QUADRATIC FORMS AND MANIFOLDS

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Quadratic forms

- Es gilt $f - f \sim 0$, denn diese Form hat die Koeffizienten $a_i, -a_i$, und nach dem Hilfssatz kann jedes Paar $a_i, -a_i$ durch $1, -1$ ersetzt werden. Wie jetzt leicht zu sehen ist, gilt

  Satz 6. *Die Klassen ähnlicher Formen bilden einen Ring.*

  E. Witt, *Theorie der quadratischen Formen in beliebigen Körpern* (Crelle, 1936)
- The **Witt group** $W(K)$ of a field K is the group of stable isomorphism classes of quadratic forms over K, i.e. vector spaces V over K with a nonsingular symmetric bilinear pairing

$$\phi \ : \ V \times V \to K \ , \ \phi(x,y) = \phi(y,x) \ .$$

- Stable = hyperbolic forms $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are equivalent to 0.
- The Witt group $W(\mathbb{R})$ of quadratic forms over $K = \mathbb{R}$ is isomorphic to $\mathbb{Z}$ by the isomorphism

$$\sigma \ : \ W(\mathbb{R}) \to \mathbb{Z} \ ; \ (V, \phi) \mapsto \text{signature}(V, \phi) \ .$$

Generalized Witt groups

- Quadratic forms on modules over a ring with involution A, with Wall groups $L_*(A)$.
- If $1/2 \in A$

$$L_0(A) \;=\; W(A)$$

  with $W(A)$ defined as for fields.
- If $1/2 \notin A$ $L_0(A)$ uses quadratic refinements of ϕ, but there is a forgetful map $L_0(A) \to W(A)$ which is an isomorphism modulo 8-torsion.
- Quadratic forms on A-module chain complexes, same $L_*(A)$
- Sheaves of quadratic forms over a topological space X, with Witt groups the generalized homology groups $H_*(X; \mathbf{L}(\mathbb{Z}))$. Here $\mathbf{L}(\mathbb{Z})$ is a spectrum with

$$\pi_*(\mathbf{L}(\mathbf{Z})) \;=\; L_*(\mathbb{Z}) \;.$$

Cobordism of manifolds

- The most direct application of the Witt group to manifolds is via the symmetric **intersection form** of a closed oriented $4k$-dimensional manifold M

$$\phi \ : \ H^{2k}(M;\mathbb{R}) \times H^{2k}(M;\mathbb{R}) \to \mathbb{R} \ ; \ (x,y) \mapsto \langle x \cup y, [M]\rangle \ .$$

- The **signature** of M

$$\sigma(M) \ = \ \sigma(H^{2k}(M;\mathbb{R}),\phi) \in W(\mathbb{R}) \ = \ \mathbb{Z}$$

  is a cobordism invariant: if $M = \partial N$ is the boundary of a $(4k+1)$-dimensional manifold N then $\sigma(M) = 0$.
- **Example** The intersection form of $M = S^{2k} \times S^{2k}$ is

$$(H^{2k}(M;\mathbb{R}),\phi) \ = \ (\mathbb{R} \oplus \mathbb{R}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \ .$$

  $M = \partial N$ with $N = S^{2k} \times D^{2k+1}$, and $\sigma(M) = 0 \in \mathbb{Z}$.

The homotopy types of topological manifolds

- An n-**dimensional topological manifold** M is a paracompact Hausdorff topological space such that each $x \in M$ has an open neighbourhood homeomorphic to $\mathbb{R}^n$.
- Will only consider compact oriented manifolds.
- The Browder-Novikov-Sullivan-Wall surgery theory developed in the 1960's for **differentiable** and **combinatorial** manifolds culminated in the 1970 Kirby-Siebenmann breakthrough on the structure theory of **topological manifolds** of dimension $n > 4$.
- The Whitney trick for removing singularities fails for $n = 4$ in general. Freedman (1982) extended the K-S theory to 4-dimensional topological manifolds, subject to fundamental group restrictions.
- *...topological manifolds bear the simplest possible relation to their underlying homotopy types. This is a broad statement worth testing.* (Siebenmann, ICM talk 1970)

