NOVIKOV ADDITIVITY OF THE SIGNATURE, WALL NON-ADDITIVITY AND THE MASLOV INDEX

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Symmetric and symplectic forms over $\mathbb{R}$

- Let $\epsilon = +1$ or -1.
- An ϵ-**symmetric form** (H, ϕ) is a finite-dimensional real vector space H together with a bilinear pairing

$$\phi \ : \ H \times H \to \mathbb{R} \ ; \ (x, y) \mapsto \phi(x, y)$$

  such that

$$\phi(x, y) \ = \ \epsilon\phi(y, x) \in \mathbb{R} \ .$$

- A **morphism** $f : (H, \phi) \to (H', \phi')$ is a linear map $f : H \to H'$ such that

$$\phi'(f(x), f(y)) \ = \ \phi(x, y) \in \mathbb{R} \ .$$

- A 1-symmetric form is called **symmetric**.
- A -1-symmetric form is called **symplectic**.

Duality

- The **dual** of a vector space H is

$$H^* \ = \ \mathrm{Hom}_{\mathbb{R}}(H, \mathbb{R}) \ .$$

  For finite-dimensional H use the natural isomorphism

$$H \xrightarrow{\cong} H^{**} \ ; \ x \mapsto (f \mapsto f(x))$$

  to identify $H = H^{**}$.
- The **dual** of a linear map $f : G \to H$ is

$$f^* \ : \ H^* \to G^* \ ; \ h \mapsto (x \mapsto h(f(x))) \ .$$

- The **adjoint** of an ϵ-symmetric form (H, ϕ) is the linear map

$$\phi \ : \ H \to H^* \ ; \ x \mapsto (y \to \phi(x, y))$$

  such that

$$\phi^* \ = \ \epsilon\phi \in \mathrm{Hom}_{\mathbb{R}}(H, H^*) \ .$$

- An ϵ-symmetric form (H, ϕ) is **nonsingular** if $\phi : H \to H^*$ is an isomorphism.

Sublagrangians and lagrangians

- Given an ϵ-symmetric form (H, ϕ) and a subspace $L \subseteq H$ define a linear map
$$H \to L^* \; ; \; x \mapsto (y \mapsto \phi(x, y))$$
and let
$$L^\perp \;=\; \ker(H \to L^*) \; , \; L_\perp \;=\; \mathrm{coker}(H \to L^*) \; .$$
- **Note** If (H, ϕ) is nonsingular then $L_\perp = 0$.
- A **sublagrangian** for an (H, ϕ) is a subspace $L \subseteq H$ such that
$$L_\perp \;=\; 0 \text{ and } L \subseteq L^\perp \; .$$
- A **lagrangian** of an ϵ-symmetric form (H, ϕ) is a sublagrangian such that $L = L^\perp$, in which case (H, ϕ) is nonsingular.

The hyperbolic form $H_\epsilon(L)$

- The **hyperbolic ϵ-symmetric form** is defined for any finite-dimensional real vector space L by

$$H_\epsilon(L) \;=\; (L\oplus L^*,\phi)\ ,$$
$$\phi \;=\; \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}\ :\ L\oplus L^*\times L\oplus L^*\to\mathbb{R}\ ;$$
$$((x,f),(y,g))\mapsto g(x)+\epsilon f(y)\ .$$

- L is a lagrangian of $H_\epsilon(L)$.
- **Proposition** For any nonsingular ϵ-symmetric form (H,ϕ) there is defined an isomorphism

$$\begin{pmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{-\phi^{-1}}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & \dfrac{\phi^{-1}}{\sqrt{2}} \end{pmatrix}\ :\ H_\epsilon(H)\xrightarrow{\ \cong\ }(H,\phi)\oplus(H,-\phi)$$

sending $H\oplus 0$ to the diagonal lagrangian $\Delta=\{(x,x)\,|\,x\in H\}$ in $(H,\phi)\oplus(H,-\phi)$.

