BLANCHFIELD AND SEIFERT ALGEBRA FOR HIGH-DIMENSIONAL BOUNDARY LINKS

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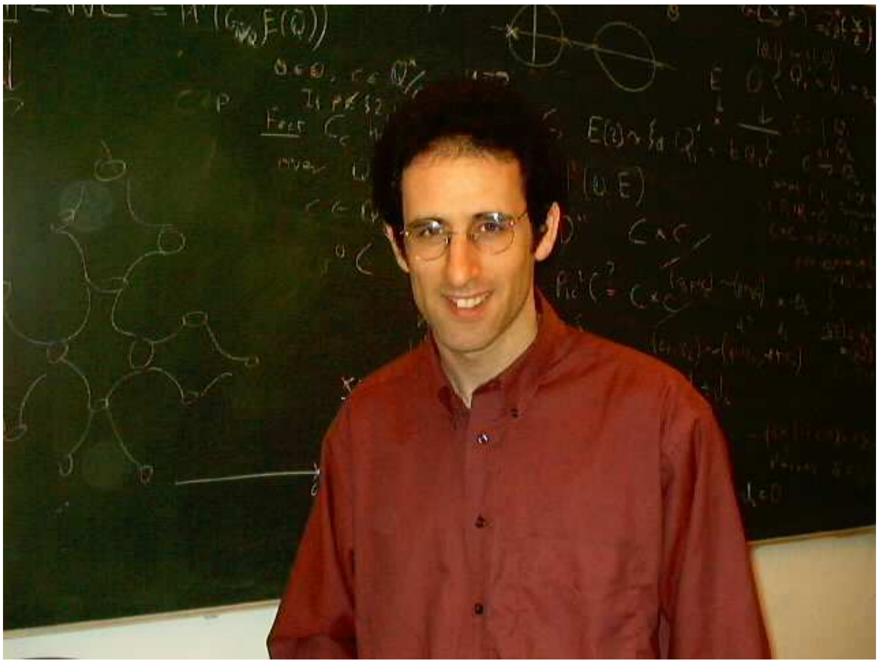


13th November, 1974 – 25th March, 2005

From “Cats cradle song”

Its monstrous, horrid, shocking,
Beyond the power of thinking,
Not to know, interlocking
Is no mere form of linking.

James Clerk Maxwell

Outline

- For $\mu \geqslant 1$ let $F_\mu = \langle z_1, z_2, \ldots, z_\mu \rangle$ be the infinite free group on μ generators. The geometry of high-dimensional μ-component boundary links motivates pure algebra, for any ring A :

  - decompositions of the algebraic K- and L-theory of $A[F_\mu]$ and a noncommutative localization $\Sigma^{-1}A[F_\mu]$ in terms of the algebraic K- and L-theory of A.

- Applications to boundary links for $A = \mathbb{Z}$:

  - Blanchfield algebra = homological $\mathbb{Z}[F_\mu]$-module invariants of the exteriors of μ-component boundary links.

  - Seifert algebra = homological $\mathbb{Z}$-module invariants of the μ-component Seifert surfaces of μ-component boundary links.

Boundary links

- Definition A μ-component link is a locally flat embedding

$$\mathcal{L} \ : \ \bigsqcup_{\mu} S^n \subset S^{n+2} \ .$$

For $\mu = 1$ a knot $S^n \subset S^{n+2}$.

- Every μ-component link $\mathcal{L}$ admits a Seifert surface, a codimension 1 submanifold $V^{n+1} \subset S^{n+2}$ with boundary

