groups $K_*(X,x)$ in the exact sequences

$$\begin{aligned}\dots \to H^{n-r-1}(\{x\}) \xrightarrow{[X]_x \cap -} & H_{r+1}(X, X\backslash\{x\}) \\ & \to K_r(X,x) \to H^{n-r}(\{x\}) \to \dots\end{aligned}$$

- X is a homology manifold if and only if $K_*(X,x) = 0$ ($x \in X$).
- Roughly speaking, the total surgery obstruction $s(X) \in \mathbb{S}_n(X)$ is the cobordism class of a sheaf over X of chain complexes with quadratic Poincaré duality over $\mathbb{Z}$ with $K_*(X,x)$ the stalk at $x \in X$.
- Chain complex with quadratic Poincaré duality
  = chain complex with quadratic structure
  = generalization of quadratic form.

Bringing in the sheaves

(From *The Night of the Hunter*)

- The book
  A.R. **Algebraic L-theory and manifolds** (CUP, 1992) developed the theory for simplicial complexes K, with an assembly map

  $$A \; : \; \{(\mathbb{Z}, K)\text{-modules}\} \to \{\mathbb{Z}[\pi_1(K)]\text{-modules}\}$$

  to provide the passage from local to global in algebra. This is sufficient for applications, since every Poincaré duality space is homotopy equivalent to one which is triangulated.
- Unfortunately, have not yet been able to develop the necessary sheaf theory. However, the paper
  A.R.+Michael Weiss **On the construction and topological invariance of the Pontryagin classes** (Geometriae Dedicata 2010) points in the direct direction!

Rings with involution

- An **involution** on a ring A is a function

$$A \to A \; ; \; a \mapsto \overline{a}$$

  such that

$$\overline{a+b} \;=\; \overline{a}+\overline{b} \;,\; \overline{ab} \;=\; \overline{b}\,\overline{a} \;,\; \overline{\overline{a}} \;=\; a \; (a, b \in A) \; .$$

- **Example 1** A commutative ring A, with $\overline{a} = a$.
- **Example 2** A group ring $A = \mathbb{Z}[\pi]$ with $\overline{g} = g^{-1}$ $(g \in \pi)$.
- Regard a left A-module P as a right A-module with

$$P \times A \to P \; ; \; (x, a) \mapsto \overline{a}x \; .$$

- The tensor product of left A-modules P, Q is the abelian group defined by

$$P \otimes_A Q \;=\; P \otimes_{\mathbb{Z}} Q / \{ax \otimes y - x \otimes \overline{a}y \,|\, a \in A, x \in P, y \in Q\}$$

  with transposition isomorphism

$$P \otimes_A Q \to Q \otimes_A P \; ; \; x \otimes y \mapsto y \otimes x \; .$$

Duality over a ring with involution

- The **dual** of a left A-module P is the left A-module

$$P^* \ = \ \mathrm{Hom}_A(P,A) \ , \ A \times P^* \to P^* \ ; \ (a,f) \mapsto (x \mapsto f(x)\overline{a}) \ .$$

- The natural A-module morphism

$$P \to P^{**} \ ; \ x \mapsto (f \mapsto \overline{f(x)})$$

  is an isomorphism for f.g. free P.
- For A-modules P, Q the abelian group morphisms

$$P^* \otimes_A Q \to \mathrm{Hom}_A(P,Q) \ ; \ f \otimes y \mapsto (x \mapsto \overline{f(x)}y) \ ,$$
$$* : \mathrm{Hom}_A(P,Q) \to \mathrm{Hom}_A(Q^*,P^*); f \mapsto (f^* : g \mapsto (x \mapsto g(f(x))))$$

  are isomorphisms for f.g. free P, Q.

Quadratic forms on chain complexes I.

