QUADRATIC FORMS, MANIFOLDS AND BAGELS

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Sylvester's Law of Inertia

- A demonstration of the theorem that every homogeneous quadratic polynomial is reducible by real orthogonal substitutions to the form of a sum of positive and negative squares.
  Philosophical Magazine IV, 138–142 (1852)

- demonstration. My knowledge of the fact of this equivalence is, as I have stated, deduced from that remarkable but simple law to which I have adverted, which affirms the invariability of the number of the positive and negative signs between all linearly equivalent functions of the form $\Sigma \pm c_r x^r$ (subject, of course, to the condition that the equivalence is expressible by means of equations into which only real quantities enter); a law to which my view of the physical meaning of quantity of matter inclines me, upon the ground of analogy, to give the name of the Law of Inertia for Quadratic Forms, as expressing the fact of the existence of an invariable number inseparably attached to such forms.

Symmetric and symplectic forms over $\mathbb{R}$

- Let $\epsilon = +1$ or -1.
- An ϵ-**symmetric form** (K, ϕ) is a finite dimensional real vector space K together with a bilinear pairing

$$\phi \ : \ K \times K \to \mathbb{R} \ ; \ (x, y) \mapsto \phi(x, y)$$

  such that

$$\phi(x, y) \ = \ \epsilon\phi(y, x) \in \mathbb{R} \ .$$

- The pairing ϕ can be identified with the adjoint linear map to the dual vector space

$$\phi \ : \ K \to K^* \ = \ \mathrm{Hom}_{\mathbb{R}}(K, \mathbb{R}) \ ; \ x \mapsto (y \mapsto \phi(x, y))$$

  such that $\phi^* = \epsilon\phi$.
- The form (K, ϕ) is **nonsingular** if $\phi : K \to K^*$ is an isomorphism.
- A 1-symmetric form is called **symmetric**.
- A (-1)-symmetric form is called **symplectic**.

Lagrangians and hyperbolic forms I.

- **Definition** A **lagrangian** of a nonsingular form (K, ϕ) is a subspace $L \subset K$ such that $L = L^\perp$, that is

$$L \;=\; \{x \in K \,|\, \phi(x, y) = 0 \text{ for all } y \in L\} \;.$$

- **Definition** The **hyperbolic ϵ-symmetric form** is defined for any finite-dimensional real vector space L by

$$H_\epsilon(L) \;=\; (L \oplus L^*, \phi = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}) \;,$$

$$\phi \;:\; L \oplus L^* \times L \oplus L^* \to \mathbb{R} \;;\; ((x, f), (y, g)) \mapsto g(x) + \epsilon f(y)$$

  with lagrangian L.

- The **graph** of a $(-\epsilon)$-symmetric form (L, λ) is the lagrangian of $H_\epsilon(L)$

$$\Gamma_{(L,\lambda)} \;=\; \{(x, \lambda(x)) \,|\, x \in L\} \subset L \oplus L^* \;.$$

Lagrangians and hyperbolic forms II.

- **Proposition** The inclusion $L \to K$ of a lagrangian in a nonsingular ϵ-symmetric form (K, ϕ) extends to an isomorphism

$$H_\epsilon(L) \xrightarrow{\cong} (K, \phi) \ .$$

- **Example** For any nonsingular ϵ-symmetric form (K, ϕ) the inclusion of the diagonal lagrangian in $(K, \phi) \oplus (K, -\phi)$

$$\Delta \ : \ K \to K \oplus K \ ; \ x \mapsto (x, x)$$

extends to the isomorphism

$$\begin{pmatrix} 1 & \dfrac{-\phi^{-1}}{2} \\ 1 & \dfrac{\phi^{-1}}{2} \end{pmatrix} \ : \ H_\epsilon(K) \xrightarrow{\cong} (K, \phi) \oplus (K, -\phi) \ .$$

The classification of symmetric forms over $\mathbb{R}$

- **Proposition** Every symmetric form (K, ϕ) is isomorphic to

$$\bigoplus_p (\mathbb{R}, 1) \oplus \bigoplus_q (\mathbb{R}, -1) \oplus \bigoplus_r (\mathbb{R}, 0)$$

  with $p + q + r = \dim_{\mathbb{R}}(K)$. Nonsingular if and only if $r = 0$.
- Two forms are isomorphic if and only if they have the same p, q, r.
- **Definition** The **signature** (or the **index of inertia**) of (K, ϕ) is

$$\sigma(K, \phi) \;=\; p - q \in \mathbb{Z} \;.$$

- **Proposition** The following conditions on a nonsingular form (K, ϕ) are equivalent:
  - $\sigma(K, \phi) = 0$, that is $p = q$,
  - (K, ϕ) admits a lagrangian L,
  - (K, ϕ) is isomorphic to $\bigoplus_p (\mathbb{R}, 1) \oplus \bigoplus_p (\mathbb{R}, -1) \cong H_+(\mathbb{R}^p)$.