The total surgery obstruction

- **Theorem** (A.R., 1978 –) (i) For any space X there is an exact sequence of generalized Witt groups

$$\cdots \to H_n(X;\mathbf{L}(\mathbb{Z})) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \to \mathcal{S}_n(X) \to H_{n-1}(X;\mathbf{L}(\mathbb{Z})) \to \ldots$$

  with A the **assembly** map.
  (ii) A compact polyhedron X with n-dimensional Poincaré duality has a **total surgery obstruction** $s(X) \in \mathcal{S}_n(X)$ such that $s(X) = 0$ if (and for $n > 4$ only if) X is homotopy equivalent to an n-dimensional manifold.
- Roughly speaking, $s(X)$ is the algebraic cobordism (i.e. Witt) class of a sheaf over X of quadratic forms over $\mathbb{Z}$ with the stalk at $x \in X$ the failure of the local homology groups $H_*(X, X\backslash\{x\})$ to be

$$H_*(\mathbb{R}^n,\mathbb{R}^n\backslash\{0\}) = \begin{cases} \mathbb{Z} & \text{for } * = n \\ 0 & \text{for } * \neq n\ . \end{cases}$$

The triangulation of manifolds

- Given a simplicial complex K let $|K|$ be the polyhedron.
- A **triangulation** of a topological manifold M is a finite simplicial complex K with a homeomorphism $M \cong |K|$.
- Every manifold M is homotopy equivalent to a compact polyhedron (Kirby+Siebenmann 1970).
- A combinatorial manifold is automatically triangulable. Differentiable manifolds have a canonical combinatorial structure.
- There are non-triangulable 4-dimensional manifolds (Freedman+Casson 1990).
- It is not known if there exist non-triangulable n-dimensional manifolds for $n > 4$.

Poincaré duality

- The homology and cohomology groups of an n-dimensional manifold M are related by the **Poincaré duality isomorphisms**

$$[M]\cap - \; : \; H^*(M) \;\cong\; H_{n-*}(M)$$

  with $[M] \in H_n(M)$ the fundamental class.
- An n-dimensional manifold with boundary $(M, \partial M)$ has **Poincaré-Lefschetz duality isomorphisms**

$$[M]\cap - \; : \; H^*(M) \;\cong\; H_{n-*}(M, \partial M)$$

  with $[M] \in H_n(M, \partial M)$ the fundamental class.
- Working with the universal cover $\widetilde{M}$ of M there are also $\mathbb{Z}[\pi_1(M)]$-coefficient Poincaré and Poincaré-Lefschetz duality isomorphisms.

The Browder-Novikov-Sullivan-Wall surgery theory from the modern point of view I. Manifold structures

- **Existence problem** When is a compact polyhedron X with n-dimensional $\mathbb{Z}[\pi_1(X)]$-coefficient Poincaré duality

$$H^{n-*}(\widetilde{X}) \cong H_*(\widetilde{X})$$

  homotopy equivalent to an n-dimensional manifold?
- Yes for $n = 2$, but no for $n > 2$ in general.
- BNSW+KS surgery theory provides a 2-stage obstruction for $n > 4$, working outside of X, involving maps $f : M \to X$ from manifolds M.
- Primary obstruction in topological K-theory of vector bundles to the existence of f.
- Secondary obstruction in algebraic L-theory of quadratic forms over $\mathbb{Z}[\pi_1(X)]$ to making f a homeomorphism by surgery/cobordism.

Manifold structures from the modern point of view

- **Modern existence theorem** For $n > 4$ a compact polyhedron X is homotopy equivalent to an n-dimensional manifold if and only if X has sufficient Poincaré duality.
- Can see the total surgery obstruction $s(X)$ inside X as failures of local Poincaré duality on the simplicial chain level, although still need to work outside X for the proofs.
- (A.R.) **The total surgery obstruction** (Aarhus Proceedings, Springer, 1979)
- (A.R.) **Algebraic L-theory and Topological Manifolds** (Tract, Cambridge, 1992)
- Would prefer to develop obstruction theory for any space X, using singular chains, but there are technical difficulties, see: (A.R.+M.Weiss) **On the construction and topological invariance of the Pontryagin classes** (Geometriae Dedicata, 2010)

How much Poincaré duality is sufficient?