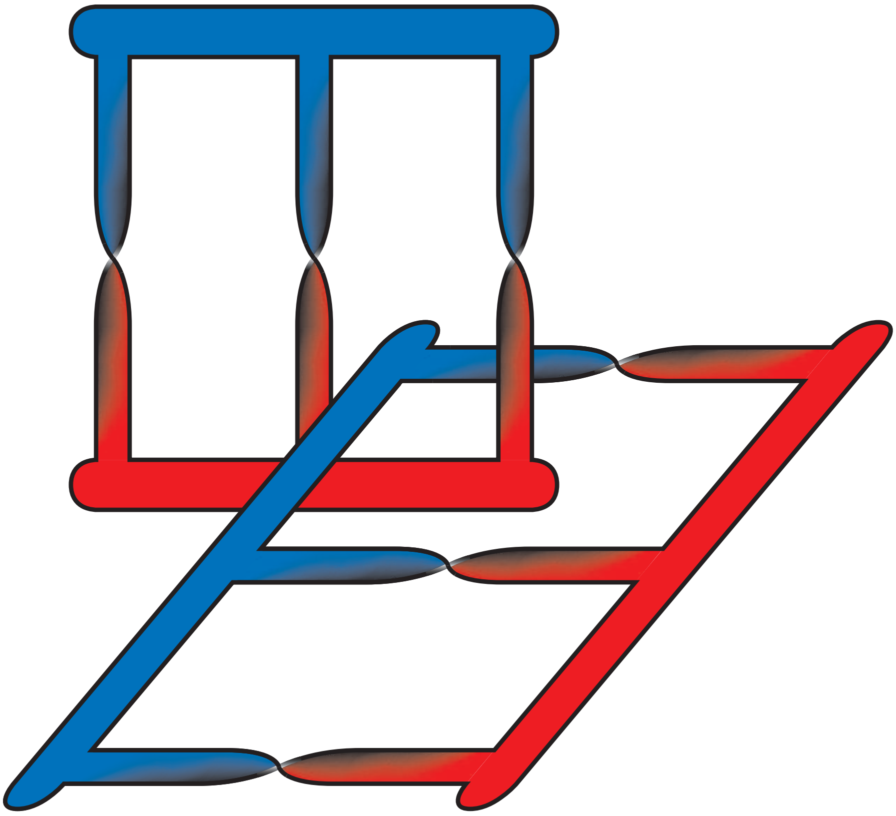
$$\partial V \ = \ \mathcal{L}(\bigsqcup_{\mu} S^n) \subset S^{n+2} \ .$$

- Definition $\mathcal{L}$ is a boundary link if there exists a μ-component Seifert surface

$$V \ = \ \bigsqcup_{i=1}^{\mu} V_i \subset S^{n+2} \ ,$$

$V_1, V_2, \dots, V_\mu \subset S^{n+2}$ disjoint and connected.

A 2-component boundary link with a 2-component Seifert surface



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A brief history of boundary links

- Definition and basic properties of boundary 1-links, 1966 (Fox, Smythe)

- Characterization of high-dimensional boundary links, 1972 (Gutierrez)

- Formulation in terms of high-dimensional homology surgery theory, 1980, 1987 (Cappell-Shaneson, Ko)

- Isotopy classification of high-dimensional simple boundary links, 1977 (Liang) 1992 (Farber)

- Computation of high-dimensional boundary link cobordism, 2003 (Sheiham)

The exterior

- Every n-link $\mathcal{L} : \bigsqcup_{\mu} S^n \subset S^{n+2}$ has a tubular neighbourhood $N = \mathcal{L}\Big(\bigsqcup_{\mu} S^n\Big) \times D^2 \subset S^{n+2}$.

- Definition The exterior of $\mathcal{L}$ is the compact $(n+2)$-dimensional manifold

$$X = \text{closure}(S^{n+2} \backslash N)$$

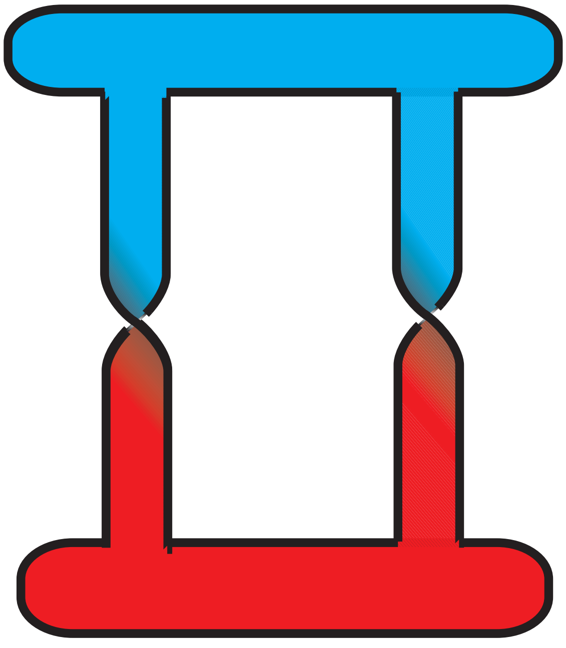
with boundary

$$\partial X = \partial N = \mathcal{L}(\bigsqcup_{\mu} S^n) \times S^1 .$$

- Theorem (Smythe 1966 for $n = 1$, Gutierrez 1972 for $n \geqslant 2$)
An n-link $\mathcal{L} : \bigsqcup_{\mu} S^n \subset S^{n+2}$ is a boundary link if and only if there exists a surjection $c : \pi_1(X) \to F_\mu$ sending the meridians $\{i\} \times S^1 \subset X$ $(1 \leqslant i \leqslant \mu)$ to $z_1, z_2, \ldots, z_\mu$.