The classification of symplectic forms over $\mathbb{R}$

- **Theorem** Every symplectic form (K, ϕ) is isomorphic to

$$H_-(\mathbb{R}^p) \oplus \bigoplus_r (\mathbb{R}, 0)$$

with $2p + r = \dim_{\mathbb{R}}(K)$. Nonsingular if and only if $r = 0$.
- Two forms are isomorphic if and only if they have the same p, r.
- **Proposition** Every nonsingular symplectic form (K, ϕ) admits a lagrangian.
- **Proof** By induction on $\dim_{\mathbb{R}}(K)$.
For every $x \in K$ have $\phi(x, x) = 0$. If $x \neq 0 \in K$ the linear map

$$K \to \mathbb{R} \; ; \; y \mapsto \phi(x, y)$$

is onto, so there exists $y \in K$ with $\phi(x, y) = 1 \in \mathbb{R}$. The subform $(\mathbb{R}x \oplus \mathbb{R}y, \phi|)$ is isomorphic to $H_-(\mathbb{R})$, and

$$(K, \phi) \;\cong\; H_-(\mathbb{R}) \oplus (K', \phi')$$

with $\dim_{\mathbb{R}}(K') = \dim_{\mathbb{R}}(K) - 2$.

Poincaré duality

- H.P. Analysis Situs and its Five Supplements (1892–1904) (English translation by John Stillwell, 2009)
-

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Donc

$$P_p = P_{h-p}.$$

Par conséquent, pour une variété fermée, les nombres de Betti également distants des extrêmes sont égaux.

Ce théorème n'a, je crois, jamais été énoncé; il était cependant connu de plusieurs personnes qui en ont même fait des applications.

Voyons maintenant ce qui se passe quand h est pair pour le nombre moyen $P_{\frac{h}{2}}$. Supposons que h soit multiple de $4+2$, de telle façon que $\frac{h}{2}$ soit impair.

The $(-)^n$-symmetric form of a $2n$-dimensional manifold

- Manifolds will be oriented.
- Homology and cohomology will be with $\mathbb{R}$-coefficients.
- The **intersection form** of a $2n$-dimensional manifold with boundary $(M, \partial M)$ is the $(-)^n$-symmetric form given by the evaluation of the cup product on the fundamental class $[M] \in H_{2n}(M, \partial M)$

$$(H^n(M, \partial M) \, , \, \phi_M : (x, y) \mapsto \langle x \cup y, [M] \rangle \,) \ .$$

- By Poincaré duality and universal coefficient isomorphisms

$$H^n(M, \partial M) \; \cong \; H_n(M) \ , \ H^n(M, \partial M) \; \cong \; H_n(M, \partial M)^*$$

  the adjoint linear map ϕ_M fits into an exact sequence

$$\dots \to H_n(\partial M) \to H_n(M) \xrightarrow{\phi_M} H_n(M, \partial M) \to H_{n-1}(\partial M) \to \dots \ .$$

- The isomorphism class of the form is a homotopy invariant of $(M, \partial M)$.
- If M is closed, $\partial M = \emptyset$, then $(H^n(M, \partial M), \phi_M)$ is nonsingular.
- The intersection form of $S^n \times S^n$ is $H_{(-)^n}(\mathbb{R})$.

The lagrangian of a $(2n+1)$-dimensional manifold with boundary

- **Proposition** If (N^{2n+1}, M^{2n}) is a $(2n+1)$-dimensional manifold with boundary then

$$L = \ker(H_n(M) \to H_n(N)) = \operatorname{im}(H^n(N) \to H^n(M)) \subset H^n(M)$$

is a lagrangian of the $(-)^n$-symmetric intersection form $(H^n(M), \phi_M)$.

- **Proof** Consider the commutative diagram

$$\begin{array}{ccccc} H^n(N) & \longrightarrow & H^n(M) & \longrightarrow & H^{n+1}(N,M) \\ \downarrow \cong & & \phi_M \downarrow \cong & & \downarrow \cong \\ H_{n+1}(N,M) & \longrightarrow & H_n(M) & \longrightarrow & H_n(N) \end{array}$$

with $H^n(N) \cong H_{n+1}(N,M)$, $H^{n+1}(N,M) \cong H_n(N)$ the Poincaré-Lefschetz duality isomorphisms.