- If X is homotopy equivalent to an n-dimensional manifold then it has $\mathbb{Z}[\pi_1(X)]$-coefficient Poincaré duality
$$H^{n-*}(\widetilde{X}) \cong H_*(\widetilde{X})$$
with $\widetilde{X}$ the universal cover of X. So $\mathbb{Z}[\pi_1(X)]$-coefficient Poincaré duality is necessary for X to be homotopy equivalent to an n-dimensional manifold.
- Since the 1960's it has been know that there exist X for each $n > 2$ with $\mathbb{Z}[\pi_1(X)]$-coefficient Poincaré duality which are not homotopy equivalent to an n-dimensional manifold.
- Thus $\mathbb{Z}[\pi_1(X)]$-coefficient Poincaré duality is in general not sufficient for X to be homotopy equivalent to an n-dimensional manifold.
- In order to make precise how much Poincaré duality is sufficient need to study the generalized Witt groups of quadratic forms on chain complexes indexed by simplicial complexes.

The Browder-Novikov-Sullivan-Wall surgery theory from the modern point of view II. Rigidity

- **Uniqueness problem** When is a homotopy equivalence of n-dimensional manifolds $f : M \to N$ homotopic to a homeomorphism?
- Again, yes for $n = 2$, but for $n > 2$ no in general.
- The 3-dimensional lens spaces provided the first examples of homotopy equivalent manifolds which are not homeomorphic.
- BNSW+KS provided a 2-stage obstruction theory for $n > 4$.
- **Modern uniqueness theorem** For $n > 4$ a homotopy equivalence $f : M \to N$ of n-dimensional manifolds is homotopic to a homeomorphism if and only if the point inverses $f^{-1}(x) \subset M$ $(x \in N)$ are sufficiently acyclic.

Uniqueness = relative existence

- The mapping cylinder of a map $f : M \to N$ is the space

$$X = (M \times [0,1] \sqcup N)/\{(x,1) \sim f(x) \,|\, x \in M\}$$

homotopy equivalent to N, with subspace

$$\partial X = M \times \{0\} \sqcup N \subset X .$$

- If $f : M \to N$ is a homotopy equivalence of n-dimensional manifolds then $(X, \partial X)$ has the $\mathbb{Z}[\pi_1(X)]$-coefficient Poincaré-Lefschetz duality

$$H^*(\widetilde{X}) \cong H_{n+1-*}(\widetilde{X}, \widetilde{\partial X})$$

of an $(n+1)$-dimensional manifold with boundary.
- For $n > 4$ f is homotopic to a homeomorphism if and only if $(X, \partial X)$ is homotopy equivalent rel ∂ to an $(n+1)$-dimensional manifold M with boundary $\partial M = \partial X$. Same obstruction theory as for the uniqueness problem.
- Will concentrate on uniqueness problem.

Rings with involution

- An **involution** on a ring A is a function

$$A \to A \; ; \; a \mapsto \overline{a}$$

such that

$$\overline{a+b} \;=\; \overline{a} + \overline{b} \;,\; \overline{ab} \;=\; \overline{b}\,\overline{a} \;,\; \overline{\overline{a}} \;=\; a \;(a, b \in A) \;.$$

- **Example 1** A commutative ring A, with $\overline{a} = a$.
- **Example 2** A group ring $A = \mathbb{Z}[\pi]$ with $\overline{g} = g^{-1}$ $(g \in \pi)$.
- Regard a left A-module P as a right A-module with

$$P \times A \to P \; ; \; (x, a) \mapsto \overline{a}x \;.$$

- The tensor product of left A-modules P, Q is the abelian group defined by

$$P \otimes_A Q \;=\; P \otimes_{\mathbb{Z}} Q / \{ax \otimes y - x \otimes \overline{a}y \,|\, a \in A, x \in P, y \in Q\}$$

with transposition isomorphism

$$P \otimes_A Q \to Q \otimes_A P \; ; \; x \otimes y \mapsto y \otimes x \;.$$

Duality over a ring with involution

- The **dual** of a left A-module P is the left A-module

  $$P^* = \mathrm{Hom}_A(P,A)\ ,\ A \times P^* \to P^*\ ;\ (a,f) \mapsto (x \mapsto f(x)\bar{a})\ .$$

- The natural A-module morphism

  $$P \to P^{**}\ ;\ x \mapsto (f \mapsto \overline{f(x)})$$

  is an isomorphism for f.g. free P.
- For A-modules P, Q the abelian group morphisms

  $$P^* \otimes_A Q \to \mathrm{Hom}_A(P,Q)\ ;\ f \otimes y \mapsto (x \mapsto \overline{f(x)}y)\ ,$$
  $$*\ :\ \mathrm{Hom}_A(P,Q) \to \mathrm{Hom}_A(Q^*,P^*)\ ;\ f \mapsto (f^* : g \mapsto (x \mapsto g(f(x))))$$

  are isomorphisms for f.g. free P, Q.