- A.R. **The algebraic theory of surgery I., II.** (1980, Proc. LMS)
- The *n*-**dual** of a f.g. free A-module chain complex

$$C \ : \ \dots \to C_r \xrightarrow{\ d\ } C_{r-1} \to \dots \to C_1 \xrightarrow{\ d\ } C_0 \to \dots$$

  is the f.g. free A-module chain complex

$$C^{n-*} \ : \ \dots \to C^0 \xrightarrow{\ d^*\ } C^1 \to \dots \to C^{r-1} \xrightarrow{\ d^*\ } C^r \to \dots$$

  with $C^r = C_r^*$.
- An 'algebraic Poincaré complex' is a f.g. free A-module chain complex C with a chain equivalence $C^{n-*} \simeq C$ satisfying extra conditions. There are two flavours: **symmetric** and **quadratic**. Will ignore the difference today, using **algebraic** for both!

Quadratic forms on chain complexes II.

- For any f.g. free A-module chain complex C there is defined an isomorphism of A-module chain complexes

$$C \otimes_A C \to \mathrm{Hom}_A(C^{-*}, C) \; ; \; x \otimes y \mapsto (f \mapsto \overline{f(x)}.y) \; .$$

The homology group

$$H_n(C \otimes_A C) \;=\; H_0(\mathrm{Hom}_A(C^{n-*}, C))$$

is the group of chain homotopy classes of chain maps $\phi : C^{n-*} \to C$.

- The action of $T \in \mathbb{Z}_2$ by the **transposition involution**

$$T \;:\; C \otimes_A C \to C \otimes_A C \; ; \; x \otimes y \mapsto (-)^{pq} y \otimes x \; (x \in C_p, y \in C_q)$$

corresponds to the **duality involution**

$$T \;:\; \mathrm{Hom}_A(C^{-*}, C) \to \mathrm{Hom}_A(C^{-*}, C) \; ; \; f \mapsto (-)^{pq} f^* \; ,$$

$$(f : C^p \to C_q) \mapsto ((-)^{pq} f^* : C^q \to C_p) \; , \; y(f^*(x)) = x(f(y)) \; .$$

Algebraic Poincaré cobordism

- An n-**dimensional algebraic Poincaré complex over** A (C,ϕ) is an n-dimensional f.g. free A-module chain complex C together with a chain equivalence $\phi : C^{n-*} \to C$ such that there exists a chain homotopy $T\phi \simeq \phi : C^{n-*} \to C$.
- If $1/2 \notin A$ need additional structure: either symmetric or quadratic.
- A cobordism $(L; M, M')$ of n-dimensional manifolds has Poincaré-Lefschetz duality

$$[L] \cap - : \; H^{n+1-*}(L, M) \;\cong\; H_*(L, M') \; .$$

- **Proposition** (Mishchenko, R., 1970's) The Wall group $L_n(A)$ is the group of cobordism classes of n-dimensional algebraic Poincaré complexes (C,ϕ) over A, with $(C,\phi) \sim (C',\phi')$ if $C \oplus C' \subset D$ for an $(n+1)$-dimensional f.g. free A-module chain complex D such that $H^{n+1-*}(D, C) \cong H_*(D, C')$.

The polyhedron of a simplicial complex

- A **simplicial complex** K is a collection of finite subsets $\sigma \subseteq K^{(0)}$ of an ordered **vertex set** $K^{(0)}$ such that:
  (a) $v \in K$ for each $v \in K^{(0)}$,
  (b) if $\sigma \in K$ and $\tau \subseteq \sigma$ then $\tau \in K$.
- The **dimension** of $\sigma \in K$ is

$$|\sigma| \ = \ (\text{no. of vertices in } \sigma) - 1$$

  Let $K^{(n)}$ denote the set of n-simplexes in K.
- The **polyhedron** of K is the usual identification space

$$|K| \ = \ (\coprod_{n=0}^{\infty} \Delta^n \times K^{(n)})/\sim$$

  with Δ^n the convex hull of $(0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$.