The Hopf link is not a boundary link

- Let $\eta : S^3 \to S^1$ be the Hopf map. The 2-component Hopf link
$$\mathcal{L}_2 : \eta^{-1}(*_1 \cup *_2) = S^1 \sqcup S^1 \subset S^3$$
is not a boundary link, with
$$X = T^2 \times [0,1] \; , \; \pi_1(X) = \mathbb{Z}^2 \; .$$

- 

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The trivial link

- Definition The trivial μ-component link $\mathcal{L}_0 : \bigsqcup_\mu S^n \subset S^{n+2}$ is the connected sum of μ copies of the trivial knot

$$S^n \subset (S^n \times D^2) \cup (D^{n+1} \times S^1) \; = \; S^{n+2}$$

- Lemma $\mathcal{L}_0$ is a boundary link with μ-component Seifert surface and exterior given by

$$V = \bigsqcup_\mu D^{n+1} \, , \; X_0 = \#_\mu D^{n+1} \times S^1 \simeq \bigvee_\mu S^1 \vee \bigvee_{\mu-1} S^{n+1}$$

- Theorem (Gutierrez 1972) For every μ-component boundary link $\mathcal{L} : \bigsqcup_\mu S^n \subset S^{n+2}$ there exists a degree 1 map $c : X \to X_0$ which induces homology isomorphisms

$$c_* \; : \; H_*(X) \; \cong \; H_*(X_0) \; .$$

For $n \geqslant 2$ c is a homotopy equivalence if and only if $\mathcal{L}$ is unlinked.