The Witt extension theorem

- **Theorem** (W., 1937) The inclusion of a sublagrangian $(L, 0) \to (H, \phi)$ extends to an isomorphism

$$H_\epsilon(L) \oplus (L^\perp/L, [\phi]) \xrightarrow{\cong} (H, \phi) .$$

- **Corollary** The inclusion of a lagrangian $(L, 0) \to (H, \phi)$ extends to an isomorphism

$$f \ : \ H_\epsilon(L) \xrightarrow{\cong} (H, \phi) .$$

If $f_1, f_2 : H_\epsilon(L) \cong (H, \phi)$ are two such extensions

$$(f_2)^{-1} f_1 \ = \ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \ : \ H_\epsilon(L) \xrightarrow{\cong} H_\epsilon(L)$$

for some $(-\epsilon)$-symmetric form (L^*, λ).

The isomorphism classification of symmetric forms

- **Theorem** (Sylvester, 1852) Every symmetric form (H, ϕ) is isomorphic to
$$\bigoplus_p (\mathbb{R}, 1) \oplus \bigoplus_q (\mathbb{R}, -1) \oplus \bigoplus_r (\mathbb{R}, 0)$$
with $p + q + r = \dim_{\mathbb{R}}(H)$.
- Two forms are isomorphic if and only if they have the same (p, q, r).
- **Definition** The **signature** of (H, ϕ) is
$$\sigma(H, \phi) \; = \; p - q \in \mathbb{Z} \; .$$
- **Example** $\sigma(H_+(L)) = 0$, with $(p, q, r) = (\dim_{\mathbb{R}} L, \dim_{\mathbb{R}} L, 0)$.
- (H, ϕ) is nonsingular if and only if $r = 0$.
- (H, ϕ) admits a lagrangian if and only if $p = q$, $r = 0$, if and only if it is nonsingular with signature 0.

The isomorphism classification of symplectic forms

- **Theorem** Every symplectic form (H, ϕ) is isomorphic to

$$H_-(\mathbb{R}^p) \oplus (\mathbb{R}^r, 0)$$

  with $2p + r = \dim_{\mathbb{R}}(H)$.
- Two forms are isomorphic if and only if they have the same (p, r).
- Nonsingular if and only if $r = 0$.
- Every nonsingular symplectic form admits a lagrangian.

Formations

- **Definition** An ϵ-**symmetric formation** $(H, \phi; L_1, L_2)$ is an ϵ-symmetric form (H, ϕ) with an ordered pair of lagrangians L_1, L_2.
- **Example** The **boundary** of a $(-\epsilon)$-symmetric form (L, λ) is the ϵ-symmetric formation

$$\partial(L, \lambda) \ = \ (H_\epsilon(L); L, \Gamma_{(L,\lambda)})$$

with

$$\Gamma_{(L,\lambda)} \ = \ \{(x, \lambda(x)) \in L \oplus L^* \,|\, x \in L\}$$

the graph lagrangian of $H_\epsilon(L)$.
- **Definition** (i) An **isomorphism** of ϵ-symmetric formations $f : (H, \phi; L_1, L_2) \to (H', \phi'; L'_1, L'_2)$ is an isomorphism of forms $f : (H, \phi) \to (H', \phi')$ such that $f(L_1) = L'_1$, $f(L_2) = L'_2$.
- (ii) A **stable isomorphism** of ϵ-symmetric formations $[f] : (H, \phi; L_1, L_2) \to (H', \phi'; L'_1, L'_2)$ is an isomorphism of formations of the type

$$f \ : \ (H, \phi; L_1, L_2) \oplus (H_\epsilon(L); L, L^*) \to (H', \phi'; L'_1, L'_2) \oplus (H_\epsilon(L'); L', L'^*) \ .$$