The simplicial chain complex

- The **simplicial chain complex** $C(K)$ has

$$d \;:\; C(K)_n = \mathbb{Z}[K^{(n)}] \to C(K)_{n-1} = \mathbb{Z}[K^{(n-1)}] \;;$$

$$(v_0 v_1 \dots v_n) \mapsto \sum_{i=0}^{n} (-)^i (v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$$

$$(v_0 < v_1 < \dots < v_n)$$

- The homology and cohomology groups of the polyhedron are the same as those of the simplicial complex

$$H_*(|K|) \;=\; H_*(K) \;=\; H_*(C(K)) \;,$$
$$H^*(|K|) \;=\; H^*(K) \;=\; H^*(C(K)) \;.$$

- For any simplicial complexes K, L $H_n(|K| \times |L|)$ is the group of chain homotopy classes of chain maps $C(K)^{n-*} \to C(L)$.

Polyhedral Poincaré complexes

- A **triangulated n-dimensional Poincaré space** is a finite simplicial complex K with universal cover $\widetilde{K}$ and a homology class $[K] \in H_n(K)$ satisfying the equivalent conditions:
- (a) the cap products

$$[K] \cap - \ : \ H^{n-*}(\widetilde{K}) \ = \ H_*(C(\widetilde{K})^{n-*}) \to H_*(\widetilde{K})$$

  are $\mathbb{Z}[\pi_1(K)]$-module isomorphisms.
- (b) The image $\Delta[K] \in H_n(X)$ under the diagonal map

$$\Delta \ : \ |K| \to X = |\widetilde{K}| \times_{\pi_1(K)} |\widetilde{K}| \ ; \ x \mapsto (\widetilde{x}, \widetilde{x})$$

  is a chain homotopy class of $\mathbb{Z}[\pi_1(K)]$–module chain equivalences $\phi = \Delta[K] : C(\widetilde{K})^{n-*} \to C(\widetilde{K})$.
- (c) The cap product $[X] \cap - : H^n(X) \to H_n(X)$ is an isomorphism, with $\Delta[K]^* \in H^n(X)$ a $\mathbb{Z}[\pi_1(K)]$-module chain homotopy inverse $\phi^{-1} : C(\widetilde{K}) \to C(\widetilde{K})^{n-*}$.
- $(C(\widetilde{K}), \phi)$ is an n-dimensional algebraic Poincaré complex over $\mathbb{Z}[\pi_1(K)]$.

Dual cells

- The **barycentric subdivision** of K is the simplicial complex K' with $K'^{(0)} = K$ and
$$K'^{(n)} \;=\; \{(\sigma_0, \sigma_1, \ldots, \sigma_n) \,|\, \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_n\}\ .$$
Homeomorphic polyhedron $|K'| \cong |K|$.
- The **dual cells** of K are the contractible subcomplexes
$$D(\sigma) \;=\; \{(\sigma_0, \sigma_1, \ldots, \sigma_n) \in K' \,|\, \sigma_0 \subseteq \sigma\} \subseteq K'\ .$$
- The **boundary** of the dual cell $D(\sigma)$ is
$$\partial D(\sigma) \;=\; \{(\sigma_0, \sigma_1, \ldots, \sigma_n) \in D(\sigma) \,|\, \sigma_0 \neq \sigma\}\ .$$
- **Proposition** The local homology groups of $|K|$ at $x \in |K|$ are the homology groups of the dual cells relative to boundaries
$$H_*(|K|, |K|\backslash\{x\}) \;=\; H_{*-|\sigma|}(D(\sigma), \partial D(\sigma))\,(x \in \text{interior}(\sigma), \sigma \in K)\ .$$
For each $\sigma \in K$ and $x \in \text{interior}(\sigma)$ there are natural maps
$$\partial_\sigma \;:\; H_*(|K|) = H_*(K) \to H_*(|K|, |K|\backslash\{x\}) = H_{*-|\sigma|}(D(\sigma), \partial D(\sigma))$$

