SMSTC (2007/08)

Geometry and Topology

Lecture 4: The fundamental group and covering spaces

Andrew Ranicki, University of Edinburgh[a]

www.smstc.ac.uk

Contents

November 7, 2007

4.1 Introduction

4.1.1 Books

Allen Hatcher's downloadable book *Algebraic Topology*

http : //www.math.cornell.edu/ hatcher/AT/ATpage.html

is an excellent introduction to algebraic topology. Whenever possible I have included a page reference to the book, in the form [ATn].

My own book *Algebraic and geometric surgery*

http : //www.maths.ed.ac.uk/ aar/books/surgery.pdf

describes the application of algebraic topology to the classification of manifolds. The reviews of foundational material it includes might be found useful.

Warning/promise: both books go far beyond the syllabus of the SMSTC course.

4.1.2 Topological invariants

How does one recognize topological spaces, and distinguish between them? In the first instance, it is not even clear if the Euclidean spaces $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \ldots$ are topologically distinct. Standard linear algebra shows that they are all non-isomorphic as vector spaces: it follows that $\mathbb{R}^m$ is diffeomorphic to $\mathbb{R}^n$ if and only if $m = n$, since the differential of a diffeomorphism is an isomorphism of vector spaces. In 1878 Cantor constructed bijections $\mathbb{R} \to \mathbb{R}^n$ for $n \geqslant 2$, which however were not continuous. In 1890 Peano constructed continuous surjections $\mathbb{R} \to \mathbb{R}^n$ for $n \geqslant 2$, the 'space-filling curves'. Thus there might also be continuous bijections with continuous inverses, i.e. homeomorphisms. It was only proved in 1910 by Brouwer that $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ if and only if $m = n$.

[a] a.ranicki@ed.ac.uk

Algebraic topology deals with topological invariants of spaces, that is functions I which associate to a topological space X an object $I(X)$ which may be either a number or an algebraic structure such as a group. The essential requirement is that homeomorphic spaces X, Y have the same invariant $I(X) = I(Y)$, where $=$ means 'isomorphic to' for algebraic invariants. Thus if X, Y are such that $I(X) \neq I(Y)$ then X, Y are not homeomorphic.

Here are some examples:

4–1. The dimension of a Euclidean space $\mathbb{R}^n$, $I(\mathbb{R}^n) = n$.

4–2. The genus of an orientable surface Σ. an integer $g(\Sigma) \geqslant 0$ (1850's). [AT51]

4–3. The Betti numbers (1860's). [AT130]

4–4. The fundamental group $\pi_1(X)$ (Poincaré. 1895). [AT26]

4–5. The homology groups $H_*(X)$ (1920's). [AT160]

4–6. The cohomology ring $H^*(X)$ (1930's). [AT191]

4–7. The higher homotopy groups $\pi_*(X)$ (1930's). [AT340]

Given a topological space X, the first thing one might ask about its topology is whether any two points can be joined by a path: given $x_0, x_1 \in X$ does there exist a continuous map $\alpha : I = [0,1] \to X$ from $\alpha(0) = x_0 \in X$ to $\alpha(1) = x_1 \in X$? Such a function is called a 'path' in X from x_0 to x_1. The relation defined on X by $x_0 \sim x_1$ if there exists a path from x_0 to x_1 is an equivalence relation. An equivalence class is called a 'path component' of X, and the set of path components is denoted by $\pi_0(X)$. The number of path-components in a space X

$$|\pi_0(X)| \in \{0, 1, 2, 3, \ldots, \infty\}$$

is perhaps the simplest topological invariant: if $m \neq n$ a space with m path-components cannot be homeomorphic to a space with n path-components. By definition, a space X is path-connected if $|\pi_0(X)| = 1$, i.e. if for any $x_0, x_1 \in X$ there exists a path from x_0 to x_1.

More generally, given two continuous maps $f_0, f_1 : X \to Y$ one can ask if there is a continuous choice of path from $f_0(x) \in Y$ to $f_1(x) \in Y$ for each $x \in X$. A 'homotopy' from f_0 to f_1 is a continuous family of continuous maps

$$\{f_t : X \to Y \,|\, 0 \leqslant t \leqslant 1\}$$

sliding from f_0 to f_1. For any spaces X, Y homotopy is an equivalence relation on the set of continuous maps $X \to Y$, denoted by $f_0 \simeq f_1$. A continuous map $f : X \to Y$ is a 'homotopy equivalence' if there exist a continuous map $g : Y \to X$ such that $gf \simeq 1_X : X \to X$ and $fg \simeq 1_Y : Y \to Y$. In particular, a homeomorphism is a homotopy equivalence.

Regard S^1 as the unit circle in the complex plane $\mathbb{C}$. A 'loop' in a space X at a point $x \in X$ is a continuous map $\omega : S^1 \to X$ such that $\omega(1) = x \in X$. The fundamental group $\pi_1(X, x)$ of X at $x \in X$ is defined geometrically to be the set of homotopy classes of loops $\omega : S^1 \to X$ at x, with the homotopies $\{\omega_t \,|\, 0 \leqslant t \leqslant 1\}$ required to be such that $\omega_t(1) = x$.

If $x_0, x_1 \in X$ are in the same path component (i.e. joined by a path) then $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic. For a path-connected space X $\pi_1(X)$ denotes any one of the isomorphic groups $\pi_1(X, x)$ $(x \in X)$.

Here are the key properties of the fundamental group:

4–8. A continuous map $f : X \to Y$ induces a group morphism $f_* : \pi_1(X) \to \pi_1(Y)$ which depends only on the homotopy class of f.

4–9. For any space X the identity function $1_X : X \to X$ induces the identity morphism

$$(1_X)_* \;=\; 1_{\pi_1(X)} \;:\; \pi_1(X) \to \pi_1(X) \;.$$

4–10. For any continuous maps $f : X \to Y$, $g : Y \to Z$

$$(gf)_* \;=\; g_* f_* \;:\; \pi_1(X) \to \pi_1(Z) \;.$$

4–11. If f is a homotopy equivalence then f_* is an isomorphism. Thus spaces with non-isomorphic fundamental groups cannot be homotopy equivalent, and a fortiori cannot be homeomorphic.

The isomorphism class of $\pi_1(X)$ is a topological invariant of X. A space X is 'simply-connected' if it is path-connected and $\pi_1(X) = \{1\}$, i.e. every loop is homotopic to a constant loop.

In many cases it is possible to actually compute $\pi_1(X)$, and to use the fundamental group to make interesting statements about topological spaces. Here are some examples:

4–12. The Euclidean spaces $\mathbb{R}^n$ ($n \geqslant 1$) are all simply-connected, with $\pi_1(\mathbb{R}^n) = \{1\}$.

