COBORDISM IN ALGEBRA AND TOPOLOGY

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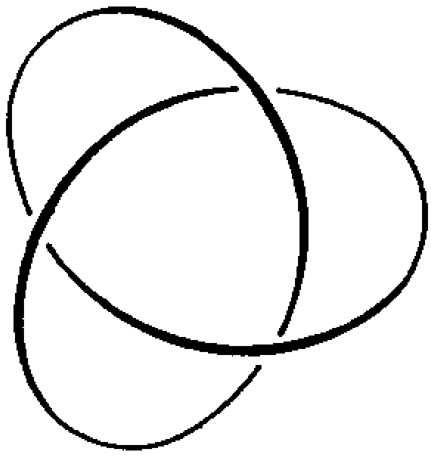


Dedicated to
Robert Switzer
and
Desmond Sheiham

Göttingen, 13th May, 2005

Cobordism

- There is a cobordism equivalence relation on each of the following 6 classes of mathematical structures, which come in 3 matching pairs of topological and algebraic types:
  - (manifolds, quadratic forms)
  - (knots, Seifert forms)
  - (boundary links, partitioned Seifert forms)

- The cobordism groups are the abelian groups of equivalence classes, with forgetful morphisms

  $\{\text{topological cobordism}\} \to \{\text{algebraic cobordism}\}$

- How large are these groups? To what extent are these morphisms isomorphisms?

Matrices and forms

- An $r \times r$ matrix $A = (a_{ij})$ has entries $a_{ij} \in \mathbb{Z}$ with $1 \leqslant i, j \leqslant r$.

- The direct sum of A and an $s \times s$ matrix $B = (b_{k\ell})$ is the $(r+s) \times (r+s)$ matrix

$$A \oplus B \;=\; \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \; .$$

- The transpose of A is the $r \times r$ matrix

$$A^T \;=\; (a_{ji}) \; .$$

- A quadratic form is an $r \times r$ matrix A which is symmetric and invertible

$$A^T \;=\; A \; , \; \det(A) = \pm 1 \; .$$

  A symplectic form is an $r \times r$ matrix A which is (-1)-symmetric and invertible

$$A^T \;=\; -A \; , \; \det(A) = \pm 1 \; .$$

Cobordism of quadratic forms

- Quadratic forms A, A' are congruent if $A' = U^T A U$ for an invertible matrix U.

- A quadratic form A is null-cobordant if it is congruent to $\begin{pmatrix} 0 & P \\ P^T & Q \end{pmatrix}$ with P an invertible $s \times s$ matrix, and Q a symmetric $s \times s$ matrix.

- Example $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is null-cobordant.

- Quadratic forms A, A' (which may be of different sizes) are cobordant if $A \oplus B$ is congruent to $A' \oplus B'$ for null-cobordant B, B'.

- Similarly for symplectic forms.

Calculation of the cobordism group of quadratic forms

- The Witt group $W(\mathbb{Z})$ is the abelian group of cobordism classes of quadratic forms, with addition by direct sum $A \oplus A'$.

- Definition (Sylvester, 1852) The signature of a quadratic form A is
$$\sigma(A) = r_+ - r_- \in \mathbb{Z}$$
with r_+ the number of positive eigenvalues of A, r_- the number of negative eigenvalues of A.

- $\sigma(1) = 1$, $\sigma(-1) = -1$, $\sigma(H) = 0$.

- Theorem Signature defines isomorphism
$$\sigma : W(\mathbb{Z}) \to \mathbb{Z} \; ; \; A \mapsto \sigma(A) \; .$$

- The Witt group of symplectic forms $= 0$.

Manifolds

- An <u>n-manifold</u> M is a topological space such that each $x \in M$ has a neighbourhood $U \subset M$ which is homeomorphic to Euclidean n-space $\mathbb{R}^n$. Will assume differentiable structure.

- The solution set $M = f^{-1}(0)$ of equation $f(x) = 0 \in \mathbb{R}^m$ for function $f : \mathbb{R}^{m+n} \to \mathbb{R}^m$ is generically an n-manifold.

- The n-sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ is an n-manifold.

- A surface is a 2-manifold, e.g. sphere S^2, torus $S^1 \times S^1$.

- Will only consider oriented manifolds: no Möbius bands, Klein bottles etc.

Cobordism of manifolds

- An <u>$(n+1)$-manifold with boundary</u> <u>$(W, \partial W \subset W)$</u> has $W \backslash \partial W$ an $(n+1)$-manifold and ∂W an n-manifold.