The $(\mathbb{Z}, K)$-category I. Modules

- A.R.+M.Weiss **Chain complexes and assembly** Math. Z. (1999)
- A $(\mathbb{Z}, K)$-**module** is a f.g. free $\mathbb{Z}$-module M with splitting
$$M \;=\; \sum_{\sigma \in K} M(\sigma)\ .$$
- A **morphism** of $(\mathbb{Z}, K)$-modules $f : M \to N$ is a $\mathbb{Z}$-module morphism such that
$$f(M(\sigma)) \subseteq \sum_{\tau \geqslant \sigma} N(\tau)\ (\sigma \in K)\ .$$
- **Proposition** A $(\mathbb{Z}, K)$-module morphism $f : M \to N$ is an isomorphism if and only if each
$$f(\sigma, \sigma)\ :\ M(\sigma) \to N(\sigma)\ (\sigma \in K)$$
is a $\mathbb{Z}$-module isomorphism.

Assembly

- Let $p : \widetilde{K} \to K$ be the universal cover of a connected simplicial complex K. The **assembly** functor

$$A \ : \ \{(\mathbb{Z}, K)\text{-modules}\} \to \{\text{f.g. free } \mathbb{Z}[\pi_1(K)]\text{-modules}\}$$

is defined by

$$A(M) \ = \ \sum_{\widetilde{\sigma} \in \widetilde{K}} M(p(\widetilde{\sigma})) \ .$$

- Local $\Longrightarrow$ global.
- **Example** For finite K the simplicial chain complex $C(K')$ is a $(\mathbb{Z}, K)$-module chain complex with

$$C(K')(\sigma) \ = \ C(D(\sigma), \partial D(\sigma)) \ (\sigma \in K)$$

The assembly is the simplicial $\mathbb{Z}[\pi_1(K)]$-module chain complex of $\widetilde{K}'$

$$A(C(K')) \ = \ C(\widetilde{K}') \ .$$

The algebraic Vietoris theorem

- Let $f : L \to K'$ be a simplicial map with K, L finite.
- Regard $C(L)$ as a $(\mathbb{Z}, K)$-module chain complex by

$$C(L)(\sigma) \;=\; C(f^{-1}D(\sigma), f^{-1}\partial D(\sigma))\ (\sigma \in K)\ .$$

- **Proposition** f has acyclic point inverses if and only if

$$f \;:\; C(L) \to C(K')$$

  is a $(\mathbb{Z}, K)$-module chain equivalence.
- **Corollary** If f has acyclic point inverses then

$$\widetilde{f} \;:\; C(\widetilde{L}) \to C(\widetilde{K}')$$

  is a $\mathbb{Z}[\pi_1(K)]$-module chain equivalence

The $(\mathbb{Z}, K)$-category II. Products

- The **product** of $(\mathbb{Z}, K)$-modules A, B is the $(\mathbb{Z}, K)$-module

$$A \otimes_{(\mathbb{Z},K)} B = \sum_{\lambda,\mu \in K, \lambda \cap \mu \neq \emptyset} A(\lambda) \otimes_{\mathbb{Z}} B(\mu) \subseteq A \otimes_{\mathbb{Z}} B \text{ with}$$

$$(A \otimes_{(\mathbb{Z},K)} B)(\sigma) = \sum_{\lambda,\mu \in K, \lambda \cap \mu = \sigma} A(\lambda) \otimes_{\mathbb{Z}} B(\mu) \ .$$

- **Example** For simplicial maps $f : L \to K'$, $g : M \to K'$ the pullback polyhedron

$$L \times_K M = \{(x, y) \in |L| \times |M| \mid f(x) = g(y) \in |K|\}$$

has homology

$$H_*(L \times_K M) = H_*(C(L) \otimes_{(\mathbb{Z},K)} C(M))$$

with

$$C(L)(\sigma) = C(f^{-1}D(\sigma), f^{-1}\partial D(\sigma)) \ ,$$
$$C(M)(\sigma) = C(g^{-1}D(\sigma), g^{-1}\partial D(\sigma)) \ .$$