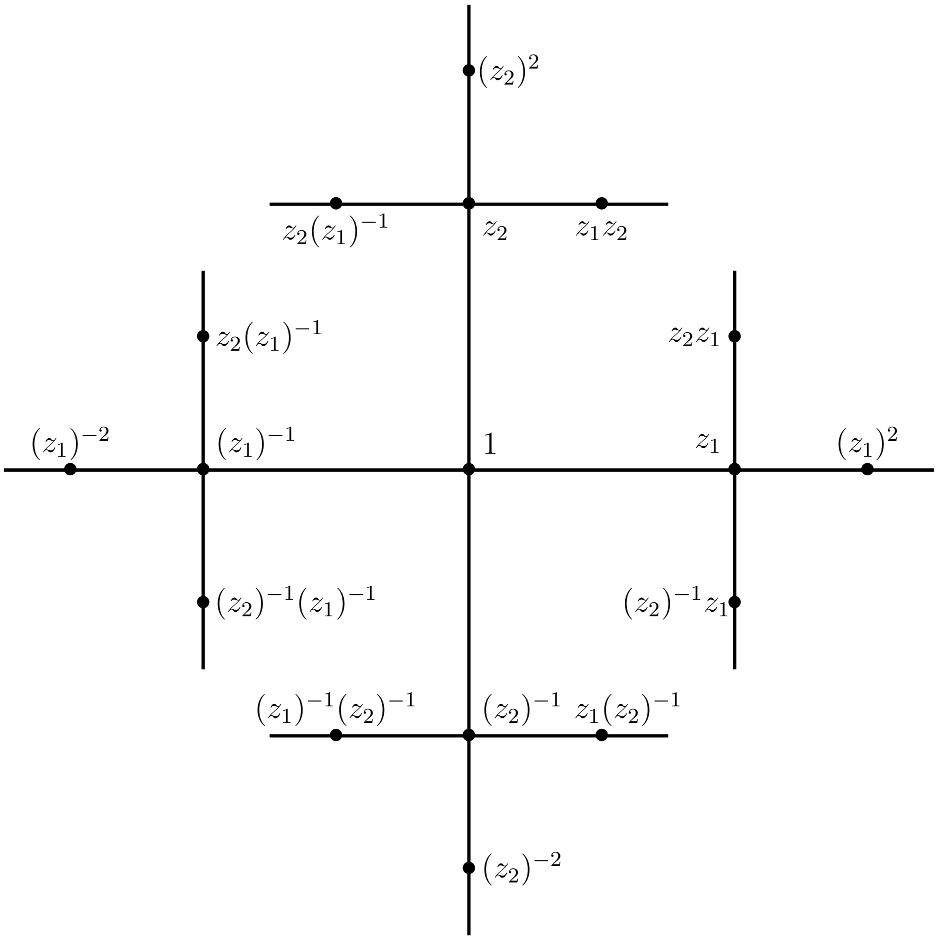
The Cayley tree G_μ

- The universal cover of $BF_\mu = \bigvee_\mu S^1$ is the Cayley tree $EF_\mu = G_\mu$ of F_μ with respect to the generators $\{z_1, z_2, \ldots, z_\mu\}$. One vertex for each $g \in F_\mu$, and one edge $[g, gz_i]$ for each pair (g, gz_i) $(g \in F_\mu, 1 \leqslant i \leqslant \mu)$.

- EF_μ is a contractible space with a free F_μ-action, and fundamental domain
$$H_\mu \ = \ \bigcup_{i=1}^{\mu} [1, z_i] \cup \bigcup_{i=1}^{\mu} [z_i^{-1}, 1] \subset EF_\mu \ .$$

- For a connected space X every group morphism $c : \pi_1(X) \to F_\mu$ is realized by a map $c : X \to BF_\mu = \bigvee_\mu S^1$. The pullback F_μ-cover
$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{c}} & EF_\mu \\ \downarrow & & \downarrow \\ X & \xrightarrow{c} & BF_\mu \end{array}$$
has fundamental domain $\tilde{c}^{-1}(H_\mu) \subset \widetilde{X}$.

The Cayley tree G_2 of $F_2 = \langle z_1, z_2 \rangle$



F_μ-covers (I.)

- <u>Proposition</u> For any compact connected manifold X every group morphism $c : \pi_1(X) \to F_\mu$ is realized by a map

$$c \ : \ X \to BF_\mu \ = \ \bigvee_\mu S^1$$

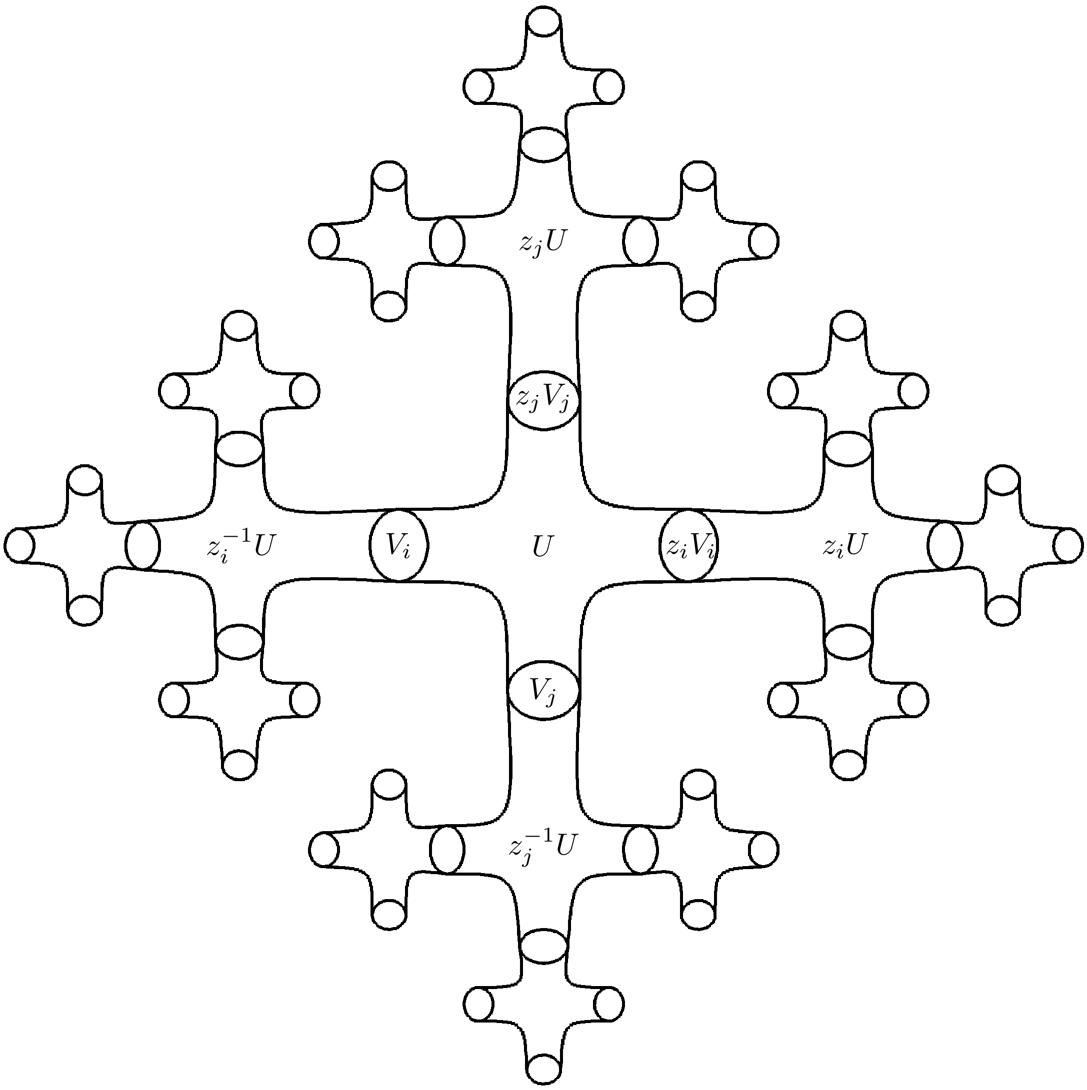
which is transverse regular at $\{1, 2, \ldots, \mu\} \subset BF_\mu$, with $V_i = c^{-1}(i) \subset X$ disjoint connected codimension 1 submanifolds.