The signature of a manifold I.

- Analisis situs combinatorio
  H.Weyl, Rev. Mat. Hispano-Americana 5, 390–432 (1923)
-

Así se obtiene para m *impar una «base canónica» para las cadenas* m*-dimensionales cerradas, tal como Riemann la construyó para las superficies bidimensionales* (m = 1); formada por pares

$$(V', V''), \ldots\ldots, (V^{(p-1)}, V^{(p)}),$$

de tal modo que dos cadenas de pares distintos tienen la característica cero (no se cortan) y las de un mismo par tienen la característica ± 1. *Si* m *es par*, se puede sustituir la forma bilineal simétrica $\mathbf{s}(\xi\,\eta)$ por la cuadrática $\mathbf{s}(\xi\,\xi)$: la característica de una cadena cerrada V_m respecto a sí misma. Aquí no existe ninguna forma normal unitaria; sino que las formas cuadráticas con coeficientes enteros y determinante ± 1 se descomponen en varias clases no equivalentes, de formas no transformables unas en otras mediante sustituciones unimodulares. Por tanto, con m par, la clase a que pertenece la forma característica de grado m (en particular su índice de inercia) constituye una nueva peculiaridad de las superficies $2\,m$-dimensionales respecto al Análisis-Situs.

- Published in Spanish in South America to spare the author the shame of being regarded as a topologist.

The signature of a manifold II.

- The **signature** of a $4k$-dimensional manifold with boundary $(M, \partial M)$ is
$$\sigma(M) \ = \ \sigma(H^{2k}(M, \partial M), \phi_M) \in \mathbb{Z} \ .$$
- **Theorem** (Thom, 1954) If a $4k$-dimensional manifold M is the boundary $M = \partial N$ of a $(4k+1)$-dimensional manifold N then
$$\sigma(M) \ = \ \sigma(H^{2k}(M), \phi_M) \ = \ 0 \in \mathbb{Z} \ .$$
Cobordant manifolds have the same signature.

> V.11 de [27]: Si une variété V^{4k} orientée est une variété-bord, l'index τ de la forme quadratique définie par le cup-produit sur $H^{2k}(V^{4k})$ est nul, va donner:
>
> **Théorème IV.1.** *Si deux variétés V, V', orientées, de dimension $4k$, sont cobordantes, les formes quadratiques définies par le cup-produit sur $H^{2k}(V)$ resp. $H^{2k}(V')$ ont même index τ.*
>
> (Rappelons que l'index d'une forme quadratique est ici l'excès du nombre des carrés positifs sur celui des carrés négatifs — en coefficients réels ou rationnels.)

- The signature map $\sigma : \Omega_{4k} \to \mathbb{Z}$ is onto for $k \geqslant 1$, with $\sigma(\mathbb{C}\,\mathbb{P}^{2k}) = 1 \in \mathbb{Z}$. Isomorphism for $k = 1$.

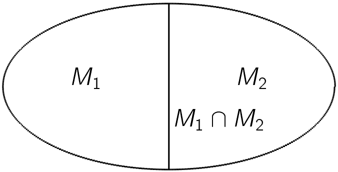
Novikov additivity of the signature

- Let M^{4k} be a closed $4k$-dimensional manifold which is a union of $4k$-dimensional manifolds with boundary M_1, M_2

$$M^{4k} = M_1 \cup M_2$$

with intersection a separating hypersurface

$$(M_1 \cap M_2)^{4k-1} = \partial M_1 = \partial M_2 \subset M .$$

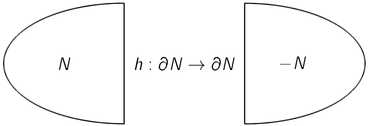


- **Theorem** (N., 1967) The union has signature

$$\sigma(M) = \sigma(M_1) + \sigma(M_2) \in \mathbb{Z} .$$

Twisted doubles

- **Definition** The **twisted double** of an n-dimensional with boundary $(N, \partial N)$ with respect to an automorphism $h : \partial N \to \partial N$ is the closed n-dimensional manifold $N \cup_h -N$.