The $(\mathbb{Z}, K)$-category III. Duality

- The **dual** of a $(\mathbb{Z}, K)$-module M is the $(\mathbb{Z}, K)$-module chain complex TM with

$$TM(\sigma)_r \;=\; \begin{cases} \sum\limits_{\tau \geqslant \sigma} M(\tau)^* & \text{if } r = -|\sigma| \\ 0 & \text{otherwise.} \end{cases}$$

- The dual of a $(\mathbb{Z}, K)$-module chain complex C is a $(\mathbb{Z}, K)$-module chain complex TC. Analogue of Verdier duality for sheaves.
- **Example** The dual of $C(K')$ is $(\mathbb{Z}, K)$-equivalent to the cochain complex of K

$$TC(K') \simeq C(K)^{-*} \;,\; C(K)^r(\sigma) \;=\; \begin{cases} \mathbb{Z} & \text{if } r = -|\sigma| \\ 0 & \text{otherwise.} \end{cases}$$

- For any $(\mathbb{Z}, K)$-module chain complexes C, D

$$H_*(C \otimes_{(\mathbb{Z},K)} D) \;=\; H_*(\mathrm{Hom}_{(\mathbb{Z},K)}(TC, D)) \;.$$

The assembly map

- **Proposition** (i) The generalized homology group $H_n(K; \mathbf{L}(\mathbb{Z}))$ is the cobordism group of n-dimensional algebraic Poincaré complexes $(C, \phi : TC_{*-n} \to C)$ in the $(\mathbb{Z}, K)$-module category.
- (ii) The assembly functor

$$A \ : \ \{(\mathbb{Z}, K)\text{-modules}\} \to \{\mathbb{Z}[\pi_1(K)]\text{-modules}\}$$

induces assembly maps in algebraic L-theory

$$A \ : \ H_n(K; \mathbf{L}(\mathbb{Z})) \to L_n(\mathbb{Z}[\pi_1(K)])$$

- (iii) $\mathbb{S}_n(K)$ is the cobordism group of $(n-1)$-dimensional algebraic Poincaré complexes (C, ϕ) in the $(\mathbb{Z}, K)$-module category such that the assembly $A(C)$ is a contractible f.g. free $\mathbb{Z}[\pi_1(K)]$-module chain complex, $H_*(A(C)) = 0$.

From local to global Poincaré duality, and back again!

- For any simplicial complex K
$$H_n(K) \;=\; H_n(\mathrm{Hom}_{(\mathbb{Z},K)}(TC(K'),C(K'))) \;.$$
The cap product with any homology class $[K] \in H_n(K)$ is a $(\mathbb{Z},K)$-module chain map
$$\phi \;=\; [K]\cap - : TC(K')_{*-n} \to C(K')$$
with diagonal components
$$\begin{aligned}\phi(\sigma,\sigma) = \partial_\sigma[K]\cap - : TC(K')_{*-n}(\sigma) = C(D(\sigma))^{n-*-|\sigma|} \\ \to C(K')(\sigma) = C(D(\sigma),\partial D(\sigma))\ (\sigma \in K)\ ,\end{aligned}$$
with assembly
$$[K]\cap - : TC(\widetilde{K}')_{*-n} \simeq C(\widetilde{K})^{n-*} \to C(\widetilde{K}') \simeq C(\widetilde{K})\ .$$
- K is a homology manifold if and only if $[K]\cap -$ is a $(\mathbb{Z},K)$-module chain equivalence. This is essentially Poincaré's original proof of duality!

The total surgery obstruction

- The **total surgery obstruction** of a polyhedral n-dimensional Poincaré duality space K is the cobordism class

$$s(K) \;=\; (\mathcal{C}(\phi)_{*+1}, \psi) \in \mathbb{S}_n(K)\ ,$$

with $\mathcal{C}(\phi)$ the $\mathbb{Z}[\pi_1(K)]$-contractible algebraic mapping cone of the $(\mathbb{Z}, K)$-module chain map

$$\phi \;=\; [K] \cap - : TC(K')_{n-*} \to C(K')\ .$$

- The image

$$t(K) \;=\; [s(K)] \in H_{n-1}(K; \mathbf{L}(\mathbb{Z}))$$

is such that $t(K) = 0$ if and only if there exists a normal map $(f, b) : M \to |K|$, M an n-dimensional topological manifold.
- $s(K) = 0$ if and only if there exists a normal map (f, b) with surgery obstruction $\sigma(f, b) = 0 \in L_n(\mathbb{Z}[\pi_1(K)])$.
- For $n \geqslant 5$ $s(K) = 0$ if and only if $|K|$ is homotopy equivalent to an n-dimensional topological manifold, by B-N-S-W theory.