Quadratic forms on chain complexes I.

- Reference **The algebraic theory of surgery** (1978, Proc. LMS)
- The *n*-**dual** of a f.g. free A-module chain complex

$$C \ : \ \cdots \to C_r \xrightarrow{d} C_{r-1} \to \cdots \to C_1 \xrightarrow{d} C_0 \to \dots$$

  is the f.g. free A-module chain complex

$$C^{n-*} \ : \ \cdots \to C^0 \xrightarrow{d^*} C^1 \to \cdots \to C^{r-1} \xrightarrow{d^*} C^r \to \dots$$

  with $C^r = C_r^*$.
- An ‘algebraic Poincaré complex’ is a f.g. free A-module chain complex C with a chain equivalence $C^{n-*} \simeq C$ satisfying extra conditions. There are two versions: **symmetric** and **quadratic**. Will ignore the difference today.

Quadratic forms on chain complexes II.

- For any f.g. free A-module chain complex C there is defined an isomorphism of A-module chain complexes

$$C \otimes_A C \to \mathrm{Hom}_A(C^{-*}, C) \; ; \; x \otimes y \mapsto (f \mapsto \overline{f(x)}.y) \; .$$

  The homology group

$$H_n(C \otimes_A C) \;=\; H_0(\mathrm{Hom}_A(C^{n-*}, C))$$

  is the group of chain homotopy classes of chain maps $\phi : C^{n-*} \to C$.
- The action of $T \in \mathbb{Z}_2$ by the **transposition involution**

$$T \;:\; C \otimes_A C \to C \otimes_A C \; ; \; x \otimes y \mapsto (-)^{pq} y \otimes x \; (x \in C_p, y \in C_q)$$

  corresponds to the **duality involution**

$$T \;:\; \mathrm{Hom}_A(C^{-*}, C) \to \mathrm{Hom}_A(C^{-*}, C) \; ; \; f \mapsto (-)^{pq} f^* \; ,$$

$$(f : C^p \to C_q) \mapsto ((-)^{pq} f^* : C^q \to C_p) \; , \; y(f^*(x)) = x(f(y)) \; .$$

Algebraic Poincaré cobordism

- An n-**dimensional quadratic Poincaré complex** (C,ϕ) is an n-dimensional f.g. free A-module chain complex C together with a chain equivalence $\phi : C^{n-*} \to C$ such that there exists a chain homotopy $T\phi \simeq \phi : C^{n-*} \to C$. If $1/2 \notin A$ need additional quadratic structure.
- The **quadratic Poincaré cobordism group** $L_n(A)$ is the group of equivalence classes of n-dimensional quadratic Poincaré complexes (C,ϕ) with $(C,\phi) \sim (C',\phi')$ if $C \oplus C' \subset D$ for an $(n+1)$-dimensional f.g. free A-module chain complex D such that

$$H^{n+1-*}(D,C) \cong H_*(D,C') .$$

- $L_n(A) = L_{n+4}(A)$.
- $L_{4k}(A)$ is the Witt group of quadratic forms over A.
- For $A = \mathbb{Z}$ signature defines the isomorphism

$$L_{4k}(\mathbb{Z}) \xrightarrow{\cong} W(\mathbb{R}) = \mathbb{Z}\,;\ (C,\phi) \mapsto \sigma(H^{2k}(\mathbb{R}\otimes_{\mathbb{Z}} C), 1\otimes\phi) .$$

The polyhedron of a simplicial complex

- A **simplicial complex** K is a collection of finite subsets $\sigma \subseteq K^{(0)}$ of an ordered **vertex set** $K^{(0)}$ such that:
  (a) $v \in K$ for each $v \in K^{(0)}$,
  (b) if $\sigma \in K$ and $\tau \subseteq \sigma$ then $\tau \in K$.
- The **dimension** of $\sigma \in K$ is

$$|\sigma| \; = \; (\text{no. of vertices in } \sigma) - 1$$

  Let $K^{(n)}$ denote the set of n-simplexes in K.
- The **polyhedron** of K is the usual identification space

$$|K| \; = \; \Big(\coprod_{n=0}^{\infty} \Delta^n \times K^{(n)}\Big)/\!\sim$$

  with Δ^n the convex hull of $(0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$.