4–13. The fundamental group of the circle S^1 is the infinite cyclic group

$$\pi_1(S^1) \;=\; \mathbb{Z}\ .$$

Every loop $\omega : S^1 \to S^1$ is homotopic to the standard loop going round S^1 n times

$$\omega_n \;:\; S^1 \to S^1 \;;\; z \mapsto z^n \quad \text{(complex multiplication)}$$

for a unique $n \in \mathbb{Z}$ called the *degree* of ω. The function $\pi_1(S^1) \to \mathbb{Z}; \omega \mapsto \text{degree}(\omega)$ is an isomorphism. [AT29]

4–14. Every loop $\omega : S^1 \to \mathbb{C}\backslash\{0\}$ is homotopic to $\omega_n : S^1 \to S^1 \subset \mathbb{C}\backslash\{0\}$ for a unique $n \in \mathbb{Z}$ called the *winding number* of ω. Cauchy's theorem computes the winding number as a closed contour integral

$$\frac{1}{2\pi i}\oint_{\omega} \frac{dz}{z} \;=\; n\ .$$

4–15. The n-sphere S^n has $\pi_1(S^n) = \{1\}$ for $n \geqslant 2$. [AT35]

4–16. The n-dimensional projective space $\mathbb{RP}^n$ has $\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2$ for $n \geqslant 2$. [AT74]

4–17. The fundamental group of the closed orientable surface M_g of genus $g \geqslant 0$ has $2g$ generators and 1 relation

$$\pi_1(M_g) \;=\; \{a_1, b_1, \dots, a_g, b_g \,|\, [a_1, b_1] \dots [a_g, b_g]\}$$

with $[a, b] = a^{-1}b^{-1}ab$ the commutator of a, b. In particular, $M_0 = S^2$ is the sphere, with $\pi_1(M_0) = \{1\}$, and $M_1 = S^1 \times S^1$ is the torus with $\pi_1(M_1) = \mathbb{Z} \oplus \mathbb{Z}$, the free abelian group on 2 generators. Since the groups $\pi_1(M_g)$ ($g \geqslant 0$) are all non-isomorphic, the surfaces M_g are non-homeomorphic. [AT51]

4–18. If $K : S^1 \subset S^3$ is a knot then $\pi_1(S^3\backslash K(S^1))$ is a topological invariant of the knot. For example, if $K_0 : S^1 \subset S^3$ is the trivial knot and $K_1 : S^1 \subset S^3$ is the trefoil knot then

$$\pi_1(S^3\backslash K_0(S^1)) \;=\; \mathbb{Z}\ ,\ \pi_1(S^3\backslash K_1(S^1)) \;=\; \{a, b \,|\, aba = bab\} \qquad \text{[AT55]}$$

These groups are not isomorphic (since one is abelian and the other one is not abelian), so that K_0, K_1 are essentially distinct knots. In particular, this algebra shows that the trefoil cannot be unknotted.

4–19. If $L = S^1 \cup \dots \cup S^1 \subset S^3$ is a link (= knot in the case of a single S^1) then $\pi_1(S^3\backslash L(S^1 \cup \dots \cup S^1))$ is a topological invariant of the link. For example, if $L_0 : S^1 \cup S^1 \subset S^3$ is the trivial link then $\pi_1(S^3\backslash L_0(S^1 \cup S^1)) = \mathbb{Z} * \mathbb{Z}$ is the free nonabelian group on 2 generators, while if $L_1 : S^1 \cup S^1 \subset S^3$ is the simplest non-trivial link then $\pi_1(S^3\backslash L_1(S^1 \cup S^1)) = \mathbb{Z} \oplus \mathbb{Z}$. [AT24,47]

The Seifert-van Kampen Theorem states that the fundamental group of a union $X = X_1 \cup X_2$ of path-connected spaces X_1, X_2 with the intersection $Y = X_1 \cap X_2$ path-connected is the amalgamated free product $\pi_1(X) = \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2)$. [AT43]

4–20. Example: The figure 8 has $\pi_1(8) = \mathbb{Z} * \mathbb{Z}$. [AT40,77]

Every group G is the fundamental group $G = \pi_1(X)$ of some path-connected space X, and every group morphism $\phi : G \to H$ is the induced morphism $\phi = f_*$ of a continuous map $f : X \to Y$ with $\pi_1(X) = G$, $\pi_1(Y) = H$. [AT89]

Every set has the 'discrete' topology, in which every subset is open (Example 2.2.3). A 'covering' of a space X with 'fibre' a discrete space F is a continuous map $p : \widetilde{X} \to X$ such that for each $x \in X$ there exists an open subset $U \subseteq X$ with $x \in U$, and with a homeomorphism $\phi : F \times U \to p^{-1}(U)$ such that $p\phi(a, u) = u \in U \subseteq X$ for all $a \in F$, $u \in U$. As a set $\widetilde{X} = X \times F$, but it is the topology on $\widetilde{X}$ which makes the covering interesting.

Let us informally call a space 'reasonable' if it is a simplicial complex (e.g. a manifold) or more generally a 'Δ-complex' in the sense of [AT102]. A reasonable space X which is path-connected has a 'universal covering' $p : \widetilde{X} \to X$, which is a covering with $\widetilde{X}$ simply-connected. [AT64]

There are two key results for universal covers:

4–21. The fibre of a universal covering $p : \widetilde{X} \to X$ is the fundamental group $\pi_1(X)$, and there is defined an isomorphism of groups

$$\pi_1(X) \cong \text{Homeo}_p(\widetilde{X})$$

with $\text{Homeo}_p(\widetilde{X})$ the group of homeomorphisms $h : \widetilde{X} \to \widetilde{X}$ such that $ph = p : \widetilde{X} \to X$, called the 'covering translations'.

4–22. For a path-connected space X with a universal covering $p : \widetilde{X} \to X$ every subgroup $G \subseteq \pi_1(X)$ determines a covering projection

$$p_G \ : \ \widetilde{X}/G \ = \ \widetilde{X}/\{x \sim y \text{ if } y = xg \text{ for some } g \in G\} \to X \ ; \ x \mapsto p(x) \ .$$

The fibre of p_G is the set $[\pi_1(X); G]$ of left G-cosets $xG \subseteq \pi_1(X)$ $(x \in \pi_1(X))$, and

$$(p_G)_* \ = \ \text{inclusion} \ : \ \pi_1(\widetilde{X}/G) \ = \ G \to \pi_1(X) \ .$$

Moreover, if $q : Y \to X$ is an arbitrary covering of X with Y path-connected, then there exists a subgroup $G \subseteq \pi_1(X)$ such that $q = p_H$, $Y = \widetilde{X}_H$, and the fibre is $F = [\pi_1(X); G]$. There is a one-one correspondence between coverings $q : Y \to X$ with Y path-connected and the conjugacy classes of subgroups $G \subseteq \pi_1(X)$. By definition, two subgroups $G, G' \subseteq \pi_1(X)$ are *conjugate* if $G' = xGx^{-1}$ for some $x \in \pi_1(X)$. [AT67]

The simplest non-trivial example of a covering is:

4–23. The real line $\mathbb{R}$ is simply-connected and the function $p : \mathbb{R} \to S^1; t \mapsto e^{2\pi i t}$ is a universal covering, with $\text{Homeo}_p(\mathbb{R}) = \mathbb{Z}$ the infinite cyclic group generated by $\mathbb{R} \to \mathbb{R}; x \mapsto x + 1$. [AT56]

Note how much easier it is easier to compute $\text{Homeo}_p(\mathbb{R}) = \mathbb{Z}$ than $\pi_1(S^1) = \mathbb{Z}$ directly from the definition!

The 'nth homotopy group' $\pi_n(X, x)$ is defined geometrically to be the set of homotopy classes of continuous maps $\omega : S^n \to X$ such that $\omega(1) = x \in X$, just like $\pi_1(X, x)$ but for all $n \geqslant 1$. As for $n = 1$, if $x_0, x_1 \in X$ are in the same path component then $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ are isomorphic. For a path-connected space X $\pi_n(X)$ denotes any one of the isomorphic groups $\pi_n(X, x)$ $(x \in X)$.

Here are some facts about the higher homotopy groups $\pi_*(X)$:

4–24. For $n \geqslant 2$ $\pi_n(X)$ is abelian with a $\pi_1(X)$-action. [AT340]

4–25. If X is contractible (= homotopy equivalent to a point) then $\pi_*(X) = 0$, e.g. $\pi_*(\mathbb{R}^m) = 0$.