- Will only consider compact oriented manifolds with boundary (which may be empty).

- <u>Example</u> (D^{n+1}, S^n) is a compact oriented $(n+1)$-manifold with boundary, where $D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid \|x\| \leqslant 1\}$.

- Two n-manifolds M_0, M_1 are <u>cobordant</u> if the disjoint union $M_0 \sqcup -M_1$ is the boundary ∂W of an $(n+1)$-manifold W, where $-M_1$ is M_1 with reverse orientation.

- Every surface M is the boundary $M = \partial W$ of a 3-manifold W, so any two surfaces M, M' are cobordant.

The cobordism groups of manifolds

- The cobordism group Ω_n of cobordism classes of n-manifolds, with addition by disjoint union $M \sqcup M'$.
  The cobordism ring $\Omega_* = \bigoplus_n \Omega_n$ with multiplication by cartesian product $M \times N$.

- Theorem (Thom, 1952) Each cobordism group Ω_n is finitely generated with 2-torsion only. The cobordism ring is

  $$\Omega_* = \mathbb{Z}[x_4, x_8, \ldots] \oplus \bigoplus_{\infty} \mathbb{Z}_2 .$$

  $\mathbb{Z}[x_4, x_8, \ldots]$ is the polynomial algebra with one generator x_{4k} in each dimension $4k$. Note that Ω_n grows in size as n increases.

- Nice account of manifold cobordism in Switzer's book *Algebraic Topology – Homotopy and Homology* (Springer, 1975)

The signature of a $4k$-manifold

- (Poincaré, 1895) The intersection matrix $A = (a_{ij})$ of a $2q$-manifold M defined by intersection numbers $a_{ij} = z_i \cap z_j \in \mathbb{Z}$ for a basis $z_1, z_2, \dots, z_r$ of the homology group $H_q(M) = \mathbb{Z}^r \oplus$ torsion, with

  $$A^T = (-1)^q A \ , \ \det(A) = \pm 1 \ .$$

  A is a quadratic form if q is even.
  A is a symplectic form if q is odd.

- If $M = S^q \times S^q$ then $A = \begin{pmatrix} 0 & 1 \\ (-1)^q & 0 \end{pmatrix}$.

- The signature of a $4k$-manifold M^{4k} is

  $$\sigma(M) = \sigma(A) \in \mathbb{Z} \ .$$

- $\sigma(S^{4k}) = \sigma(S^{2k} \times S^{2k}) = 0$, $\sigma(x_{4k}) = 1$.

The signature morphism $\sigma : \Omega_{4k} \to W(\mathbb{Z})$

- Let M, M' be $4k$-manifolds with intersection matrices A, A'. If M and M' are cobordant then A and A' are cobordant, and
  $\sigma(M) \ = \ \sigma(A) \ = \ \sigma(A') \ = \ \sigma(M') \in \mathbb{Z}$.
  However, a cobordism of A and A' may not come from a cobordism of M and M'.

- Signature defines surjective ring morphism
  $\sigma \ : \ \Omega_{4k} \to W(\mathbb{Z}) = \mathbb{Z} \ ; \ M \mapsto \sigma(M)$
  with $x_{4k} \mapsto 1$. Isomorphism for $k = 1$.

- Example The 8-manifolds $(x_4)^2$, x_8 have same signature $\sigma = 1$, but are not cobordant, $(x_4)^2 - x_8 \neq 0 \in \ker(\sigma : \Omega_8 \to \mathbb{Z})$.

- Can determine class of $4k$-manifold M in Ω_{4k}/torsion $= \mathbb{Z}[x_4, x_8, \dots]$ from signatures $\sigma(N)$ of submanifolds $N^{4\ell} \subseteq M$ $(\ell \leqslant k)$.

Cobordism of knots

- A <u>n-knot</u> is an embedding

$$\mathcal{K} : S^n \subset S^{n+2} \ .$$

  Traditional knots are 1-knots.

- Two n-knots $\mathcal{K}_0, \mathcal{K}_1 : S^n \subset S^{n+2}$ are <u>cobordant</u> if there exists an embedding

$$\mathcal{J} : S^n \times [0,1] \subset S^{n+2} \times [0,1]$$

  such that $\mathcal{J}(x,i) = \mathcal{K}_i(x)$ $(x \in S^n, i = 0,1)$.