- The induced F_μ-cover $\widetilde{X} = c^* EF_\mu$ of X has a fundamental domain $U = \tilde{c}^{-1}(H_\mu) \subset \widetilde{X}$, a manifold with boundary

$$\partial U \ = \ \bigsqcup_{i=1}^{\mu} (V_i \sqcup z_i V_i) \ .$$

- <u>Example</u> If X is the exterior of a boundary link $\mathcal{L} : \bigsqcup_\mu S^n \subset S^{n+2}$ then V_i is a Seifert surface for ith component $S^n \subset S^{n+2}$.

F_μ-covers (II.)



There is one translate gU of U for each vertex $g \in F_\mu$ of the Cayley tree G_μ, and one translate $gV_i = g(z_i^{-1}U \cap U)$ of V_i for each edge $[g, gz_i]$.

Algebraic transversality I.

- <u>Theorem</u> For any ring A, every finite f.g. free $A[F_\mu]$-module chain complex E has a f.g. free $A[F_\mu]$-module <u>M-V presentation</u>

$$\mathcal{E} \ : \ 0 \longrightarrow \bigoplus_{i=1}^{\mu} D_i[F_\mu] \xrightarrow{f} C[F_\mu] \longrightarrow E \longrightarrow 0$$

with C, D_i finite f.g. free A-module chain complexes, $f = (f_1^+ - z_1 f_1^- \ \ldots \ f_\mu^+ - z_\mu f_\mu^-)$ for A-module chain maps $f_i^+, f_i^- : D_i \to C$.

- <u>Proof</u> Construct $\mathcal{E}$ as a subobject of the infinitely generated free $A[F_\mu]$-module M-V presentation of E defined by the E-coefficient simplicial chain complex of the Cayley tree

$$\mathcal{E}\langle\infty\rangle \ : \ 0 \longrightarrow \bigoplus_{i=1}^{\mu} E[F_\mu] \xrightarrow{\partial} E[F_\mu] \longrightarrow E \longrightarrow 0$$

with $\partial = (\zeta_1 - z_1 \ \ldots \ \zeta_\mu - z_\mu)$ for the A-module chain maps

$$\zeta_i \ : \ E \to E \ ; \ x \mapsto z_i x \ .$$

Algebraic transversality II.