4–26. $\pi_n(\mathbb{R}^{m+1}\backslash\{0\}) = \pi_n(S^m) = \begin{cases} \mathbb{Z} & \text{if } n = m \\ 0 & \text{if } n < m. \end{cases}$ [AT349,361]

4–27. $\pi_3(S^2) = \mathbb{Z}$ (Hopf, 1926) [AT474]

4–28. Although the homotopy groups $\pi_n(S^m)$ for $n > m$ have been studied intensively for the last 70 years, they are still largely unknown!

4–29. $\pi_n(X) = \pi_{n-1}(\Omega X)$ with $\Omega X = (X, x)^{S^1}$ the space of loops in X at $x \in X$, so that for $n \geqslant 2$

$$\pi_n(X) \;=\; \pi_1(\Omega^{n-1}X)$$

with $\Omega^{n-1}X = \Omega\Omega \dots \Omega X$. [AT395]

4–30. A continuous map $f : X \to Y$ induces group morphisms $f_* : \pi_*(X) \to \pi_*(Y)$ such that

$$(1_X)_* \;=\; 1_{\pi_*(X)} \;:\; \pi_*(X) \to \pi_*(X) \;,\; (gf)_* \;=\; g_*f_* \;:\; \pi_*(X) \to \pi_*(Z)$$

with $g : Y \to Z$. If f is a homotopy equivalence then the f_* are isomorphisms.

4–31. A map of reasonable path-connected spaces $f : X \to Y$ is a homotopy equivalence if and only if the morphisms $f_* : \pi_*(X) \to \pi_*(Y)$ are isomorphisms. [AT346]

Here is a consequence of **4–26**: if $m \neq n$ then the Euclidean spaces $\mathbb{R}^m, \mathbb{R}^n$ cannot be homeomorphic. For if there existed a homeomorphism then $\mathbb{R}^m\backslash\{0\}$ would be homeomorphic to $\mathbb{R}^n\backslash\{0\}$ and

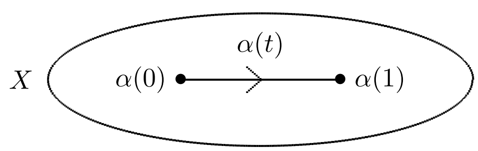
$$\pi_{m-1}(\mathbb{R}^m\backslash\{0\}) \;=\; \mathbb{Z} \;=\; \pi_{m-1}(\mathbb{R}^n\backslash\{0\}) \;=\; 0 \;,$$

a contradiction.

4.2 The fundamental group $\pi_1(X)$

As already indicated in the Introduction, the construction of the fundamental group uses paths and homotopies, which we now define.

Definition 4.2.1 (i) A *path* in a space X is a continuous map $\alpha : I = [0,1] \to X$, with starting and end points $\alpha(0), \alpha(1) \in X$.

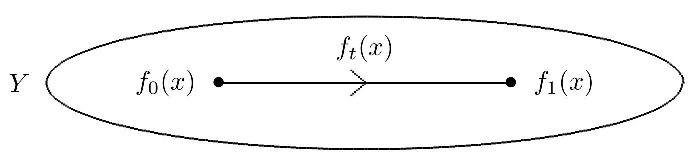


(ii) A *homotopy* between continuous maps $f_0, f_1 : X \to Y$ is a continuous map

$$f \;:\; X \times I \to Y \;;\; (x,t) \mapsto f(x,t) \;=\; f_t(x) \;.$$

□

Think of a homotopy as a single 'take' in a film, with f_t the position of the actors at time t, starting at f_0 and ending at f_1.



Example 4.2.2 If $X = \{x\}$ is a space with one element x, a continuous map $f : X \to Y$ is the same as an element $f(x) \in Y$. A homotopy $h : f \simeq g : X \to Y$ is the same as a path $h : I \to Y$ with initial point $h(0) = f(x) \in Y$ and terminal point $h(1) = g(x) \in Y$. A homotopy $h : f \simeq f : X \to Y$ is the same as a closed path $h : I \to Y$. □ □

Proposition 4.2.3 *For fixed X, Y the notion of homotopy is an equivalence relation on the set of continuous maps $f : X \to Y$.*

Proof (i) For every continuous map $f : X \to Y$ define the *constant homotopy* $h : f \simeq f : X \to Y$ by

$$h \ : \ X \times I \to Y \ ; \ (x,t) \mapsto f(x)$$

(ii) Given a homotopy $h : f \simeq g : X \to Y$ define the *reverse homotopy* $-h : g \simeq f : X \to Y$ by

$$-h \ : \ X \times I \to Y \ ; \ (x,t) \mapsto h(x, 1-t) \ .$$

(iii) Given homotopies $h_1 : f_1 \simeq f_2 : X \to Y$ and $h_2 : f_2 \simeq f_3 : X \to Y$ define the *concatenation homotopy* $h_1 \bullet h_2 : f_1 \simeq f_3 : X \to Y$

$$h_1 \bullet h_2 \ : \ X \times I \to Y \ ; \ (x,t) \mapsto \begin{cases} h_1(x, 2t) & \text{if } 0 \leqslant t \leqslant 1/2 \\ h_2(x, 2t-1) & \text{if } 1/2 \leqslant t \leqslant 1 \ . \end{cases}$$

$h_1 \bullet h_2(x,0) = f_1(x)$ $\quad$ $h_1 \bullet h_2(x,1/2) = f_2(x)$ $\quad$ $h_1 \bullet h_2(x,1) = f_3(x)$

$h_1(x,-)$ at twice the speed $\quad$ $h_2(x,-)$ at twice the speed

□

In general, geometry is used to construct homotopies, and algebra is used to show that homotopies with certain properties cannot exist.

Definition 4.2.4 Two spaces X, Y are *homotopy equivalent* if there exist continuous maps $f : X \to Y$, $g : Y \to X$ and homotopies

$$h \ : \ gf \simeq 1_X \ : \ X \to X \ , \ k \ : \ fg \simeq 1_Y \ : \ Y \to Y \ .$$

A continuous map $f : X \to Y$ is a *homotopy equivalence* if there exist such g, h, k. The continuous maps f, g are *inverse homotopy equivalences.* □

Example 4.2.5 The inclusion $f : S^n \to \mathbb{R}^{n+1} \backslash \{0\}$ is a homotopy equivalence, with homotopy inverse

$$g \ : \ \mathbb{R}^{n+1} \backslash \{0\} \to S^n \ ; \ x \mapsto \frac{x}{\|x\|} \ .$$

□

The relation defined on the set of topological spaces by

$$X \ \simeq \ Y \text{ if } X \text{ is homotopy equivalent to } Y$$

is an equivalence relation.