Formations and automorphisms of forms

- **Proposition** Given an ϵ-symmetric form (H, ϕ), a lagrangian L, and an automorphism $\alpha : (H, \phi) \to (H, \phi)$ there is defined an ϵ-symmetric formation $(H, \phi; L, \alpha(L))$.
- **Proposition** For any formation $(H, \phi; L_1, L_2)$ there exists an automorphism $\alpha : (H, \phi) \to (H, \phi)$ such that $\alpha(L_1) = L_2$.
- **Proof** By the Witt extension theorem the inclusions $(L_i, 0) \to (H, \phi)$ $(i = 1, 2)$ extend to isomorphisms $f_i : H_\epsilon(L_i) \cong (H, \phi)$. Since $\dim_{\mathbb{R}}(L_1) = \dim_{\mathbb{R}}(H)/2 = \dim_{\mathbb{R}}(L_2)$ there exists an isomorphism $g : L_1 \cong L_2$, and hence an isomorphism

$$h = \begin{pmatrix} g & 0 \\ 0 & (g^*)^{-1} \end{pmatrix} : H_\epsilon(L_1) \xrightarrow{\cong} H_\epsilon(L_2) .$$

The composite automorphism

$$\alpha : (H, \phi) \xrightarrow[\cong]{f_1^{-1}} H_\epsilon(L_1) \xrightarrow[\cong]{h} H_\epsilon(L_2) \xrightarrow[\cong]{f_2} (H, \phi)$$

is such that $\alpha(L_1) = L_2$.

The $(-)^n$-symmetric form of a $2n$-dimensional manifold

- Homology and cohomology will be with $\mathbb{R}$-coefficients.
- Manifolds will be oriented.
- The **intersection form** of a $2n$-dimensional manifold with boundary $(M, \partial M)$ is the $(-)^n$-symmetric form given by the evaluation of the cup product on the fundamental class $[M] \in H_{2n}(M, \partial M)$

$$(H, \phi_M) \;=\; (H^n(M, \partial M), (x, y) \mapsto \langle x \cup y, [M] \rangle) \;.$$

- By Poincaré duality and universal coefficient isomorphisms

$$H^n(M, \partial M) \;\cong\; H_n(M) \;,\; H^n(M, \partial M) \;\cong\; H_n(M, \partial M)^*$$

the adjoint linear map $\phi_M : H \to H^*$ fits into an exact sequence

$$\dots \to H_n(\partial M) \to H = H_n(M) \xrightarrow{\phi_M} H^* = H_n(M, \partial M) \to H_{n-1}(\partial M) \to \dots \;.$$

- The isomorphism class of the form is a homotopy invariant of M.
- If M is closed, $\partial M = \emptyset$, then (H, ϕ_M) is nonsingular.

Intersections

- **Geometric interpretation of** $(H_n(M), \phi_M)$
  If $K^n, L^n \subset M^{2n}$ are closed n-dimensional submanifolds which intersect transversely in a 0-dimensional manifold $K \cap L \subset M \backslash \partial M$ then $[K], [L] \in H_n(M) \cong H^n(M, \partial M)$ are such that
  $$\phi_M([K], [L]) \ = \ |K \cap L| \in \mathbb{Z} \ .$$
- **Example** If $(M, \partial M) = (D(\eta), S(\eta))$ is the (D^n, S^{n-1})-bundle of an n-plane bundle $\eta : S^n \to BSO(n)$ over S^n then
  $$(H_n(M), \phi_M) \ = \ (\mathbb{R}, e(\eta)) \ ,$$
  with
  $$\begin{aligned} e(\eta) \ &= \ \text{the Euler number of } \eta \\ &= \ \text{the intersection number of the zero section } S^n \subset M \\ &\quad\ \ \text{with a generic section } S^n \subset M \in \mathbb{Z} \subset H^{2n}(M, \partial M) = \mathbb{R} \\ &\quad\ \ (= 0 \text{ for } n \text{ odd}). \end{aligned}$$