- **Theorem** (P.Heegaard 1898 for $n = 1$, S.Smale 1962 ($\pi_1 = 0$) and T.Lawson 1978 for $n \geqslant 2$)
  Every $(2n+1)$-dimensional manifold is a twisted double.
- **Theorem** (H.Winkelnkemper 1972) For $k \geqslant 2$ a simply-connected $4k$-dimensional manifold M is a twisted double $N \cup_h -N$ if and only if $\sigma(M) = 0 \in \mathbb{Z}$.
- **Theorem** (F.Quinn 1979, A.R. 1998) For $n \geqslant 3$ a $2n$-dimensional manifold M is a twisted double if and only if the Witt class of an asymmetric form over $\mathbb{Z}[\pi_1(M)]$ is 0.

Formations

- **Definition** An ϵ-**symmetric formation** $(K, \phi; L_1, L_2)$ is a nonsingular ϵ-symmetric form (K, ϕ) with an ordered pair of lagrangians L_1, L_2.
- **Example** The **boundary** of a $(-\epsilon)$-symmetric form (L, λ) is the ϵ-symmetric formation

$$\partial(L, \lambda) \ = \ (H_\epsilon(L); L, \Gamma_{(L,\lambda)})$$

with $\Gamma_{(L,\lambda)} = \{(x, \lambda(x)) \,|\, x \in L\}$ the graph lagrangian of $H_\epsilon(L)$.
- **Definition** (i) An **isomorphism** of ϵ-symmetric formations $f : (K, \phi; L_1, L_2) \to (K', \phi'; L_1', L_2')$ is an isomorphism of forms $f : (K, \phi) \to (K', \phi')$ such that $f(L_1) = L_1'$, $f(L_2) = L_2'$.
- (ii) A **stable isomorphism** of ϵ-symmetric formations $[f] : (K, \phi; L_1, L_2) \to (K', \phi'; L_1', L_2')$ is an isomorphism of the type

$$f \ : \ (K, \phi; L_1, L_2) \oplus (H_\epsilon(L); L, L^*) \to (K', \phi'; L_1', L_2') \oplus (H_\epsilon(L'); L', L'^*) \ .$$

- Two formations are stably isomorphic if and only if

$$\dim_{\mathbb{R}}(L_1 \cap L_2) \ = \ \dim_{\mathbb{R}}(L_1' \cap L_2') \ .$$

Formations and automorphisms of forms

- **Proposition** Given a nonsingular ϵ-symmetric form (K, ϕ), a lagrangian L, and an automorphism $\alpha : (K, \phi) \to (K, \phi)$ there is defined an ϵ-symmetric formation $(K, \phi; L, \alpha(L))$.
- **Proposition** For any formation $(K, \phi; L_1, L_2)$ there exists an automorphism $\alpha : (K, \phi) \to (K, \phi)$ such that $\alpha(L_1) = L_2$.
- **Proof** The inclusions $(L_i, 0) \to (K, \phi)$ $(i = 1, 2)$ extend to isomorphisms $f_i : H_\epsilon(L_i) \cong (K, \phi)$. Since

$$\dim_{\mathbb{R}}(L_1) \ = \ \dim_{\mathbb{R}}(H)/2 \ = \ \dim_{\mathbb{R}}(L_2)$$

there exists an isomorphism $g : L_1 \cong L_2$. The composite automorphism

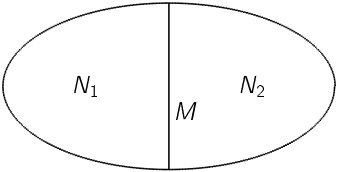
$$\alpha \ : \ (K, \phi) \xrightarrow[\cong]{f_1^{-1}} H_\epsilon(L_1) \xrightarrow[\cong]{h} H_\epsilon(L_2) \xrightarrow[\cong]{f_2} (K, \phi)$$

is such that $\alpha(L_1) = L_2$, where

$$h \ = \ \begin{pmatrix} g & 0 \\ 0 & (g^*)^{-1} \end{pmatrix} \ : \ H_\epsilon(L_1) \xrightarrow{\cong} H_\epsilon(L_2) \ .$$

The $(-)^n$-symmetric formation of a $(2n+1)$-dimensional manifold

- **Proposition** Let N^{2n+1} be a closed $(2n+1)$-dimensional manifold.



A separating hypersurface $M^{2n} \subset N = N_1 \cup_M N_2$ determines a $(-)^n$-symmetric formation

$$(K,\phi;L_1,L_2)=(H^n(M),\phi_M;\mathrm{im}(H^n(N_1)\to H^n(M)),\mathrm{im}(H^n(N_2)\to H^n(M)))$$

If $H_r(M) \to H_r(N_1) \oplus H_r(N_2)$ is onto for $r = n+1$ and one-one for $r = n-1$ then

$$L_1 \cap L_2 \ = \ H^n(N) \ = \ H_{n+1}(N) \ , \ H/(L_1+L_2) \ = \ H^{n+1}(N) \ = \ H_n(N) \ .$$

- The stable isomorphism class of the formation is a homotopy invariant of N. If $N = \partial P$ for some P^{2n+2} the class includes $\partial(H_{n+1}(P), \phi_P)$.