- Remark The algebraic transversality theorem can be proved using the Higman linearization methods of Waldhausen, applied to the Bass-Serre trees of the successive amalgamations in

$$F_\mu = F_1 * F_{\mu-1} = F_1 * (F_1 * (F_1 * (\cdots * F_1))) .$$

- Example For manifold X with $\pi_1(X) \to F_\mu$ define the inclusions

$$f_i^+, f_i^- \ : \ V_i \to U \ ; \ x \mapsto x, \ x \mapsto z_i^{-1} x \ .$$

The cellular $\mathbb{Z}[F_\mu]$-module chain complex $C(\widetilde{X})$ has the finite f.g. free $\mathbb{Z}[F_\mu]$-module M-V presentation

$$0 \longrightarrow \bigoplus_{i=1}^{\mu} C(V_i)[F_\mu] \xrightarrow{f} C(U)[F_\mu] \longrightarrow C(\widetilde{X}) \longrightarrow 0$$

with $f = (f_1^+ - z_1 f_1^- \ \ldots \ f_\mu^+ - z_\mu f_\mu^-)$.

Seifert and Blanchfield chain complexes

- Corollary A finite f.g. free $A[F_\mu]$-module chain complex E which is A-acyclic (i.e. $H_*(A \otimes_{A[F_\mu]} E) = 0$) has an M-V presentation of the type

$$0 \longrightarrow \bigoplus_{i=1}^{\mu} D_i[F_\mu] \xrightarrow{f} \bigoplus_{i=1}^{\mu} D_i[F_\mu] \longrightarrow E \longrightarrow 0 \quad (*)$$

  with D_i finite f.g. free A-module chain complexes, $f = (f_1^+ - z_1 f_1^- \ \ldots \ f_\mu^+ - z_\mu f_\mu^-)$ for A-module chain maps $f_{ij}^+, f_{ij}^- : D_j \to D_i$ such that $f_{ij}^+ - f_{ij}^- = \delta_{ij}$.

- Example Let X be the exterior of a μ-component boundary link $\mathcal{L}$, with μ-component Seifert surface $V = V_1 \sqcup \cdots \sqcup V_\mu$. The Blanchfield finite f.g. free $\mathbb{Z}[F_\mu]$-module chain complex $E = \mathcal{C}(\widetilde{c} : C(\widetilde{X}) \to C(\widetilde{X}_0))$ is $\mathbb{Z}$-acyclic. E has an M-V presentation as in $(*)$, with $D_i = \mathcal{C}(C(V_i) \to C(D^{n+1}))$ the Seifert $\mathbb{Z}$-module chain complexes.

The algebraic K-theory of $A[F_\mu]$

- Theorem The Bass-Quillen algebraic K-groups of $A[F_\mu]$ split as

  $$K_n(A[F_\mu]) = K_n(A) \oplus \mathrm{Nil}_{n-1}(A, 2\mu)$$

  with

  $$\mathrm{Nil}_*(A, 2\mu) = \bigoplus_\mu K_*(A) \oplus \widetilde{\mathrm{Nil}}_*(A, 2\mu)$$

  the algebraic K-groups of the exact category $\mathrm{Nil}(A, 2\mu)$ of f.g. projective A-modules with a 2μ-component nilpotent endomorphism (= M-V presentations of 0).

- The proof uses the Waldhausen K-theory fibration theorem.

- For $\mu = 1$, $n = 1$ this is the classical Bass-Heller-Swan splitting

  $$K_1(A[z, z^{-1}]) = K_1(A) \oplus K_0(A) \oplus \widetilde{\mathrm{Nil}}_0(A) \oplus \widetilde{\mathrm{Nil}}_0(A)$$

The noncommutative localization $\Sigma^{-1}A[F_\mu]$

- Let $\Sigma^{-1}A[F_\mu]$ be the noncommutative Cohn localization of $A[F_\mu]$ inverting the set Σ of A-invertible matrices.

- Universal property of $A[F_\mu] \to \Sigma^{-1}A[F_\mu]$: a finite f.g. free $A[F_\mu]$-module chain complex E is A-acyclic if and only if the finite f.g. free $\Sigma^{-1}A[F_\mu]$-module chain complex

$$\Sigma^{-1}E \;=\; \Sigma^{-1}A[F_\mu] \otimes_{A[F_\mu]} E$$

  is acyclic.

- Example For a μ-component boundary link $\mathcal{L}$ with exterior X the finite f.g. free $\mathbb{Z}[F_\mu]$-module chain complex

$$E \;=\; \mathcal{C}(\widetilde{c} : C(\widetilde{X}) \to C(\widetilde{X_0}))$$

  is $\mathbb{Z}$-acyclic, and hence $\Sigma^{-1}\mathbb{Z}[F_\mu]$-acyclic.