The signature of a manifold

- The **signature** of a $4k$-dimensional manifold with boundary $(M^{4k}, \partial M)$ is
$$\sigma(M) \ = \ \sigma(H^{2k}(M, \partial M), \phi_M) \in \mathbb{Z}$$
- The signature of a manifold was first defined by Weyl in a 1923 paper: http://www.maths.ed.ac.uk/˜aar/papers/weyl.pdf
published in Spanish in South America to spare the author the shame of being regarded as a topologist. Full story in Beno Eckmann's 2004 lecture:
http://www.maths.ed.ac.uk/˜aar/papers/beno.pdf
- **Example** For the torus $T^{4k} = S^{2k} \times S^{2k}$
$$\sigma(T^{4k}) \ = \ \sigma(H^{2k}(T^{4k}), \phi_{T^{4k}}) \ = \ \sigma(H_+(\mathbb{R})) \ = \ 0 \in \mathbb{Z} \ .$$
- **Example** For the complex projective space $\mathbb{C}\,\mathbb{P}^{2k}$
$$\sigma(\mathbb{C}\,\mathbb{P}^{2k}) \ = \ \sigma(H^{2k}(\mathbb{C}\,\mathbb{P}^{2k}), \phi_{\mathbb{C}\,\mathbb{P}^{2k}}) \ = \ \sigma(\mathbb{R}, 1) \ = \ 1 \in \mathbb{Z} \ .$$

The lagrangian of a $(2n+1)$-dimensional manifold with boundary

- **Theorem** (Thom, 1952) If (N^{2n+1}, M^{2n}) is an $(2n+1)$-dimensional manifold with boundary then

$$\begin{aligned} L ~=~ \ker(H_n(M) \to H_n(N)) ~&=~ \mathrm{im}(H^n(N) \to H^n(M)) \\ &\subset H ~=~ H_n(M) ~=~ H^n(M) \end{aligned}$$

  is a lagrangian of the $(-)^n$-symmetric intersection form (H, ϕ_M).

- **Proof** Consider the commutative diagram

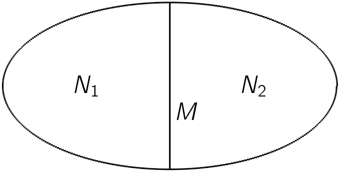
$$\begin{array}{ccccc} H^n(N) & \longrightarrow & H^n(M) & \longrightarrow & H^{n+1}(N,M) \\ \downarrow \cong & & \phi_M \downarrow \cong & & \downarrow \cong \\ H_{n+1}(N,M) & \longrightarrow & H_n(M) & \longrightarrow & H_n(N) \end{array}$$

- **Corollary** If a $4k$-dimensional manifold M is the boundary $M = \partial N$ of a $(4k+1)$-dimensional manifold N then

$$\sigma(M) ~=~ \sigma(H, \phi_M) ~=~ 0 \in \mathbb{Z} ~.$$

The $(-)^n$-symmetric formation of a $(2n+1)$-dimensional manifold

- **Proposition** Let N^{2n+1} be a closed $(2n+1)$-dimensional manifold.



A separating hypersurface $M^{2n} \subset N = N_1 \cup_M N_2$ determines a $(-)^n$-symmetric formation

$$(H, \phi; L_1, L_2) = (H^n(M), \phi_M; \mathrm{im}(H^n(N_1) \to H^n(M)), \mathrm{im}(H^n(N_2) \to H^n(M)))$$

If $H_r(M) \to H_r(N_1) \oplus H_r(N_2)$ is onto for $r = n+1$ and one-one for $r = n-1$ then

$$L_1 \cap L_2 \ = \ H^n(N) \ = \ H_{n+1}(N) \ , \ H/(L_1 + L_2) \ = \ H^{n+1}(N) \ = \ H_n(N) \ .$$

- The stable isomorphism class of the formation is a homotopy invariant of N. If $N = \partial P$ for some P^{2n+2} the class includes $\partial(H_{n+1}(P), \phi_P)$.

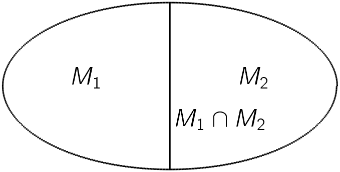
Novikov additivity for $M^{4k} = M_1 \cup M_2$ I.