The simplicial chain complex

- The **simplicial chain complex** $C(K)$ has

$$d \ : \ C(K)_n = \mathbb{Z}[K^{(n)}] \to C(K)_{n-1} = \mathbb{Z}[K^{(n-1)}] \ ;$$

$$(v_0 v_1 \dots v_n) \mapsto \sum_{i=0}^{n} (-)^i (v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$$

$$(v_0 < v_1 < \cdots < v_n)$$

- The homology and cohomology groups of the polyhedron are the same as those of the simplicial complex

$$H_*(|K|) \ = \ H_*(K) \ = \ H_*(C(K)) \ ,$$

$$H^*(|K|) \ = \ H^*(K) \ = \ H^*(C(K)) \ .$$

- For any simplicial complexes K, L $H_n(|K| \times |L|)$ is the group of chain homotopy classes of chain maps $C(K)^{n-*} \to C(L)$.

Polyhedral Poincaré complexes

- An **polyhedral *n*-dimensional Poincaré complex** is a finite simplicial complex K with universal cover $\widetilde{K}$ and a homology class $[K] \in H_n(K)$ satisfying the equivalent conditions:
  - (a) the cap products
  $$[K] \cap - \;:\; H^{n-*}(\widetilde{K}) \;=\; H_*(C(\widetilde{K})^{n-*}) \to H_*(\widetilde{K})$$
  are $\mathbb{Z}[\pi_1(K)]$-module isomorphisms.
  - (b) The image $\Delta[K] \in H_n(X)$ under the diagonal map
  $$\Delta \;:\; |K| \to X = |\widetilde{K}| \times_{\pi_1(K)} |\widetilde{K}| \;;\; x \mapsto (\widetilde{x}, \widetilde{x})$$
  is a chain homotopy class of $\mathbb{Z}[\pi_1(K)]$–module chain equivalences $\phi = \Delta[K] : C(\widetilde{K})^{n-*} \to C(\widetilde{K})$.
  - (c) the cap product $[X] \cap - : H^n(X) \to H_n(X)$ is an isomorphism, with $\Delta[K]^* \in H^n(X)$ a $\mathbb{Z}[\pi_1(K)]$-module chain homotopy inverse $\phi^{-1} : C(\widetilde{K}) \to C(\widetilde{K})^{n-*}$.
- **Example** A triangulated *n*-dimensional manifold is a polyhedral *n*-dimensional Poincaré complex, and $(C(\widetilde{K}), \phi)$ is an *n*-dimensional algebraic Poincaré complex over $\mathbb{Z}[\pi_1(K)]$.

Dual cells

- The **barycentric subdivision** of K is the simplicial complex K' with $K'^{(0)} = K$ and
$$K'^{(n)} = \{(\sigma_0, \sigma_1, \dots, \sigma_n) \,|\, \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_n\} \ .$$
Homeomorphic polyhedron $|K'| \cong |K|$.
- The **dual cells** of K are the contractible subcomplexes
$$D(\sigma) = \{(\sigma_0, \sigma_1, \dots, \sigma_n) \in K' \,|\, \sigma_0 \subseteq \sigma\} \subseteq K' \ .$$
- The **boundary** of the dual cell $D(\sigma)$ is
$$\partial D(\sigma) = \{(\sigma_0, \sigma_1, \dots, \sigma_n) \in D(\sigma) \,|\, \sigma_0 \neq \sigma\} \ .$$
- **Proposition** The local homology groups of $|K|$ at $x \in |K|$ are the homology groups of the dual cells relative to boundaries
$$H_*(|K|, |K| \backslash \{x\}) = H_{*-|\sigma|}(D(\sigma), \partial D(\sigma)) \ (x \in \text{interior}(\sigma), \sigma \in K) \ .$$
For each $\sigma \in K$ and $x \in \text{interior}(\sigma)$ there are natural maps
$$\partial_\sigma : H_*(|K|) = H_*(K) \to H_*(|K|, |K| \backslash \{x\}) = H_{*-|\sigma|}(D(\sigma), \partial D(\sigma)) \ .$$

Homology manifolds

- A **polyhedral n-dimensional homology manifold** is a finite simplicial complex K with a homology class $[K] \in H_n(K)$ such that for each $\sigma \in K$

$$\partial_\sigma[K] \cap - \;:\; H^*(D(\sigma)) \to H_{n-|\sigma|-*}(D(\sigma), \partial D(\sigma))$$

is an isomorphism, or equivalently such that for each $x \in |K|$

$$H_*(|K|, |K|\backslash\{x\}) \;=\; H_*(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\}) \;=\; \begin{cases} \mathbb{Z} & \text{if } * = n \\ 0 & \text{if } * \neq n \;. \end{cases}$$

- **Example** A triangulated n-dimensional manifold is a polyhedral n-dimensional homology manifold.
- Being a homology manifold is not a homotopy invariant property: so this is too much Poincaré duality to characterize polyhedra homotopy equivalent to a manifold.