The triple index

- **Definition** (Wall 1969) The **triple index** of lagrangians L_1, L_2, L_3 in a nonsingular symplectic form (K, ϕ) is

$$\sigma(L_1, L_2, L_3) = \sigma(L_{123}, \lambda_{123}) \in \mathbb{Z}$$

with (L_{123}, λ_{123}) the symmetric form defined by

$$\begin{aligned} L_{123} &= \ker(L_1 \oplus L_2 \oplus L_3 \longrightarrow K) \ , \\ \lambda_{123} &= \lambda_{123}^* = \begin{pmatrix} 0 & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & 0 & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & 0 \end{pmatrix} : K \longrightarrow K^* \ , \\ \lambda_{ij} &= \lambda_{ji}^* \ : \ L_j \longrightarrow K \xrightarrow{\phi} K^* \longrightarrow L_i^* \ . \end{aligned}$$

- **Motivation** A stable isomorphism of formations

$$[f] \ : \ (K, \phi; L_1, L_2) \oplus (K, \phi; L_2, L_3) \oplus (K, \phi; L_3, L_1) \to \partial(L_{123}, \lambda_{123})$$

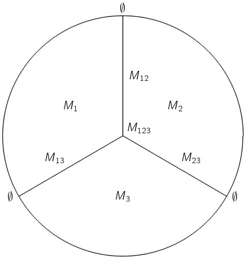
Wall non-additivity for $M^{4k} = M_1 \cup M_2 \cup M_3$ I.

- Let M^{4k} be a closed $4k$-dimensional manifold which is a triple union

$$M^{4k} \ = \ M_1 \cup M_2 \cup M_3$$

of $4k$-dimensional manifolds with boundary M_1, M_2, M_3 such that the double intersections $M_{ij}^{4k-1} = M_i \cap M_j$ $(1 \leqslant i < j \leqslant 3)$ are codimension 1 submanifolds of M. The triple intersection $M_{123}^{4k-2} = M_1 \cap M_2 \cap M_3$ is required to be a codimension 2 submanifold of M, with

$$\partial M_1 \ = \ \partial(M_2 \cup_{M_{23}} M_3) \ = \ M_{12} \cup_{M_{123}} M_{13} \text{ etc.}$$



Wall non-additivity for $M^{4k} = M_1 \cup M_2 \cup M_3$ II.

- **Theorem** (W. Non-additivity of the signature, Invent. Math. 7, 269–274 (1969))
  The signature of a triple union $M = M_1 \cup M_2 \cup M_3$ of $4k$-dimensional manifolds with boundary is

$$\sigma(M) \; = \; \sigma(M_1) + \sigma(M_2) + \sigma(M_3) + \sigma(L_1, L_2, L_3) \in \mathbb{Z}$$

with $\sigma(L_1, L_2, L_3)$ the triple index of the three lagrangians

$$L_i \; = \; \mathrm{im}(H_{2k}(M_{jk}, M_{123}) \to K) \subset K \; = \; H_{2k-1}(M_{123})$$

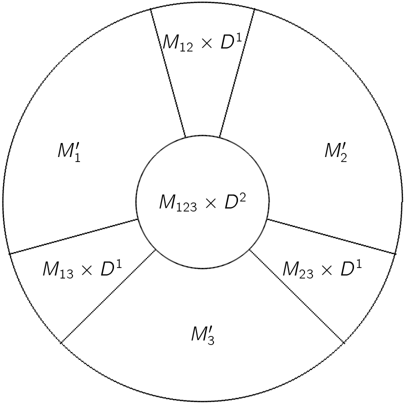
in the symplectic intersection form of M_{123}

$$(K, \phi) \; = \; (H_{2k-1}(M_{123}), \phi_{M_{123}}) \; .$$

Wall non-additivity for $M^{4k} = M_1 \cup M_2 \cup M_3$ III.