- The <u>n-knot cobordism group</u> C_n is the abelian group of cobordism classes of n-knots, with addition by connected sum. First defined for $n = 1$ by Fox and Milnor (1966).

Cobordism of Seifert surfaces

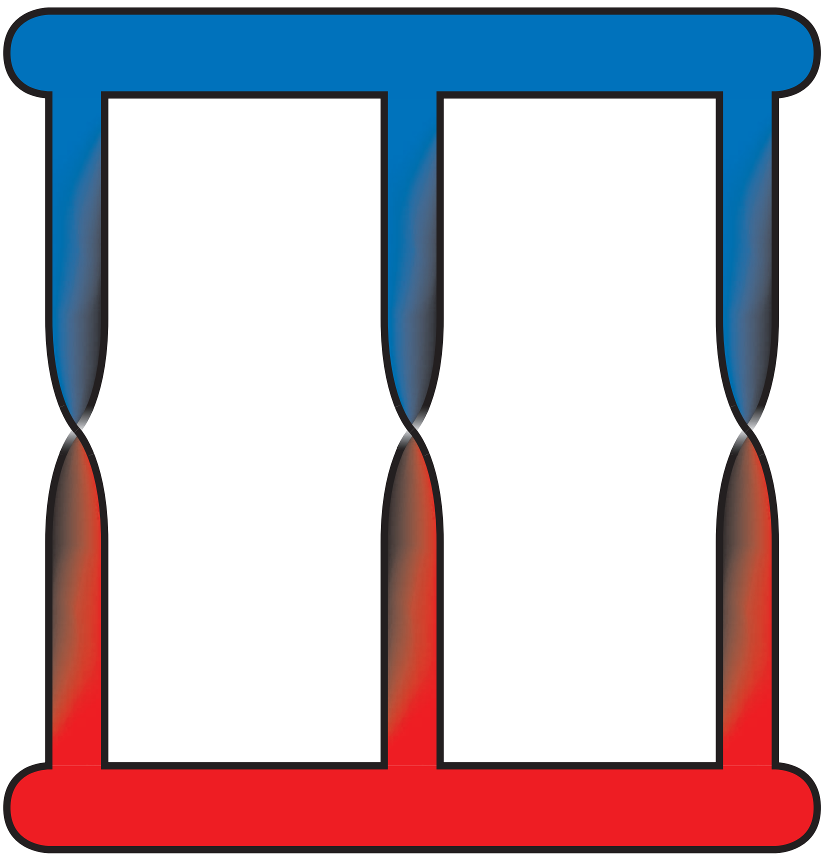
- A Seifert surface for n-knot $\mathcal{K} : S^n \subset S^{n+2}$ is a submanifold $V^{n+1} \subset S^{n+2}$ with boundary $\partial V = \mathcal{K}(S^n) \subset S^{n+2}$.

- Every n-knot $\mathcal{K}$ has Seifert surfaces V – highly non-unique!

- If $\mathcal{K}_0, \mathcal{K}_1 : S^n \subset S^{n+2}$ are cobordant n-knots, then for any Seifert surfaces $V_0, V_1 \subset S^{n+2}$ there exists a Seifert surface cobordism $W^{n+2} \subset S^{n+2} \times [0,1]$ such that
$W \cap (S^{n+2} \times \{i\}) = V_i$ $(i = 0, 1)$.

- Theorem (Kervaire 1965) $C_{2q} = 0$ $(q \geqslant 1)$
Proof: for every $\mathcal{K} : S^{2q} \subset S^{2q+2}$ and Seifert surface $V^{2q+1} \subset S^{2q+2}$ can construct null-cobordism by 'killing $H_*(V)$ by ambient surgery'.

The trefoil knot, with a Seifert surface



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Seifert forms

- A Seifert $(-1)^q$-form is an $r \times r$ matrix B such that the $(-1)^q$-symmetric matrix

$$A = B + (-1)^q B^T$$

  is invertible.

- A $(2q-1)$-knot $\mathcal{K} : S^{2q-1} \subset S^{2q+1}$ with a Seifert surface $V^{2q} \subset S^{2q+1}$ determine a Seifert $(-1)^q$-form B.

- B is the $r \times r$ matrix of linking numbers $b_{ij} = \ell(z_i, z'_j) \in \mathbb{Z}$, for any basis $z_1, z_2, \ldots, z_r \in H_q(V)$, with $z'_1, z'_2, \ldots, z'_r \in H_q(S^{2q+1} \backslash V)$ the images of the z_i's under a map $V \to S^{2q+1} \backslash V$ pushing V off itself in S^{2q+1}. $A = B + (-1)^q B^T$ is the intersection matrix of V.