- Let M^{4k} be a closed $4k$-dimensional manifold which is a union of $4k$-dimensional manifolds with boundary M_1, M_2

$$M^{4k} = M_1 \cup M_2$$

with intersection a separating hypersurface

$$(M_1 \cap M_2)^{4k-1} = \partial M_1 = \partial M_2 \subset M .$$



- **Theorem** (N., 1967) The union has signature

$$\sigma(M) = \sigma(M_1) + \sigma(M_2) \in \mathbb{Z} .$$

Novikov additivity for $M^{4k} = M_1 \cup M_2$ II.

- **Proof** Define the subspaces

$$H_i \;=\; \mathrm{im}(H_{2k}(M_i) \to H_{2k}(M)) \subset H \;=\; H_{2k}(M)\ (i = 1, 2)\ .$$

By the exact sequence

$$H_{2k}(M_1) \to H_{2k}(M) \to H_{2k}(M, M_1) \;=\; H_{2k}(M_2, M_1 \cap M_2) \;=\; H^{2k}(M_2)$$

the subforms $(H_i, \phi_i) \subset (H, \phi)$ are such that

$$(H_1^{\perp}, \phi|) \;=\; (H_2, \phi)\ ,\ (H_2^{\perp}, \phi|) \;=\; (H_1, \phi_1) \subset (H, \phi)\ .$$

The inclusion of the sublagrangian

$$(H_1, 0) \to (H_1, -\phi_1) \oplus (H, \phi)\ ;\ x \mapsto (x, x)$$

is such that up to isomorphism

$$\begin{aligned}(H_1, -\phi_1) \oplus (H, \phi) \;&=\; H_+(H_1) \oplus (H_1^{\perp}/H_1, [-\phi_1 \oplus \phi]) \\ &=\; H_+(H_1) \oplus (H_2, \phi_2)\ ,\ \text{so that}\end{aligned}$$

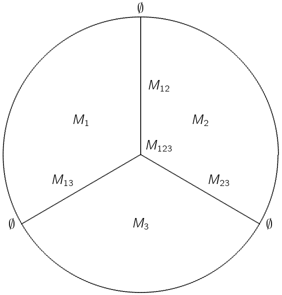
$$\sigma(M) \;=\; \sigma(H, \phi) \;=\; \sigma(H_1, \phi_1) + \sigma(H_2, \phi_2) \;=\; \sigma(M_1) + \sigma(M_2) \in \mathbb{Z}.$$

Wall non-additivity for $M^{4k} = M_1 \cup M_2 \cup M_3$ I.

- Let M^{4k} be a closed $4k$-dimensional manifold which is a triple union

$$M^{4k} \ = \ M_1 \cup M_2 \cup M_3$$

of $4k$-dimensional manifolds with boundary M_1, M_2, M_3 such that the double intersections $M_{ij}^{4k-1} = M_i \cap M_j$ $(1 \leqslant i < j \leqslant 3)$ are codimension 1 submanifolds of M and the triple intersection $M_{123}^{4k-2} = M_1 \cap M_2 \cap M_3 \subset M$ is a codimension 2 submanifold of M, with $\partial M_1 = \partial(M_2 \cup_{M_{23}} M_3) = M_{12} \cup_{M_{123}} M_{13}$ etc.



Wall non-additivity for $M^{4k} = M_1 \cup M_2 \cup M_3$ II.

- **Theorem** (W. Non-additivity of the signature, Invent. Math. 7, 269–274 (1969)) The signature of a triple union $M = M_1 \cup M_2 \cup M_3$ is

$$\sigma(M) \;=\; \sigma(M_1) + \sigma(M_2) + \sigma(M_3) + \sigma(N) \in \mathbb{Z}$$

with $(N^{4k}, \partial N)$ a manifold neighbourhood of $M_{12} \cup M_{13} \cup M_{13} \subset M$

$$N \;=\; (M_{12} \cup M_{23} \cup M_{13}) \times D^1 \cup (M_{123} \times D^2) \;.$$

Let $(H, \phi) = (H_{2k-1}(M_{123}), \phi_{M_{123}})$ be the symplectic form of M_{123}. The non-additivity term $\sigma(N) = \sigma(K, \lambda) \in \mathbb{Z}$ is the signature of the symmetric form (K, λ) defined using the three lagrangians $L_i = \mathrm{im}(H_{2k}(M_{jk}, M_{123}) \to H) \subset H$ of (H, ϕ) determined by the three null-cobordisms M_{ij} of M_{123}, with