- **Idea of proof** $\sigma(L_1, L_2, L_3) = \sigma(N) \in \mathbb{Z}$ is the signature of a manifold neighbourhood $(N^{4k}, \partial N)$ of $M_{12} \cup M_{13} \cup M_{13} \subset M$

$$N = (M_{12} \cup M_{23} \cup M_{13}) \times D^1 \cup (M_{123} \times D^2) .$$



The space of lagrangians $\Lambda(n)$

- **Definition** For $n \geqslant 1$ let $\Lambda(n)$ be the spaces of lagrangians $L \subset H_-(\mathbb{R}^n)$.
- Use the complex structure on $H_-(\mathbb{R}^n)$

$$J \ : \ \mathbb{R}^n \oplus \mathbb{R}^n \to \mathbb{R}^n \oplus \mathbb{R}^n \ ; \ (x,y) \mapsto (-y,x)$$

  to associate to every lagrangian $L \in \Lambda(n)$ a canonical complement $JL \in \Lambda(n)$ with $L \oplus JL = \mathbb{R}^n \oplus \mathbb{R}^n$.
- For every $L \in \Lambda(n)$ there exists a unitary matrix $A \in U(n)$ such that

$$A(\mathbb{R}^n \oplus \{0\}) \ = \ L \in \Lambda(n) \ .$$

  If $A' \in U(n)$ is another such unitary matrix then

$$(A')^{-1}A \ = \ \begin{pmatrix} B & 0 \\ 0 & B^t \end{pmatrix} \ (B \in O(n))$$

  with $(b_{jk})^t = (b_{kj})$ the transpose.

Maslov index : $\pi_1(\Lambda(n)) \cong \mathbb{Z}$

- **Proposition** (Arnold, 1967) (i) The function

$$U(n)/O(n) \to \Lambda(n) \; : \; A \mapsto A(\mathbb{R}^n \oplus \{0\})$$

is a diffeomorphism. $\Lambda(n)$ is a compact manifold of dimension

$$\dim \Lambda(n) \; = \; \dim U(n) - \dim O(n) \; = \; n^2 - \frac{n(n-1)}{2} \; = \; \frac{n(n+1)}{2} \; .$$

The graphs $\{\Gamma_{(\mathbb{R}^n,\phi)} \,|\, \phi^* = \phi\} \subset \Lambda(n)$ define a chart at $\mathbb{R}^n \in \Lambda(n)$.

- (ii) The square of the determinant function

$$\det{}^2 \; : \; \Lambda(n) \to S^1 \; ; \; L = A(\mathbb{R}^n \oplus \{0\}) \mapsto \det(A)^2$$

induces the **Maslov index** isomorphism

$$\det{}^2 \; : \; \pi_1(\Lambda(n)) \xrightarrow{\cong} \pi_1(S^1) \; = \; \mathbb{Z} \; .$$

- **Proposition** (Kashiwara and Schapira, 1992) The triple index $\sigma(L_1, L_2, L_3) \in \mathbb{Z}$ of $L_1, L_2, L_3 \in \Lambda(n)$ is the Maslov index of a loop $S^1 \to \Lambda(n)$ passing through L_1, L_2, L_3.

The algebraic η-invariant

- **Definition/Proposition** (Atiyah-Patodi-Singer 1974, Cappell-Lee-Miller 1994, Bunke 1995)
  (i) The **algebraic η-invariant** of $L_1, L_2 \in \Lambda(n)$ is

$$\eta(L_1, L_2) = \sum_{j=1, \theta_j \neq 0}^{n} (1 - 2\theta_j/\pi) \in \mathbb{R}$$

  with $\theta_1, \theta_2, \ldots, \theta_n \in [0, \pi)$ such that $\pm e^{i\theta_1}, \pm e^{i\theta_2}, \ldots, \pm e^{i\theta_n}$ are the eigenvalues of any $A \in U(n)$ with $A(L_1) = L_2$.
- (ii) The algebraic η-invariant is a cocycle for the triple index of $L_1, L_2, L_3 \in \Lambda(n)$

$$\sigma(L_1, L_2, L_3) = \eta(L_1, L_2) + \eta(L_2, L_3) + \eta(L_3, L_1) \in \mathbb{Z} \subset \mathbb{R} .$$

The real signature I.