Quadratic Poincaré complexes

- (R. 1980) The Wall surgery obstruction groups $L_n(A)$ of a ring with involution A are the cobordism groups n-dimensional quadratic Poincaré complexes C over A, with C a f.g. free A-module chain complex and

$$\psi_s \ : \ C^r \ = \mathrm{Hom}_A(C_r, A) \to C_{n-r-s}$$

A-module morphisms satisfying

$$d\psi_s + \psi_s d^* + \psi_{s+1} + \psi^*_{s+1} = 0 \quad (s \geqslant 0)$$

with

$$\psi_0 + \psi_0^* \ : \ C^{n-*} \to C$$

a chain equivalence.

- Abstract Poincaré duality isomorphisms

$$H^{n-*}(C) \ \cong \ H_*(C) \ .$$

Blanchfield and Seifert duality complexes

- A Blanchfield duality complex (E,ψ) over $A[F_\mu]$ is an A-acyclic quadratic Poincaré complex over $A[F_\mu]$.

- A Seifert duality complex (D,θ) over A is a quadratic Poincaré complex over A with $D = \bigoplus_{i=1}^{\mu} D(i)$, and θ given by a chain map $\theta_0 : D^{n-*} \to D$, with $\theta_s = 0$ for $s \geqslant 1$.

- Example A μ-component boundary link $\mathcal{L} : \bigsqcup_{i=1}^{\mu} S^n \subset S^{n+2}$ with exterior X and μ-component Seifert surface $V = V_1 \sqcup \cdots \sqcup V_\mu$ determines an $(n+2)$-dimensional Blanchfield duality complex (E,ψ) over $\mathbb{Z}[F_\mu]$ with $E = \mathcal{C}(\widetilde{c} : C(\widetilde{X}) \to C(\widetilde{X}_0))$, and also an $(n+1)$-dimensional Seifert duality complex $(\bigoplus_{i=1}^{\mu} C(V_i),\theta)$ over $\mathbb{Z}$.

The algebraic L-theory of $A[F_\mu]$ and $\Sigma^{-1}A[F_\mu]$

- <u>Theorem</u> (Cappell, 1976 for $A = \mathbb{Z}$, R. 2005 for all A)

  $$L_n(A[F_\mu]) \ = \ L_n(A) \oplus \bigoplus_\mu L'_{n-1}(A)$$

  where $L' = L$ with change of K-theory decoration. For $\mu = 1$ this is the Shaneson-Novikov-R. splitting of $L_n(A[z, z^{-1}])$.

- <u>Theorem</u> (Vogel 1982, Neeman-R. 2001) The algebraic L-groups of $A[F_\mu]$ and $\Sigma^{-1}A[F_\mu]$ fit into the noncommutative localization exact sequence

  $$\begin{aligned}\cdots \to L_n(A[F_\mu]) &\to L_n(\Sigma^{-1}A[F_\mu]) \\ &\to L_n(A[F_\mu], \Sigma) \to L_{n-1}(A[F_\mu]) \to \dots\end{aligned}$$

  with $L_n(A[F_\mu], \Sigma)$ the cobordism group of the A-acyclic $(n-1)$-dimensional Blanchfield duality complexes over $A[F_\mu]$.

The cobordism of Blanchfield and Seifert duality complexes

- <u>Theorem</u> (R.+S.) For any ring with involution A the cobordism group $L_{n+3}(A[F_\mu],\Sigma)$ of $(n+2)$-dimensional Blanchfield duality complexes over $A[F_\mu]$ is isomorphic to the cobordism group $L\mathrm{Sei}_{n+1}(A,\mu)$ of $(n+1)$-dimensional Seifert duality complexes.

- Proof by algebraic transversality, mimicking in algebra the construction of μ-component Seifert surfaces for boundary links.

- For $n \geqslant 2$ the cobordism class of a μ-component boundary link $\mathcal{L} : \bigsqcup_{i=1}^{\mu} S^n \subset S^{n+2}$ with Blanchfield duality complex (E,ψ) and Seifert duality complex (D,θ) is

$$[\mathcal{L}] = (E,\psi) = (D,\theta) \in C_n(F_\mu) = L_{n+3}(\mathbb{Z}[F_\mu],\Sigma) = L\mathrm{Sei}_{n+1}(\mathbb{Z},\mu) \ .$$