Definition 4.2.6 A space X is *contractible* if it is homotopy equivalent to $\{\text{pt.}\}$. □

Example 4.2.7 (i) A subset $X \subseteq \mathbb{R}^n$ is convex if for any $x, y \in X$ the line segment

$$[x, y] \ = \ \{(1-t)x + ty \,|, \ 0 \leqslant t \leqslant 1\}$$

is contained in X. Then X is contractible.
(ii) The n-dimensional Euclidean space $\mathbb{R}^n$ is contractible, by (i).
(iii) The unit n-ball $D^n = \{x \in \mathbb{R}^n \,|\, \|x\| \leqslant 1\}$ is contractible, by (i). □

Definition 4.2.8 (i) A *closed path at* $x \in X$ is a path $\alpha : I \to X$ such that $\alpha(0) = \alpha(1) = x \in X$.
(ii) A *loop at* $x \in X$ is a continuous map $\omega : S^1 \to X$ such that $\omega(1) = x \in X$. □

Use the homeomorphism

$$[0,1]/(0 \sim 1) \to S^1 \ ; \ [t] \mapsto e^{2\pi i t} \ .$$

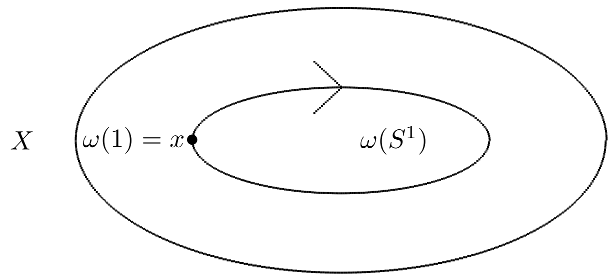
We have that a closed path $\alpha : I \to X$ at $x \in X$ is essentially the same as a loop $\omega : S^1 \to X$ at $x \in X$, with

$$\alpha(t) \ = \ \omega(e^{2\pi i t}) \in X \ .$$

Definition 4.2.9 (i) A *based space* (X, x) is a space with a base point $x \in X$.
(ii) A *based continuous map* $f : (X, x) \to (Y, y)$ is a continuous map $f : X \to Y$ such that $f(x) = y \in Y$.
(iii) A *based homotopy* $h : f \simeq g : (X, x) \to (Y, y)$ is a homotopy $h : f \simeq g : X \to Y$ such that $h(x, t) = y \in Y$ $(t \in I)$. □

For any based spaces (X, x), (Y, y) based homotopy is an equivalence relation on the set of based continuous maps $f : (X, x) \to (Y, y)$.

Definition 4.2.10 *A* based loop *is a based continuous map* $\omega : (S^1, 1) \to (X, x)$ *where* $1 = (1, 0) \in S^1$.



□

Homotopy theory uses the topological properties of closed paths $I \to X$ and loops $S^1 \to X$ and the algebraic properties of groups to decide whether topological spaces are homotopy equivalent. Since I is contractible any two paths $I \to X$ are homotopic. It is necessary to keep the endpoints fixed!
The fundamental group $\pi_1(X, x)$ will be defined, for any space X and point $x \in X$, to be the set of 'rel $\{0, 1\}$ homotopy classes' of closed paths $\alpha : [0, 1] \to X$ such that

$$\alpha(0) \;=\; \alpha(1) \;=\; x \in X \;,$$

with appropriate group law and inversion. What does 'rel $\{0, 1\}$' mean?

Definition 4.2.11 If $f, g : X \to Y$ are continuous maps and $A \subseteq X$ is a subspace such that

$$f(a) \;=\; g(a) \in Y \quad (a \in A)$$

then a *homotopy rel* A (or *relative* to A) is a homotopy $h : f \simeq g : X \to Y$ such that

$$h(a, t) \;=\; f(a) \;=\; g(a) \in Y \quad (a \in A, t \in I) \;.$$

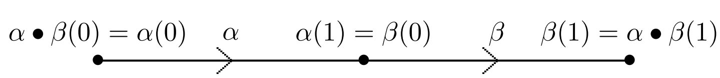
□

The rel $\{0, 1\}$ homotopy classes of closed paths $\alpha : I \to X$ such that $\alpha(0) = \alpha(1) = x \in X$ are in one-one correspondence with the rel $\{1\}$ homotopy classes of loops $\omega : S^1 \to X$ with $\omega(1) = x \in X$.

Definition 4.2.12 The *concatenation* of paths $\alpha : I \to X$, $\beta : I \to X$ with $\alpha(1) = \beta(0) \in X$ is the path

$$\alpha \bullet \beta \;:\; I \to X \;;\; t \mapsto \begin{cases} \alpha(2t) & \text{if } 0 \leqslant t \leqslant 1/2 \\ \beta(2t - 1) & \text{if } 1/2 \leqslant t \leqslant 1 \end{cases}$$

which starts at $\alpha(0)$, follows along α at twice the speed in the first half, switching at $\alpha(1) = \beta(0)$ (at half-time) to follow β at twice the speed in the second half.

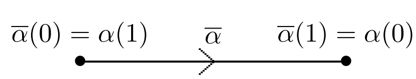


□

Definition 4.2.13 The *reverse* of a path $\alpha : I \to X$ is the path

$$\overline{\alpha} \;:\; I \to X \;;\; t \mapsto \alpha(1 - t)$$

retracing α, with



□

Definition 4.2.14 The *fundamental group* $\pi_1(X,x)$ is the set of rel $\{0,1\}$ homotopy classes $[\alpha]$ of closed paths $\alpha : I \to X$ such that

$$\alpha(0) \;=\; \alpha(1) \;=\; x \in X$$

with group law

$$\pi_1(X,x) \times \pi_1(X,x) \to \pi_1(X,x) \;;\; ([\alpha],[\beta]) \mapsto [\alpha][\beta] = [\alpha \bullet \beta] \;,$$

inversion by

$$\pi_1(X,x) \to \pi_1(X,x) \;;\; [\alpha] \mapsto [\alpha]^{-1} \;=\; [\overline{\alpha}]$$

and neutral element $[e_x] \in \pi_1(X,x)$ the class of the constant path

$$e_x \;:\; I \to X \;;\; t \mapsto x \;.$$

□

It is of course also possible to regard $\pi_1(X,x)$ as the set of rel $\{1\}$ homotopy classes $[\omega]$ of loops $\omega : S^1 \to X$ such that $\omega(1) = x \in X$. The path formulation is more convenient for algebra, while the loops are more geometric.

Theorem 4.2.15 *The fundamental group $\pi_1(X,x)$ is a group.*

Proof that $[\alpha][e_x] = [\alpha] \in \pi_1(X,x)$.
Define a rel $\{0,1\}$ homotopy

$$h \;:\; \alpha \bullet e_x \simeq \alpha \;:\; I \to X$$

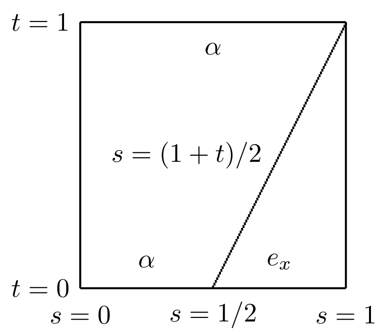
by

$$h \;:\; I \times I \to X \;;\; (s,t) \mapsto \begin{cases} \alpha(2s/(1+t)) & \text{if } s \leqslant (1+t)/2 \\ p & \text{if } s \geqslant (1+t)/2 \;. \end{cases}$$

To make sense of this formula draw the unit square in the (s,t)-plane and join the point $(1/2,0)$ to the point $(1,1)$ by the line $s = (1+t)/2$. Think what happens at each time $t \in I$: the continuous map

$$h_t \;:\; I \to X \;;\; s \mapsto h_t(s) = h(s,t)$$

starts by going along α at $2/(1+t)$ the speed on $[0,(1+t)/2]$, and then stays put at x on $[(1+t)/2,1]$. The homotopy h starts at $h_0 = \alpha \bullet e_x$ and ends at $h_1 = \alpha$.