Cobordism of Seifert forms

- The cobordism of Seifert $(-1)^q$-forms defined as for quadratic forms, with cobordism group $G_{(-1)^q}(\mathbb{Z})$.

- Depends only on $q(\bmod 2)$.

- Theorem (Levine, 1969) The morphism

  $$C_{2q-1} \to G_{(-1)^q}(\mathbb{Z}) \ ; \ \mathcal{K} \mapsto B \quad (\text{any } V)$$

  is an isomorphism for $q \geqslant 2$ and surjective for $q = 1$. Thus for $q \geqslant 2$

  knot cobordism C_{2q-1}

  $$= \text{algebraic cobordism } G_{(-1)^q}(\mathbb{Z}) \ .$$

- For $q \geqslant 2$ can realize Seifert $(-1)^q$-form cobordisms by Seifert surface and $(2q-1)$-knot cobordisms!

The calculation of the knot cobordism group C_{2q-1}

- Theorem (Levine 1969) For $q \geqslant 2$

  $$C_{2q-1} = G_{(-1)^q}(\mathbb{Z}) = \bigoplus_{\infty} \mathbb{Z} \oplus \bigoplus_{\infty} \mathbb{Z}_2 \oplus \bigoplus_{\infty} \mathbb{Z}_4 \ .$$

  Countably infinitely generated.

- The $\mathbb{Z}$'s are signatures, one for each algebraic integer $s \in \mathbb{C}$ (= root of monic integral polynomial) with Re$(s) = 1/2$ and Im$(s) > 0$, so that $s + \bar{s} = 1$.

- The $\mathbb{Z}_2$'s and $\mathbb{Z}_4$'s are Hasse-Minkowski invariants, as in the Witt group of rational quadratic forms

  $$W(\mathbb{Q}) = \mathbb{Z} \oplus \bigoplus_{\infty} \mathbb{Z}_2 \oplus \bigoplus_{\infty} \mathbb{Z}_4 \ .$$

- Corollary For $q \geqslant 2$ an algorithm for deciding if two $(2q-1)$-knots are cobordant.

The Milnor-Levine knot signatures

- For an $r \times r$ Seifert $(-1)^q$-form B define the complex vector space $K = \mathbb{C}^r$ and the linear map $J = A^{-1}B : K \to K$ with $A = B + (-1)^q B^T$. The eigenvalues of J are algebraic integers, the roots $s \in \mathbb{C}$ of the characteristic monic integral polynomial $\det(sI - J)$ of J. K and A split as

$$K = \bigoplus_s K_s \ , \ A = \bigoplus_s A_s$$

  with $K_s = \bigcup_{n=0}^{\infty} \ker(sI - J)^n$ the generalized eigenspace. For each s with $s + \bar{s} = 1$ (K_s, A_s) has signature $\sigma_s(B) = \sigma_{\bar{s}}(B) \in \mathbb{Z}$.

- The morphism

$$G_{(-1)^q}(\mathbb{Z}) \to \bigoplus_s \mathbb{Z} \ ; \ B \mapsto \bigoplus_s \sigma_s(B)$$

  is an isomorphism modulo 4-torsion, with s running over all the algebraic integers $s \in \mathbb{C}$ with $\mathrm{Re}(s) = 1/2$ and $\mathrm{Im}(s) > 0$.

The cobordism class of the trefoil knot

- The trefoil knot $\mathcal{K} : S^1 \subset S^3$ has a Seifert surface $V^2 = (S^1 \times S^1)\backslash D^2$, with
$$H_1(V) \ = \ \mathbb{Z} \oplus \mathbb{Z}$$
and Seifert (-1)-form $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, with
$$J \ = \ (B - B^T)^{-1}B \ = \ \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

- The characteristic polynomial of J
$$\det(sI - J) \ = \ s^2 - s + 1$$
has roots the algebraic integers
$$s_+ = (1 + \sqrt{3}i)/2 \ , \ s_- = (1 - \sqrt{3}i)/2 \ .$$
The Milnor-Levine signature is
$$\sigma_{s_+}(B) \ = \ 1 \in \mathbb{Z} \subset G_{-1}(\mathbb{Z})$$
so that $\mathcal{K}$ is not cobordant to the trivial knot, $\mathcal{K} \neq 0 \in C_1$.