McCrory's theorem

- **Theorem** (McC., 1977) A polyhedral n-dimensional Poincaré complex K is an n-dimensional homology manifold if and only if the polyhedral $2n$-dimensional Poincaré complex

$$(X, [X]) \ = \ (|K| \times |K|, [K] \otimes [K])$$

is such that the Poincaré dual $\Delta[K]^* \in H^n(X)$ of the diagonal class $\Delta[K] \in H_n(X)$ is supported near the diagonal $\Delta_{|K|} \subset X$, i.e.

$$\begin{aligned} \Delta[K]^* \in \operatorname{im}(H^n(X, X \backslash \Delta_{|K|}) \to H^n(X)) \\ = \ \ker(H^n(X) \to H^n(X \backslash \Delta_{|K|})) \ . \end{aligned}$$

- (A.R.) **Singularities, double points, controlled topology and chain duality** Doc. Math. (1999)
Interpretation of Theorem in terms of the $(\mathbb{Z}, K)$-module category.

The $(\mathbb{Z}, K)$-category I. Modules

- (A.R.+M.Weiss) **Chain complexes and assembly**, Math.Z.(1999)
- A $(\mathbb{Z}, K)$-**module** is a f.g. free $\mathbb{Z}$-module A with splitting
$$A = \sum_{\sigma \in K} A(\sigma) .$$
- A **morphism** of $(\mathbb{Z}, K)$-modules $f : A \to B$ is a $\mathbb{Z}$-module morphism such that
$$f(A(\sigma)) \subseteq \sum_{\tau \supseteq \sigma} B(\tau) \ (\sigma \in K) .$$
- **Example** The simplicial chain complex $C(K')$ is a $(\mathbb{Z}, K)$-module chain complex with
$$C(K')(\sigma) = C(D(\sigma), \partial D(\sigma)) \ (\sigma \in K) .$$
- **Proposition** A $(\mathbb{Z}, K)$-module morphism $f : A \to B$ is an isomorphism if and only if each diagonal component $f(\sigma, \sigma) : A(\sigma) \to B(\sigma)$ $(\sigma \in K)$ is a $\mathbb{Z}$-module isomorphism.

The $(\mathbb{Z}, K)$-category II. Products

- The **product** of $(\mathbb{Z}, K)$-modules A, B is the $(\mathbb{Z}, K)$-module

$$A \otimes_{(\mathbb{Z},K)} B \;=\; A \otimes_{\mathbb{Z}} B \;,$$
$$(A \otimes_{(\mathbb{Z},K)} B)(\sigma) \;=\; \sum_{\lambda,\mu \in K, \lambda \cap \mu = \sigma} A(\lambda) \otimes_{\mathbb{Z}} B(\mu) \;.$$

- **Example** For simplicial maps $f : L \to K'$, $g : M \to K'$ the pullback polyhedron

$$L \times_K M \;=\; \{(x,y) \in |L| \times |M| \,|\, f(x) = g(y) \in |K|\}$$

has homology

$$H_*(L \times_K M) \;=\; H_*(C(L) \otimes_{(\mathbb{Z},K)} C(M))$$

with

$$C(L)(\sigma) \;=\; C(f^{-1}D(\sigma), f^{-1}\partial D(\sigma)) \;,$$
$$C(M)(\sigma) \;=\; C(g^{-1}D(\sigma), g^{-1}\partial D(\sigma)) \;.$$

The $(\mathbb{Z}, K)$-category III. Duality

- The **dual** of a $(\mathbb{Z}, K)$-module A is the $(\mathbb{Z}, K)$-module chain complex TC with

$$TC(\sigma)_r = \begin{cases} \sum\limits_{\tau \supseteq \sigma} A(\tau)^* & \text{if } r = -|\sigma| \\ 0 & \text{otherwise.} \end{cases}$$