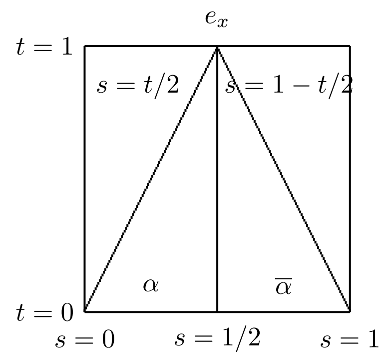


(Work out the corresponding formula for $[e_x][\alpha] = [\alpha] \in \pi_1(X,x)$.)
Proof that $[\alpha][\overline{\alpha}] = [e_x] \in \pi_1(X,x)$
Define a rel $\{0,1\}$ homotopy

$$h \;:\; \alpha \bullet \overline{\alpha} \simeq e_x \;:\; I \to X$$

by

$$h \;:\; I \times I \to X \;;\; (s,t) \mapsto \begin{cases} x & \text{if } 0 \leqslant s \leqslant t/2 \\ \alpha(2s-t) & \text{if } t/2 \leqslant s \leqslant 1/2 \\ \alpha(2-2s-t) & \text{if } 1/2 \leqslant s \leqslant 1-t/2 \\ x & \text{if } 1-t/2 \leqslant s \leqslant 1 \;. \end{cases}$$



Again, think what happens at each time $t \in I$: the path

$$h_t \ : \ I \to X \ ; \ s \mapsto h_t(s) = h(s,t)$$

is constant on $[0, t/2]$, goes along the restriction $\alpha| : [0, 1-t] \to X$ (i.e. using only a part of α) at twice the speed on $[t/2, 1/2]$, then along the restriction $\overline{\alpha}| : [t, 1] \to X$ at twice the speed on $[1/2, 1-t/2]$, and stays constant on $[1-t/2, 1]$. Note that $\alpha(1-t) = \overline{\alpha}(t)$ is essential for continuity. The homotopy h starts at $h_0 = \alpha \bullet \overline{\alpha}$ and ends at $h_1 = e_x$. (Work out the corresponding formula for $[\overline{\alpha}][\alpha] = [e_x]$.)

Proof that $([\alpha][\beta])[\gamma] = [\alpha]([\beta][\gamma]) \in \pi_1(X, x)$ **(associativity of multiplication)**

Let $\alpha, \beta, \gamma : I \to X$ be paths which send each endpoint to $x \in X$. For $0 < \lambda < \mu < 1$ let $c(\lambda, \mu) : I \to X$ be the path defined by

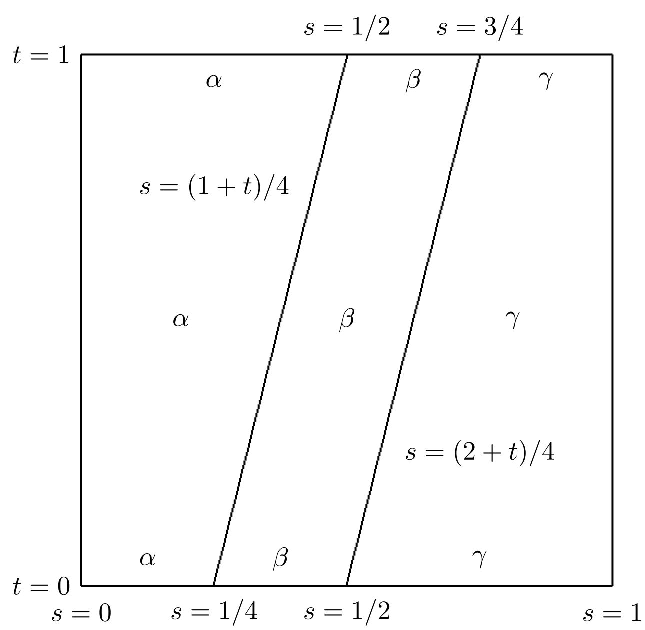
$$c(\lambda,\mu) \ : \ I \to X \ ; \ s \mapsto \begin{cases} \alpha(s/\lambda) & \text{if } 0 \leqslant s \leqslant \lambda \\ \beta((s-\lambda)/(\mu-\lambda)) & \text{if } \lambda \leqslant s \leqslant \mu \\ \gamma((s-\mu)/(1-\mu)) & \text{if } \mu \leqslant s \leqslant 1 \ . \end{cases}$$

The path starts by going along α at $1/\lambda$ the speed on $[0, \lambda]$, followed by going along β at $1/(\mu - \lambda)$ the speed on $[\lambda, \mu]$, and finish by going along γ at $1/(1-\mu)$ the speed on $[\mu, 1]$. From the definitions

$$([\alpha][\beta])[\gamma] \ = \ c(1/4, 1/2) \ : \ I \to X \ ; \ s \mapsto \begin{cases} \alpha(4s) & \text{if } 0 \leqslant s \leqslant 1/4 \\ \beta(4s-1) & \text{if } 1/4 \leqslant s \leqslant 1/2 \\ \gamma(2s-1) & \text{if } 1/2 \leqslant s \leqslant 1 \end{cases}$$

and

$$[\alpha]([\beta][\gamma]) \ = \ c(1/2, 3/4) \ : \ I \to X \ ; \ s \mapsto \begin{cases} \alpha(2s) & \text{if } 0 \leqslant s \leqslant 1/2 \\ \beta(4s-2) & \text{if } 1/2 \leqslant s \leqslant 3/4 \\ \gamma(4s-3) & \text{if } 3/4 \leqslant s \leqslant 1 \ . \end{cases}$$



Finally, construct a homotopy rel $\{0,1\}$

$$h \ : \ ([\alpha][\beta])[\gamma] \simeq [\alpha]([\beta][\gamma]) \ : \ I \to X$$

by

$$\begin{aligned} h_t &= c((1-t)/4 + t/2, (1-t)/2 + t(3/4)) \\ &= c((1+t)/4, (2+t)/4) \ : \ I \to X \end{aligned}$$

with $h_0 = c(1/4, 1/2)$, $h_1 = c(1/2, 3/4)$. End of proof of 4.2.15.

The fundamental group $\pi_1(X,x)$ of a space X at a point $x \in X$ is defined geometrically, in terms of paths $\alpha : I \to X$ such that $\alpha(0) = \alpha(1) = x$, or equivalently in terms of loops $\omega : S^1 \to X$ such that $\omega(1) = x \in X$. A calculation of $\pi_1(X,x)$ is an algebraic description. In general, it is quite difficult to compute $\pi_1(X,x)$, unless there is a geometric reason for it to be the trivial group $\{1\}$.

A space determines a group. A continuous map $f : X \to Y$ induces a group morphism

$$f_* \ : \ \pi_1(X,x) \to \pi_1(Y, f(x)) \ ; \ [\alpha] \mapsto [f\alpha]$$

for any $x \in X$.