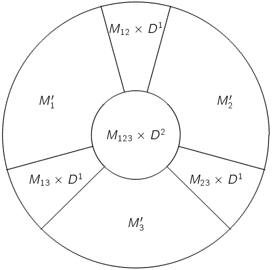
$$K \;=\; \ker(L_1 \oplus L_2 \oplus L_3 \to H) \;,\; \lambda_{ij} \;=\; \lambda_{ji}^* \;:\; L_j \longrightarrow H \xrightarrow{\phi} H^* \longrightarrow L_i^* \;,$$

$$\lambda \;=\; \lambda^* \;=\; \begin{pmatrix} 0 & \lambda_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;:\; K \to K^* \;.$$

Wall non-additivity for $M^{4k} = M_1 \cup M_2 \cup M_3$ III.

- **Idea of proof** The closure of the complement of $N \subset M$ is a disjoint union of 3 copies M_1', M_2', M_3' of M_1, M_2, M_3

$$\overline{M \backslash N} \;=\; M_1' \sqcup M_2' \sqcup M_3'$$



- By Novikov additivity applied to $M = \overline{M \backslash N} \cup_{\partial N} N$

$$\begin{aligned} \sigma(M) &= \sigma(M_1' \cup M_2' \cup M_3') + \sigma(N) \\ &= \sigma(M_1) + \sigma(M_2) + \sigma(M_3) + \sigma(N) \in \mathbb{Z} \; . \end{aligned}$$

Wall non-additivity for $M^{4k} = M_1 \cup M_2 \cup M_3$ IV.

- **Idea of proof** (contd.) Consider the Mayer-Vietoris exact sequence

$$\dots \to H_{2k}(N) \to H_{2k}(M_{23}, M_{123}) \oplus H_{2k}(M_{13}, M_{123}) \oplus H_{2k}(M_{12}, M_{123}) \to H_{2k-1}(M_{123}) \to \dots \ .$$

By algebraic surgery below the middle dimension it may be assumed that $H_{2k}(M_{123}) = 0$, so that the symmetric intersection form of N is

$$\begin{aligned} & (H_{2k}(N), \phi_N) \\ = \ & (\ker(H_{2k}(M_{23}, M_{123}) \oplus H_{2k}(M_{13}, M_{123}) \oplus H_{2k}(M_{12}, M_{123}) \to H_{2k-1}(M_{123})), \lambda) \\ = \ & (K, \lambda)\ ,\ K \ = \ \ker(L_1 \oplus L_2 \oplus L_3 \to H) \end{aligned}$$

and the signature is

$$\sigma(N) \ = \ \sigma(H_{2k}(N), \phi_N) \ = \ \sigma(K, \lambda) \in \mathbb{Z} \ .$$

Algebraic interpretation of the Maslov index

- For any nonsingular symplectic form (H, ϕ) and three lagrangians L_1, L_2, L_3 there is defined a stable isomorphism of formations

$$(H, \phi; L_1, L_2) \oplus (H, \phi; L_2, L_3) \oplus (H, \phi; L_3, L_1) \;\cong\; \partial(K, \lambda)$$

with (K, λ) the symmetric form defined by

$$K \;=\; \ker(L_1 \oplus L_2 \oplus L_3 \longrightarrow H)\ ,$$

$$\lambda \;=\; \lambda^* \;=\; \begin{pmatrix} 0 & \lambda_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;:\; K \longrightarrow K^*\ ,$$

$$\lambda_{ij} \;=\; \lambda^*_{ji} \;:\; L_j \longrightarrow H \xrightarrow{\phi} H^* \longrightarrow L_i^*\ .$$

- (Kashiwara and Schapira, 1992) The Wall non-additivity term is the Maslov index

$$\sigma(N) \;=\; \tau(L_1, L_2, L_3) \;=\; \text{signature}(K, \lambda) \in \mathbb{Z}\ .$$

- (K, λ) is nonsingular if and only if L_1, L_2, L_3 are pairwise complements.