The symmetric signature

- The **symmetric signature** of a triangulated n-dimensional Poincaré space K is the algebraic Poincaré cobordism class

$$\sigma(K) \ = \ (C(\widetilde{K}), \phi) \in L_n(\mathbb{Z}[\pi_1(K)]) \ .$$

- The symmetric signature is a homotopy invariant, generalizing the signature.
- Modulo 2-torsion, the total surgery obstruction is the image

$$s(K) \ = \ [\sigma(K)] \in \mathrm{im}(L_n(\mathbb{Z}[\pi_1(K)]) \to \mathbb{S}_n(K)) \ .$$

- **Theorem** (A.R., 1992) Modulo 2-torsion, if $n \geqslant 5$ $|K|$ is homotopy equivalent to an n-dimensional topological manifold if and only if $s(K) = 0 \in \mathbb{S}_n(K)$, if and only if

$$\sigma(K) \in \mathrm{im}(A : H_n(K; \mathbf{L}(\mathbb{Z})) \to L_n(\mathbb{Z}[\pi_1(K)])) \ .$$

- For $n = 4k$, $\pi_1(K) = \{1\}$ this is just Browder's converse of the Hirzebruch signature theorem.

The homotopy types of topological manifolds

- For $n \geqslant 5$ the homotopy types of n-dimensional topological manifolds M fit into a fibre square

$$\begin{tikzcd}
\text{topological manifolds} \arrow[r] \arrow[d] & \text{cobordism of local APC's} \arrow[d, "A"] \\
\text{PD spaces} \arrow[r] & \text{cobordism of global APC's}
\end{tikzcd}$$

  with PD = Poincaré duality, APC = algebraic Poincaré complexes, A = assembly.
- Local = in the $(\mathbb{Z}, K)$-module category, for a finite simplicial complex K with a surjection $|K| \to M$ with acyclic point inverses, and $\pi_1(|K|) \cong \pi_1(M)$,
- Global = in the $\mathbb{Z}[\pi]$-module category, $\pi = \pi_1(|K|) = \pi_1(M)$.

Three conjectures

- The **Novikov conjecture** (1969) on the homotopy invariance of the higher signatures of manifolds with fundamental group π is equivalent to the injectivity of the local $\Longrightarrow$ global assembly map $1 \otimes A : H_*(B\pi; \mathbf{L}(\mathbb{Z})) \otimes \mathbb{Q} \to L_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q}$.
  History and survey of the Novikov conjecture.
- The **Borel conjecture** (1953) on the existence and rigidity of topological manifold structures on aspherical Poincaré complexes $B\pi$ is essentially equivalent to the assembly map $A : H_*(B\pi; \mathbf{L}(\mathbb{Z})) \to L_*(\mathbb{Z}[\pi])$ being an isomorphism, so that local $\Longleftrightarrow$ global.
  1953 letter from Borel to Serre.
- The **Farrell-Jones conjecture** (1982) that a generalized assembly map from equivariant homology to the L-theory of $\mathbb{Z}[\pi]$ is an isomorphism for all groups π.

Conclusion

- Starting with Novikov himself, many authors in the last 40 years have proved many special cases of the Novikov, Borel and Farrell-Jones conjectures, using a wide variety of algebraic, geometric and analytic methods.
- Some (though not all) have used the algebraic L-theory assembly map defined here.
- There is still much work to be done to understand the relationship between all these methods of proof, and maybe even prove new results!