Definition 4.2.16 Let X be a space, and $x \in X$. A continuous map $f : X \to Y$ is a *homotopy equivalence rel* $\{x\}$ if there exists a continuous map $g : Y \to X$ such that $g(f(x)) = x$, a homotopy rel $\{x\}$ $h : gf \simeq 1_X : X \to X$ (with $h(x,t) = f(x)$ for $t \in I$) and a homotopy rel $\{f(x)\}$ $k : fg \simeq 1_Y : Y \to Y$. □

Proposition 4.2.17 *(i) If $f, g : X \to Y$ are continuous maps which are related by a rel $\{x\}$ homotopy $h : f \simeq g : X \to Y$ then*

$$f_* \ = \ g_* \ : \ \pi_1(X,x) \to \pi_1(Y, f(x)) \ .$$

(ii) If $f : X \to Y$ is a homotopy equivalence rel $\{x\}$ then f_* is an isomorphism, with inverse

$$(f_*)^{-1} \ = \ g_* \ : \ \pi_1(Y, f(x)) \to \pi_1(X,x) \ .$$

Remark 4.2.18 If $f : X \to Y$ is a homotopy equivalence (not just rel $\{x\}$) then $f_* : \pi_1(X,x) \to \pi_1(Y, f(x))$ is an isomorphism. □

4.3 Covering spaces

Definition 4.3.1 A *covering space* of a space X with *fibre* the discrete space F is a space $\widetilde{X}$ with a *covering projection* continuous map $p : \widetilde{X} \to X$ such that for each $x \in X$ there exists an open subset $U \subseteq X$ with $x \in U$, and with a homeomorphism $\phi : F \times U \to p^{-1}(U)$ such that

$$p\phi(a,u) \ = \ u \in U \subseteq X \quad (a \in F, u \in U) \ .$$

In particular, for each $x \in X$ $p^{-1}(x)$ is homeomorphic to F. [AT56]□

A covering projection $p : \widetilde{X} \to X$ is a 'local homeomorphism': for each $\widetilde{x} \in \widetilde{X}$ there exists an open subset $U \subseteq \widetilde{X}$ such that $\widetilde{x} \in U$ and $U \to p(U); u \mapsto p(u)$ is a homeomorphism, with $p(U) \subseteq X$ an open subset.

Definition 4.3.2 Given a covering projection $p : \widetilde{X} \to X$ let $\mathrm{Homeo}_p(\widetilde{X})$ be the subgroup of $\mathrm{Homeo}(\widetilde{X})$ consisting of the homeomorphisms $h : \widetilde{X} \to \widetilde{X}$ such that $ph = p : \widetilde{X} \to X$, i.e. such that the diagram

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "h"] \arrow[dr, "p"'] & & \widetilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

is commutative. Such h are called *covering translations*. □

Definition 4.3.3 A covering projection $p : \widetilde{X} \to X$ with fibre F is *trivial* if there exists a homeomorphism $\phi : F \times X \to \widetilde{X}$ such that

$$p\phi(a,x) \;=\; x \in X \quad (a \in F,\ x \in X)\ .$$

A particular choice of ϕ is a *trivialisation* of p. □

Example 4.3.4 (i) For any space X and discrete space F the covering projection

$$p \;:\; \widetilde{X} \;=\; F \times X \to X \;;\; (a,x) \mapsto x$$

is trivial, with the identity trivialization $\phi = 1 : F \times X \to \widetilde{X}$.
(ii) The continuous map

$$p \;:\; \mathbb{R} \to S^1 \;;\; x \mapsto e^{2\pi i x}$$

is a covering projection with fibre $\mathbb{Z}$. Note that p is not trivial, since $\mathbb{R}$ is not homeomorphic to $\mathbb{Z} \times S^1$ (although there does exist a bijection $\phi : \mathbb{Z} \times S^1 \cong \mathbb{R}$ such that $p\phi : \mathbb{Z} \times S^1 \to S^1$ is the projection). The group of covering translations is the infinite cyclic group $\mathbb{Z}$. □

Definition 4.3.5 Let $p : \widetilde{X} \to X$ be a covering projection. A *lift* of a continuous map $f : Y \to X$ is a continuous map $\widetilde{f} : Y \to \widetilde{X}$ such that

$$p(\widetilde{f}(y)) \;=\; f(y) \in X \quad (y \in Y)$$

so that there is defined a commutative diagram

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

□

Example 4.3.6 For the trivial covering projection $p : \widetilde{X} = F \times X \to X$ of Example 4.3.3 define a lift of any continuous map $f : Y \to X$ by choosing a point $a \in F$ and setting

$$\widetilde{f}_a \;:\; Y \to \widetilde{X} \;=\; F \times X \;;\; y \mapsto (a, f(y))\ .$$

If Y is path-connected every lift of f is of this type, and the function $a \mapsto \widetilde{f}_a$ defines a one-one correspondence between the points $a \in F$ and the lifts $\widetilde{f}$ of f. □

Theorem 4.3.7 (Path lifting property) *Let $p : \widetilde{X} \to X$ be a covering projection with fibre F. Let $x_0 \in X$, $\widetilde{x}_0 \in \widetilde{X}$ be such that $p(\widetilde{x}_0) = x_0 \in X$.*
(i) *Every path $\alpha : I \to X$ with $\alpha(0) = x_0 \in X$ has a unique lift to a path $\widetilde{\alpha} : I \to \widetilde{X}$ such that $\widetilde{\alpha}(0) = \widetilde{x}_0 \in \widetilde{X}$.*
(ii) *Let $\alpha, \beta : I \to X$ be paths with $\alpha(0) = \beta(0) = x_0 \in X$, and let $\widetilde{\alpha}, \widetilde{\beta} : I \to \widetilde{X}$ be the lifts with $\widetilde{\alpha}(0) = \widetilde{\beta}(0) = \widetilde{x}_0 \in \widetilde{X}$ given by* (i). *Every rel $\{0,1\}$ homotopy*

$$h \;:\; \alpha \;\simeq\; \beta \;:\; I \to X$$

has a unique lift to a rel $\{0,1\}$ homotopy

$$\widetilde{h} \;:\; \widetilde{\alpha} \;\simeq\; \widetilde{\beta} \;:\; I \to \widetilde{X}$$

and in particular

$$\widetilde{\alpha}(1) \;=\; \widetilde{h}(1,t) \;=\; \widetilde{\beta}(1) \in \widetilde{X} \quad (t \in I)\ .$$ [AT60]

Definition 4.3.8 Given a covering projection $p : \widetilde{X} \to X$ and a path $\alpha : I \to X$ use the path lifting property (4.3.7) to define the *fibre transport* bijection

$$\alpha_{\#} \;:\; p^{-1}(\alpha(0)) \to p^{-1}(\alpha(1)) \;;\; \widetilde{x} \mapsto \widetilde{\alpha}_{\widetilde{x}}(1)$$

where $\widetilde{\alpha}_{\widetilde{x}} : I \to \widetilde{X}$ is the unique lift of α with

$$\widetilde{\alpha}_{\widetilde{x}}(0) \;=\; \widetilde{x} \in \widetilde{X}\ .$$ □

Example 4.3.9 (i) For any discrete space F and permutation (= self-bijection) $\sigma : F \to F$ define a covering $p : \widetilde{S}^1 \to S^1$ with fibre F by

$$p \ : \ \widetilde{S}^1 \ = \ F \times I/\{(x,0) \sim (\sigma(x),1)\} \to S^1 = I/\{0 \sim 1\} \ ; \ [x,t] \mapsto [t] \ .$$

In fact, every covering $p : \widetilde{S}^1 \to S^1$ arises in this way: define the closed path

$$\alpha \ : \ I \to S^1 \ ; \ t \mapsto e^{2\pi i t}$$

with $\alpha(0) = \alpha(1) = 1 \in S^1$, and note that the fibre transport is a bijection

$$\alpha_{\#} \ : \ F \ = \ p^{-1}(1) \to F \ = \ p^{-1}(1)$$

such that

$$p \ : \ \widetilde{S}^1 \ = \ F \times I/\{(x,0) \sim (\alpha_{\#}(x),1)\} \to S^1 = I/\{0 \sim 1\} \ ; \ [x,t] \mapsto [t] \ .$$

(ii) Exercise: verify that the covering $p : \widetilde{S}^1 \to S^1$ corresponding to the cyclic permutation

$$\sigma \ : \ F \ = \ \{1,2,\dots,n\} \to F \ ; \ x \mapsto (x+1)(\mathrm{mod}\, n)$$

is just

$$p \ : \ \widetilde{S}^1 \ = \ S^1 \to S^1 \ ; \ z \mapsto z^n \ .$$

□

Proposition 4.3.10 *A covering projection $p : Y \to X$ of path-connected spaces induces an injective group morphism $p_* : \pi_1(Y) \to \pi_1(X)$.*

Proof If $\omega : S^1 \to Y$ is a loop at $y \in Y$ such that there exists a homotopy $h : p\omega \simeq e_{p(y)} : S^1 \to X$ rel 1, then h can be lifted to a homotopy $\widetilde{h} : \omega \simeq e_y : S^1 \to Y$ rel 1, by the relative version of 4.3.7. □

Recall that a subgroup $H \subseteq G$ is *normal* if $xH = Hx$ for all $x \in G$, in which case there is defined a quotient group G/H with a canonical surjection $G \to G/H$.