- The dual of a $(\mathbb{Z}, K)$-module chain complex C is a $(\mathbb{Z}, K)$-module chain complex TC. (Analogue of Verdier duality for sheaves).
- **Example** The dual of $C(K')$ is $(\mathbb{Z}, K)$-equivalent to the cochain complex of K

$$TC(K') \simeq C(K)^{-*} , \ C(K)^r(\sigma) = \begin{cases} \mathbb{Z} & \text{if } r = -|\sigma| \\ 0 & \text{otherwise.} \end{cases}$$

- For any $(\mathbb{Z}, K)$-module chain complexes A, B

$$H_*(A \otimes_{(\mathbb{Z},K)} B) = H_*(\mathrm{Hom}_{(\mathbb{Z},K)}(TA, B)) .$$

The assembly map

- **Proposition** (i) The generalized homology group $H_n(K;\mathbf{L}(\mathbb{Z}))$ is the cobordism group of n-dimensional quadratic Poincaré complexes (C,ϕ) in the $(\mathbb{Z},K)$-module category.
- (ii) The assembly map $A : H_n(K;\mathbf{L}(\mathbb{Z})) \to L_n(\mathbb{Z}[\pi_1(K)])$ is induced by the functor

$$A \;:\; \{(\mathbb{Z},K)\text{-modules}\} \to \{\text{f.g. free } \mathbb{Z}[\pi_1(K)]\text{-modules}\}\;;$$
$$B = \sum_{\sigma\in K} B(\sigma) \mapsto A(B) = \sum_{\widetilde{\sigma}\in\widetilde{K}} B(p(\widetilde{\sigma}))$$

  with $p : \widetilde{K} \to K$ the universal covering projection.
- (iii) $\mathcal{S}_n(K)$ is the cobordism group of $(n-1)$-dimensional quadratic Poincaré complexes (C,ϕ) in the $(\mathbb{Z},K)$-module category such that the assembly $A(C)$ is a contractible f.g. free $\mathbb{Z}[\pi_1(K)]$-module chain complex, $H_*(A(C)) = 0$.

From local to global Poincaré duality, and back again!

- For any simplicial complex K

$$H_n(K) \;=\; H_n(\mathrm{Hom}_{(\mathbb{Z},K)}(TC(K'),C(K')))\ .$$

The cap product with any homology class $[K] \in H_n(K)$ is a $(\mathbb{Z},K)$-module chain map $\phi = [K] \cap - : TC(K')_{*-n} \to C(K')$ with diagonal components

$$\begin{aligned}\phi(\sigma,\sigma) = \partial_\sigma[K] \cap - : TC(K')_{*-n}(\sigma) = C(D(\sigma))^{n-*-|\sigma|}\\ \to C(K')(\sigma) = C(D(\sigma),\partial D(\sigma))\ (\sigma \in K)\ ,\end{aligned}$$

assembly $[K] \cap - : TC(\widetilde{K}')_{*-n} \simeq C(\widetilde{K})^{n-*} \to C(\widetilde{K}') \simeq C(\widetilde{K})$.
- K is a homology manifold if and only if $[K] \cap -$ is a $(\mathbb{Z},K)$-module chain equivalence, in which case it is a Poincaré complex. (Essentially Poincaré's original proof!)
- The total surgery obstruction of a Poincaré complex K is

$$s(K) \;=\; (C(\phi)_{*+1},\psi) \in \mathcal{S}_n(K)\ ,$$

with $C(\phi)$ the algebraic mapping cone of ϕ.

Conclusion

- The **Novikov conjecture** on the homotopy invariance of the higher signatures of manifolds with fundamental group π is equivalent to the injectivity of
$$1 \otimes A : H_*(B\pi; \mathbf{L}(\mathbb{Z})) \otimes \mathbb{Q} \to L_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q} .$$
- The **Borel conjecture** on the existence and rigidity of topological manifold structures on aspherical Poincaré complexes $B\pi$ is essentially equivalent to $A : H_*(B\pi; \mathbf{L}(\mathbb{Z})) \to L_*(\mathbb{Z}[\pi])$ being an isomorphism.
- Starting with Novikov himself, many authors in the last 40 years have proved many special cases of the Novikov and Borel conjectures, and the related **Farrell-Jones isomorphism conjecture**, using algebraic, geometric and analytic methods.
- Some (though not all) have used the algebraic L-theory assembly map defined here. There is still much work to be done to understand the relationship between all these methods of proof, and maybe even prove new results!