- Let M be a $4k$-dimensional manifold with a decomposed boundary $\partial M = N_1 \cup_P N_2$, where $P \subset \partial M$ is a separating codimension 1 submanifold. Let $(H_{2k-1}(P), \phi_P)$ be the nonsingular symplectic intersection form, and $n = \dim_{\mathbb{R}}(H_{2k-1}(P))/2$.
- Given a choice of isomorphism

$$J \ : \ (H_{2k-1}(P), \phi_P) \ \cong \ H_-(\mathbb{R}^n)$$

(or just a complex structure on $(H_{2k-1}(P), \phi_P)$) define the **real signature**

$$\sigma_J(M, N_1, N_2, P) \ = \ \sigma(M) + \eta(JL_1, JL_2) \in \mathbb{R}$$

using the lagrangians

$$L_j \ = \ \ker(H_{2k-1}(P) \to H_{2k-1}(N_j)) \subset (H_{2k-1}(P), \phi_P) \ .$$

- **Proposition** The real signature is additive

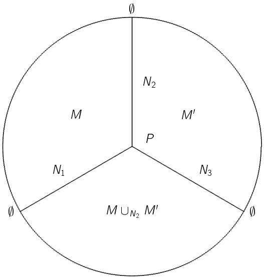
$$\sigma_J(M \cup_{N_2} M'; N_1, N_3, P) = \sigma_J(M; N_1, N_2, P) + \sigma_J(M'; N_2, N_3, P) \in \mathbb{R} \ .$$

The real signature II.

- **Proof** Apply the Wall non-additivity formula to the union

$$M \cup M' \cup -(M \cup_{N_2} M') \;=\; \partial((M \cup_{N_2} M') \times I)\ ,$$

which is an (un)twisted double with signature $\sigma = 0$.



- **Note** Analogue of the additivity of $\int_M L\text{-genus} = \sigma(M) + \eta(\partial M) \in \mathbb{R}$ in the Atiyah-Patodi-Singer signature theorem.
- **Note** In general $\sigma_J(M; N_1, N_2, P) \in \mathbb{R}$ depends on the choice of complex structure J on $(H_{2k-1}(P), \phi_P)$.

Real and complex vector bundles

- In view of the fibration

$$\Lambda(n) \; = \; U(n)/O(n) \to BO(n) \to BU(n)$$

  $\Lambda(n)$ classifies real n-plane bundles β with a trivialisation $\delta\beta : \mathbb{C} \otimes \beta \cong \epsilon^n$ of the complex n-plane bundle $\mathbb{C} \otimes \beta$.

- The canonical real n-plane bundle γ over $\Lambda(n)$ is

$$E(\gamma) \; = \; \{(L, x) \,|\, L \in \Lambda(n),\; x \in L\} \; .$$

  The complex n-plane bundle $\mathbb{C} \otimes \gamma$

$$E(\mathbb{C} \otimes \gamma) \; = \; \{(L, z) \,|\, L \in \Lambda(n),\; z \in \mathbb{C} \otimes_{\mathbb{R}} L\}$$

  is equipped with the canonical trivialisation $\delta\gamma : \mathbb{C} \otimes \gamma \cong \epsilon^n$ defined by

$$\delta\gamma \; : \; E(\mathbb{C} \otimes \gamma) \xrightarrow{\cong} E(\epsilon^n) \; = \; \Lambda(n) \times \mathbb{C}^n \; ;$$

$$(L, z) \mapsto (L, (x, y)) \text{ if } z = (x, y) = x + iy \in \mathbb{C} \otimes_{\mathbb{R}} L = L \oplus JL = \mathbb{C}^n \; .$$

The Maslov index, whichever way you slice it! I.

- The lagrangians $L \in \Lambda(1)$ are parametrized by $\theta \in \mathbb{R}$
$$L(\theta) \;=\; \{(r\cos\theta, r\sin\theta)\,|\,r \in \mathbb{R}\} \subset \mathbb{R} \oplus \mathbb{R}$$
with indeterminacy $L(\theta) = L(\theta + \pi)$. The map
$$\det{}^2 \;:\; \Lambda(1) \;=\; U(1)/O(1) \to S^1 \;;\; L(\theta) \mapsto e^{2i\theta}$$
is a diffeomorphism.
- The canonical $\mathbb{R}$-bundle γ over $\Lambda(1)$
$$E(\gamma) \;=\; \{(L, x)\,|\,L \in \Lambda(1)\,,\; x \in L\}$$
is nontrivial $=$ infinite Möbius band. The induced $\mathbb{C}$-bundle over $\Lambda(1)$
$$E(\mathbb{C} \otimes_{\mathbb{R}} \gamma) \;=\; \{(L, z)\,|\,L \in \Lambda(1)\,,\; z \in \mathbb{C} \otimes_{\mathbb{R}} L\}$$
is equipped with the canonical trivialisation $\delta\gamma : \mathbb{C} \otimes_{\mathbb{R}} \gamma \cong \epsilon$ defined by
$$\delta\gamma \;:\; E(\mathbb{C} \otimes_{\mathbb{R}} \gamma) \xrightarrow{\;\cong\;} E(\epsilon) \;=\; \Lambda(1) \times \mathbb{C}\;;$$
$$(L, z) = (L(\theta), (x + iy)(\cos\theta, \sin\theta)) \mapsto (L(\theta), (x + iy)e^{i\theta})\;.$$

The Maslov index, whichever way you slice it! II.