Definition 4.3.11 A covering projection $p : Y \to X$ of path-connected spaces is *regular* if $p_*(\pi_1(Y)) \subseteq \pi_1(X)$ is a normal subgroup. □

Here is a very general construction of regular covering projections:

Theorem 4.3.12 *Given a space Y and a subgroup $G \subseteq$ Homeo(Y) define an equivalence relation $\sim$ on Y by*

$$y_1 \sim y_2 \text{ if there exists } g \in G \text{ such that } y_2 = g(y_1)$$

and write

$$p \ : \ Y \to X \ = \ Y/{\sim} \ = \ Y/G \ ; \ y \mapsto p(y) \ = \ \text{equivalence class of } y \ .$$

Suppose that for each $y \in Y$ there exists an open subset $U \subseteq Y$ such that $y \in U$ and

$$g(U) \cap U = \emptyset \text{ for } g \neq 1 \in G \ .$$

(Such an action of a group G on a space Y is called free *and* properly discontinuous*, as in 2.4.6). Then $p : Y \to X$ is a covering projection with fibre G. Furthermore, if Y is path-connected then so is X, p is a regular covering and the group of covering translations of p is* Homeo$_p(Y) = G \subset$ Homeo(Y). [AT61,72]

Proof The subset $p(U) \subseteq X$ is open, since

$$p^{-1}p(U) \ = \ \{gu \,|\, g \in G, u \in U\} \ = \ \bigcup_{g \in G} g(U) \subseteq Y$$

is open, with an evident homeomorphism

$$\phi \ : \ G \times U \to p^{-1}p(U) \ ; \ (g,u) \mapsto gu \ .$$

If $h \in$ Homeo$_p(Y)$ then for any $y \in Y$ there is a unique $g_y \in G$ such that $h(y) = g_y(y) \in Y$ $(y \in Y)$. If Y is path-connected the continuous map $Y \to G; y \mapsto g_y$ is constant (since G is discrete), so $g_y = h \in$ Homeo$_p(Y) = G$. □

Remark 4.3.13 Every regular covering projection $p : Y \to X$ with X, Y path-connected arises as in Theorem 4.3.12 from a free action of a group $G = \pi_1(X)/p_*(\pi_1(Y))$ on Y, or equivalently from a surjection $\pi_1(X) \to G$. □

Theorem 4.3.14 *For a regular covering projection $p : Y \to X$ there is defined an isomorphism of groups*

$$\pi_1(X)/p_*(\pi_1(Y)) \ \cong \ \mathrm{Homeo}_p(Y) \ .$$

Proof Let $x_0 \in X$, $y_0 \in Y$ be base points such that $p(y_0) = x_0$. Every closed path $\alpha : I \to X$ with $\alpha(0) = \alpha(1) = x_0$ has a unique lift to a path $\widetilde{\alpha} : I \to Y$ such that $\widetilde{\alpha}(0) = y_0$. The function

$$\pi_1(X, x_0)/p_*\pi_1(Y, y_0) \to p^{-1}(x_0) \ ; \ \alpha \mapsto \widetilde{\alpha}(1)$$

is a bijection. For each $y \in p^{-1}(x_0)$ there is a unique covering translation $h_y \in \mathrm{Homeo}_p(Y)$ such that

$$h_y(y_0) \ = \ y \in Y \ .$$

The function

$$p^{-1}(x_0) \to \mathrm{Homeo}_p(Y) \ ; \ y \mapsto h_y$$

is a bijection, with inverse $h \mapsto h(\widetilde{x}_0)$. The composite bijection

$$\pi_1(X, x_0)/p_*(\pi_1(Y)) \to p^{-1}(x_0) \to \mathrm{Homeo}_p(Y)$$

is an isomorphism of groups. □

Example 4.3.15 For each $n \in \mathbb{Z}$ the translation of $\mathbb{R}$ by n units to the right defines a homeomorphism

$$h_n \ : \ \mathbb{R} \to \mathbb{R} \ ; \ x \mapsto x + n$$

with $h_n h_m = h_{m+n}$. The infinite cyclic subgroup

$$G \ = \ \{h_n \,|\, n \in \mathbb{Z}\} \subset \mathrm{Homeo}(\mathbb{R})$$

satisfies the hypothesis of Theorem 4.3.12, so that

$$p \ : \ \mathbb{R} \to \mathbb{R}/G \ = \ \mathbb{R}/\mathbb{Z} \ = \ S^1 \ ; \ x \mapsto e^{2\pi i x}$$

is a regular covering projection with fibre $G = \mathbb{Z}$ and by Theorem 4.3.14

$$\pi_1(S^1) \ = \ \mathrm{Homeo}_p(\mathbb{R}) \ = \ G \ = \ \mathbb{Z} \subset \mathrm{Homeo}(\mathbb{R}) \ .$$

Every loop $\omega : S^1 \to S^1$ can be lifted to a path $\alpha : I \to \mathbb{R}$ such that

$$\omega(e^{2\pi i t}) \ = \ e^{2\pi i \alpha(t)} \in S^1 \ \ (t \in I) \ .$$

The degree of ω is defined by

$$\mathrm{degree}(\omega) \ = \ \alpha(1) - \alpha(0) \in \mathbb{Z} \ .$$

The degree defines an isomorphism of groups

$$\pi_1(S^1) \to \mathbb{Z} \ ; \ \omega \mapsto \mathrm{degree}(\omega) \ .$$

A loop ω with $\mathrm{degree}(\omega) = n$ is homotopic to the standard loop with degree n

$$\omega_n \ : \ S^1 \to S^1 \ ; \ z \mapsto z^n$$

with lift $\alpha_n : I \to \mathbb{R}; t \mapsto nt$. □

Recall that a space X is *simply-connected* if it is path-connected and $\pi_1(X) = \{1\}$.