Boundary links

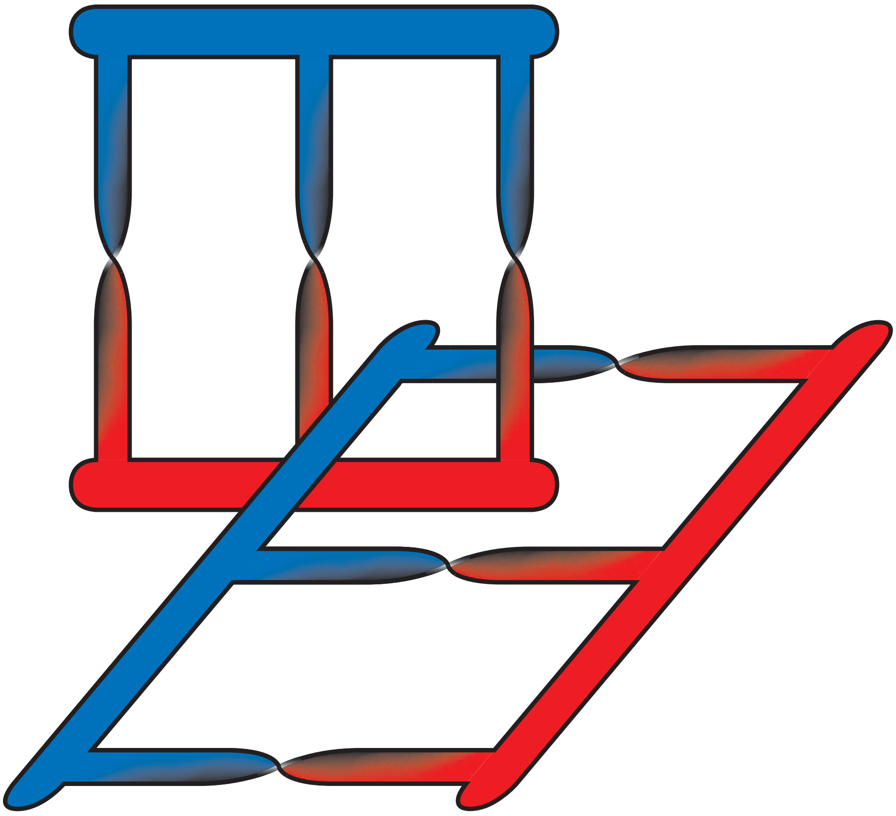
- Fix $\mu \geqslant 1$. A μ-component n-link is an embedding $\mathcal{L} : \bigsqcup_{\mu} S^n \subset S^{n+2}$. Traditional links are 1-links.

- A Seifert surface for $\mathcal{L}$ is a submanifold $V^{n+1} \subset S^{n+2}$ with $\partial V = \mathcal{L}(\bigsqcup_{\mu} S^n) \subset S^{n+2}$. Every n-link has Seifert surfaces. $\mathcal{L}$ is a boundary link if it admits a μ-component Seifert surface $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_\mu$.

- Theorem (Smythe, Gutierrez 1972) $\mathcal{L}$ is a boundary link if and only if there exists a surjection $\pi_1(S^{n+2} \backslash \mathcal{L}(\bigsqcup_{\mu} S^n)) \to F_\mu$ onto free group F_μ with μ generators.

- Trivial link is a boundary link: $\pi_1 = F_\mu$. The 2-component Hopf link is not a boundary link: $\pi_1 = \mathbb{Z} \oplus \mathbb{Z}$.

A 2-component boundary link with a 2-component Seifert surface



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μ-component Seifert forms

- A μ-component Seifert $(-1)^q$-form is a Seifert $(-1)^q$-form B with a partition into μ^2 blocks

$$B = \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1\mu} \\ B_{21} & B_{22} & \dots & B_{2\mu} \\ \vdots & \vdots & \ddots & \vdots \\ B_{\mu 1} & B_{\mu 2} & \dots & B_{\mu\mu} \end{pmatrix}$$

such that B_{ii} is a Seifert $(-1)^q$-form and $B_{ij} = (-1)^{q+1}(B_{ji})^T$ for $i \neq j$.

- A μ-component Seifert surface V for

$$\mathcal{L} = \bigsqcup_{i=1}^{\mu} \mathcal{L}_i : \bigsqcup_{i=1}^{\mu} S^{2q-1} \subset S^{2q+1}$$

determines a μ-component Seifert $(-1)^q$-form B with B_{ii} the Seifert $(-1)^q$-form of $\mathcal{L}_i : S^{2q-1} \subset S^{2q+1}$.