- Given a bagel $B = S^1 \times D^2 \subset \mathbb{R}^3$ and a map $\lambda : S^1 \to \Lambda(1) = S^1$ slice B along

$$C = \{(x,y) \in B \mid y \in \lambda(x)\} .$$

- The slicing line $(x, \lambda(x)) \subset B$ is the fibre over $x \in S^1$ of the pullback $[-1,1]$-bundle

$$[-1,1] \to C = D(\lambda^*\gamma) \to S^1$$

with boundary (where the knife goes in and out of the bagel)

$$\partial C = \{(x,y) \in C \mid y \in \partial\lambda(x)\}$$

a double cover of S^1. There are two cases:
  - C is a trivial $[-1,1]$-bundle over S^1 (i.e. an annulus), with ∂C two disjoint circles, which are linked in $\mathbb{R}^3$. The complement $B \backslash C$ has two components, with the same linking number.
  - C is a non trivial $[-1,1]$-bundle over S^1 (i.e. a Möbius band), with ∂C a single circle, which is self-linked in $\mathbb{R}^3$. The complement $B \backslash C$ is connected, with the same self-linking number ($=$ linking of ∂C and $S^1 \times \{(0,0)\} \subset C \subset \mathbb{R}^3$).

The Maslov index, whichever way you slice it! III.

- By definition, Maslov index(λ) = degree$(\lambda) \in \mathbb{Z}$.
- degree : $\pi_1(S^1) \to \mathbb{Z}$ is an isomorphism, so it may be assumed that

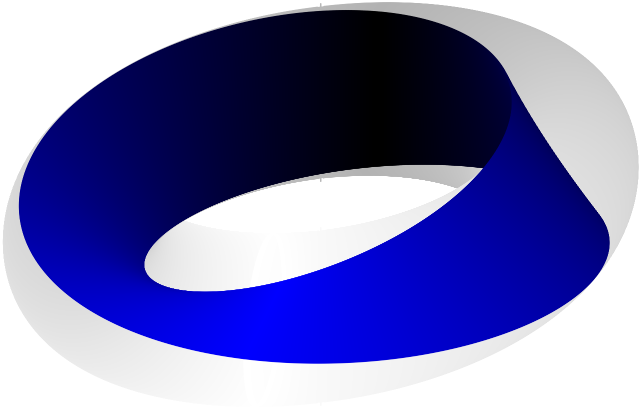
$$\lambda \ : \ S^1 \to \Lambda(1) \ ; \ e^{2i\theta} \mapsto L(n\theta)$$

  with Maslov index $= n \geqslant 0$. The knife is turned through a total angle $n\pi$ as it goes round B. It may also be assumed that the bagel B is horizontal. The projection of ∂C onto the horizontal cross-section of B consists of $n = |\lambda^{-1}(L(0))|$ points. For $n > 0$ this corresponds to the angles $\theta = j\pi/n \in [0, \pi)$ $(0 \leqslant j \leqslant n-1)$ where $L(n\theta) = L(0)$, i.e. $\sin n\theta = 0$.
- The two cases are distinguished by:
  - If $n = 2k$ then ∂C is a union of two disjoint linked circles in $\mathbb{R}^3$. Each successive pair of points in the projection contributes 1 to the linking number $n/2 = k$.
  - If $n = 2k+1$ then ∂C is a single self-linked circle in $\mathbb{R}^3$. Each point in the projection contributes 1 to the self-linking number $n = 2k+1$. (Thanks to Laurent Bartholdi for explaining this case to me.)

Maslov index = 0 , C = annulus , linking number = 0

$$\lambda \ : \ S^1 \to S^1 \ ; \ z \mapsto 1 \ .$$

Maslov index = 1 , C = Möbius band , self-linking number = 1



$$\lambda \ : \ S^1 \to S^1 \ ; \ z \mapsto z \ .$$

Thanks to Clara Löh for this picture.

Maslov index = 2 , C = annulus , linking number = 1

$$\lambda \; : \; S^1 \to S^1 \; ; \; z \mapsto z^2 \; .$$

http://www.georgehart.com/bagel/bagel.html