Proposition 4.3.16 *Every covering projection $p : \widetilde{X} \to X$ of a simply-connected space X is trivial.*

Proof Let F be the fibre. Choose a base point $x_0 \in X$, and an open neighbourhood $U_0 \subseteq X$ of x_0 with a trivialisation

$$\phi_0 \ : \ F \times U_0 \to p^{-1}(U_0)$$

of $p| : p^{-1}(U_0) \to U_0$, i.e. a homeomorphism such that

$$p\phi_0(a,u) \ = \ u \in X \quad (a \in F, u \in U_0) \ .$$

In particular, there is defined a bijection

$$F \to p^{-1}(x_0) \ ; \ a \mapsto \phi_0(a, x_0) \ .$$

For each $x \in X$ choose a path $\alpha_x : I \to X$ from $\alpha_x(0) = x_0$ to $\alpha_x(1) = x$, and use fibre transport (4.3.8) to define a homeomorphism

$$\phi \ : \ F \times X \to \widetilde{X} \ ; \ (a,x) \mapsto (\alpha_x)_{\#}(\phi_0(a,x_0)) \ .$$

The condition $\pi_1(X) = \{1\}$ is needed to prove that ϕ is independent of the choices of paths α_x. □

Example 4.3.17 Every covering $p : \widetilde{I} \to I$ is trivial, with a homeomorphism $\phi : F \times I \to \widetilde{I}$ such that $p\phi(a,x) = x$. □

Definition 4.3.18 A covering projection $p : \widetilde{X} \to X$ of a path-connected space X is *universal* if $\widetilde{X}$ is simply-connected. □

Example 4.3.19 The covering projection $p : \mathbb{R} \to S^1$ is universal. □

A space X is *locally path connected* if for each $x \in X$ and for each open subset $U \subseteq X$ with $x \in U$ there is a path-connected open subset $V \subseteq U$ with $x \in V$. (Main example: open subsets of $\mathbb{R}^n$).

Theorem 4.3.20 *Let X be a path-connected locally path-connected space with a universal covering projection $p : \widetilde{X} \to X$. Let $x_0 \in X$, $\widetilde{x}_0 \in \widetilde{X}$ be base points such that $p(\widetilde{x}_0) = x_0$.*
(i) *The function*

$$\pi_1(X, x_0) \to p^{-1}(x_0) \ ; \ \alpha \mapsto \alpha_{\#}(\widetilde{x}_0)$$

is a bijection.
(ii) *For each $y \in p^{-1}(x_0)$ there is a unique covering translation $h_y \in \mathrm{Homeo}_p(\widetilde{X})$ such that*

$$h_y(\widetilde{x}_0) \ = \ y \in \widetilde{X} \ .$$

The function

$$p^{-1}(x_0) \to \mathrm{Homeo}_p(\widetilde{X}) \ ; \ y \mapsto h_y$$

is a bijection, with inverse $h \mapsto h(\widetilde{x}_0)$. The composite bijection

$$\pi_1(X, x_0) \to p^{-1}(x_0) \to \mathrm{Homeo}_p(\widetilde{X})$$

is an isomorphism of groups. [AT61]

Remark 4.3.21 If $p : \widetilde{X} \to X$ is a universal covering projection satisfying the hypothesis of Theorem 4.3.20 then for any subgroup $G \subseteq \pi_1(X) = \mathrm{Homeo}_p(\widetilde{X})$ there is defined a universal covering projection

$$q \ : \ \widetilde{Y} \ = \ \widetilde{X} \to Y \ = \ \widetilde{X}/G$$

also satisfying the hypothesis of 4.3.20, with

$$\pi_1(Y) \ = \ \mathrm{Homeo}_q(\widetilde{Y}) \ = \ G \ .$$

The projection $r : Y \to X$ is a covering projection with

$$r_*(\pi_1(Y)) \ = \ G \subseteq \pi_1(X) \ , \ \mathrm{Homeo}_r(Y) \ = \ \pi_1(X)/N$$

where $N \subseteq \pi_1(X)$ is the smallest normal group containing G. The construction defines a one-one correspondence between the isomorphism classes of covering projections $r : Y \to X$ with Y path-connected and the conjugacy classes of subgroups $G \subseteq \pi_1(X)$. The regular covering projections correspond to the normal subgroups $G \subseteq \pi_1(X)$, with $N = G$ and

$$\mathrm{Homeo}_r(Y) \ = \ \pi_1(X)/G \ .$$

See [AT63-78] for a rather more detailed account!

Remark 4.3.22 Theorem 4.3.20 gives a geometric method for computing the fundamental group of a path-connected space X which admits a universal covering $p : \widetilde{X} \to X$, namely

$$\pi_1(X, x_0) \;=\; \mathrm{Homeo}_p(\widetilde{X}) \;=\; p^{-1}(x_0)\ .$$

For any path-connected space X and $x_0 \in X$ let $\widetilde{X}$ be the topological space of equivalence class of paths $\alpha : I \to X$ such that $\alpha(0) = x_0$, with $\alpha \sim \alpha'$ if there exists a rel $\{0,1\}$ homotopy $\beta : \alpha \simeq \alpha' : I \to X$, and

$$p \;:\; \widetilde{X} \to X\ ;\ \alpha \mapsto \alpha(1)\ .$$

It is a theorem that p is the universal covering projection of X with fibre $F = p^{-1}(x_0) = \pi_1(X, x_0)$ if X is *semi-locally simply-connected*, meaning that for every $x \in X$ there exists an open subset $U \subseteq X$ with $x \in U$ such that the inclusion $i : U \to X$ induces the trivial homomorphism $i_* = 1 : \pi_1(U, x) \to \pi_1(X, x)$ (in which case $p^{-1}(U)$ is homeomorphic to $U \times \pi_1(X, x)$). In general, this is too synthetic a construction of the universal cover to be of use in the computation of $\pi_1(X)$. In practice, a geometrically interesting space X has a geometrically interesting universal cover $\widetilde{X}$, and this can be used to compute $\pi_1(X)$. For example, a smooth atlas $\mathcal{A}$ on an m-dimensional manifold M can be used to construct a universal cover $\widetilde{M}$, which is again an m-dimensional manifold with a smooth atlas $\widetilde{\mathcal{A}}$.

4.4 The higher homotopy groups $\pi_*(X)$

The higher homotopy group $\pi_n(X, x)$ is defined for $n \geqslant 1$ to be the set of based homotopy classes of continuous maps $\omega : S^n \to X$ such that $\omega(1) = x \in X$, where $1 = (1, 0, \dots, 0) \in S^n$. In order to define the group law it is convenient to identify S^n with the quotient space $I^n/\partial I^n$, with $I^n = I \times \cdots \times I$ the unit n-cube and ∂I^n its boundary. There is an evident one-one correspondence between the continuous maps $\alpha : I^n \to X$ such that $\alpha(\partial I^n) = \{x\}$ and the continuous maps $\omega : S^n \to X$ such that $\omega(1) = x \in X$. Similarly for homotopies.

Definition 4.4.1 The *nth homotopy group* $\pi_n(X, x)$ is the set of rel ∂I^n homotopy classes of continuous maps $\alpha : I^n \to X$ such that $\alpha(\partial I^n) = \{x\}$, with the group law

$$\pi_n(X, x) \times \pi_n(X, x) \to \pi_n(X, x)\ ;\ ([\alpha], [\beta]) \mapsto [\alpha][\beta] = [\alpha \bullet \beta]$$

given by

$$\alpha \bullet \beta \;:\; I^n \to X\ ;\ (t_1, t_2, \dots, t_n) \mapsto \begin{cases} \alpha(2t_1, t_2, \dots, t_n) & \text{if } 0 \leqslant t_1 \leqslant 1/2 \\ \beta(2t_1 - 1, t_2, \dots, t_n) & \text{if } 1/2 \leqslant t_1 \leqslant 1 \end{cases}$$

and inverses by

$$\pi_n(X, x) \to \pi_n(X, x)\ ;\ [\alpha] \mapsto [-\alpha : (t_1, t_2, \dots, t_n) \mapsto \alpha(1 - t_1, t_2, \dots, t_n)]\ .$$

□

In particular, for $n = 1$ this is just the fundamental group $\pi_1(X, x)$.

The basic properties of the higher homotopy groups have already been stated in the Introduction 4.1.2.