- Cobordism as for $\mu = 1$, with group $G_{(-1)^q,\mu}(\mathbb{Z})$.

The cobordism of boundary links

- Let $C_n(F_\mu)$ be the set of cobordism classes of boundary links $\mathcal{L} : \bigsqcup_\mu S^n \subset S^{n+2}$ with a choice of surjection $\pi_1(S^{n+2}\backslash\mathcal{L}) \to F_\mu$. Abelian group for $n \geqslant 2$, with addition by connected sum.
  For knots $\mu = 1$, $C_n(F_1) = C_n$.

- Theorem (Cappell-Shaneson 1980)

$$C_{2q}(F_\mu) \ = \ 0 \ \ (q \geqslant 1) \ .$$

- Theorem (Ko, Mio 1989) For $q \geqslant 2$ boundary link cobordism $C_{2q-1}(F_\mu)$

$$= \text{algebraic cobordism } G_{(-1)^q,\mu}(\mathbb{Z}) \ .$$

  Proof: Can realize μ-component Seifert $(-1)^q$-form cobordisms by Seifert surface and boundary link cobordisms, just like in the knot case $\mu = 1$!

The calculation of the cobordism of boundary links

- Theorem (Sheiham, 2001) For $q \geqslant 2$

$$\begin{aligned} C_{2q-1}(F_\mu) &= G_{(-1)^q,\mu}(\mathbb{Z}) \\ &= \bigoplus_\infty \mathbb{Z} \oplus \bigoplus_\infty \mathbb{Z}_2 \oplus \bigoplus_\infty \mathbb{Z}_4 \oplus \bigoplus_\infty \mathbb{Z}_8 \ . \end{aligned}$$

  The $\mathbb{Z}$'s are signatures, the $\mathbb{Z}_2$'s, $\mathbb{Z}_4$'s and $\mathbb{Z}_8$'s are generalized Hasse-Minkowski invariants.

- Depends only on $q \pmod 2$.
  Countably infinitely generated.

- Corollary For $q \geqslant 2$ an algorithm for deciding if two boundary $(2q-1)$-links are cobordant.

The Sheiham boundary link signatures

- Ring with involution $P_\mu = \mathbb{Z}[s, \pi_1, \pi_2, \ldots, \pi_\mu]$

  $$\sum_{i=1}^{\mu} \pi_i = 1 \ , \ \pi_i \pi_j = \delta_{ij} \ , \ \bar{s} = 1 - s \ , \ \bar{\pi}_i = \pi_i$$

  (Farber, 1991).

- An $r \times r$ μ-component Seifert $(-1)^q$-form B is a self-dual representation of P_μ on $\mathbb{Z}^r$, a morphism of rings with involution

  $$\rho : P_\mu \to R = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^r, \mathbb{Z}^r) \ .$$

  Use $A = B + (-1)^q B^T \in R$ to define $R \to R; D \mapsto A^{-1} D^T A$, with $\rho(\pi_i) \in R$ the idempotent of the ith block in B and $\rho(s) = A^{-1} B \in R$.

- There is one Sheiham signature for each 'algebraic integer' in the moduli space of self-dual representations of P_μ on finite-dimensional complex vector spaces.

The low-dimensional case $n = 1$

- For $n \geqslant 2$ every μ-component boundary n-link $\mathcal{L} : \bigsqcup_{\mu} S^n \subset S^{n+2}$ is cobordant to one with Seifert surface $V = \bigsqcup_{i=1}^{\mu} V_i$ such that

  $\pi_1(S^{n+2} \backslash \mathcal{L}(\bigsqcup_{\mu} S^n)) = F_\mu \ , \ \pi_1(V_i) = \{1\}$

  This is not possible for $n = 1$.

- For knots $\mathcal{K} : S^1 \subset S^3$ Casson and Gordon (1975) and Cochran, Teichner, Orr (1999) used the special low-dimensional properties of the fundamental group $\pi_1(S^3 \backslash \mathcal{K}(S^1))$ and L^2-cohomology to obtain many more signatures for $C_1 = C_1(F_1)$, almost calculating the torsion-free part completely.

- Next step: compute the cobordism set $C_1(F_\mu)$ of boundary links $\mathcal{L} : \bigsqcup_{\mu} S^1 \subset S^3$ for $\mu \geqslant 2$!