The complex projective plane $\mathbb{CP}^2$ I.

- The Hopf bundle $\eta : S^2 \to BSO(2)$ has clutching function
$$S^1 \to SO(2) \ ; \ x \mapsto (y \mapsto xy)$$
with
$$(D^2, S^1) \to (D(\eta), S(\eta)) \to S^2 \ = \ D^2 \cup_{S^1} D^2 \ ,$$
$$S(\eta) \ = \ S^3 \ , \ e(\eta) \ = \ 1 \in H^2(S^2) \ = \ \mathbb{R} \ .$$
- $M = \mathbb{CP}^2 = M_1 \cup M_2 \cup M_3$ is a closed 4-dimensional manifold with
$$(M_1 \cup_{M_{12}} M_2, M_{13} \cup_{M_{123}} M_{23}) \ = \ (D(\eta), S(\eta)) \ ,$$
$$M_1 \ = \ D^2 \times D^2 \ , \ M_2 \ = \ D^2 \times D^2 \ , \ M_3 \ = \ D^2 \times D^2 \ ,$$
$$M_{12} \ = \ \{((x,y),(x,xy)) \in M_1 \times M_2 \,|\, x \in S^1, y \in D^2\} \ ,$$
$$M_{23} \ = \ \{((x,y),(x,y)) \in M_2 \times M_3 \,|\, x \in S^1, y \in D^2\} \ ,$$
$$M_{13} \ = \ \{((x,y),(y,x)) \in M_1 \times M_3 \,|\, x \in S^1, y \in D^2\} \ ,$$
$$M_{123} \ = \ M_{12} \cap M_{23} \cap M_{13} \ = \ S^1 \times S^1 \ .$$

The complex projective plane $\mathbb{C}\mathbb{P}^2$ II.

- The symmetric intersection form of $M = \mathbb{C}\mathbb{P}^2$ is

$$(H^2(M), \phi_M) = (\mathbb{R}, 1)$$

so the signature is

$$\sigma(M) = \sigma(H^2(M), \phi_M) = 1 \in \mathbb{Z} .$$

- Since each of M_1, M_2, M_3 is contractible,

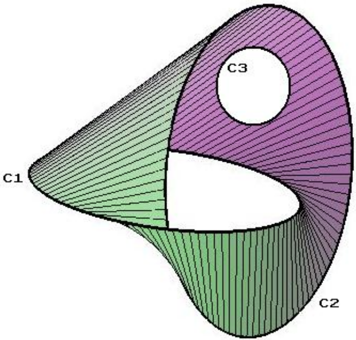
$$\sigma(M_1) = \sigma(M_2) = \sigma(M_3) = 0 \in \mathbb{Z} .$$

In this case, Wall's non-additivity term must be

$$\sigma(N) = \sigma(M) = 1 \in \mathbb{Z} .$$

The complex projective plane $\mathbb{C}\mathbb{P}^2$ III.

- N is a Hopf pair of pants, distinguished by the signature from the thrice-punctured S^4.
- N is a 4-dimensional analogue of the 2-dimensional pair of pants cobordism



The complex projective plane $\mathbb{CP}^2$ IV.

- The symplectic intersection form of $M_{123} = S^1 \times S^1$ is the hyperbolic form
$$(H^1(M_{123}), \phi) \;=\; H_-(\mathbb{R}) \;=\; (\mathbb{R} \oplus \mathbb{R}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$$
- The three lagrangians of $H_-(\mathbb{R})$ are
$$L_1 \;=\; \{(x,x) \,|\, x \in \mathbb{R}\}\,,\; L_2 \;=\; \{(x,0) \,|\, x \in \mathbb{R}\}\,,\; L_3 \;=\; \{(0,x) \,|\, x \in \mathbb{R}\}\,.$$
- The symmetric intersection form of N is
$$(H_2(N), \phi_N) \;=\; (\ker(L_1 \oplus L_2 \oplus L_3 \to \mathbb{R} \oplus \mathbb{R}), \begin{pmatrix} 0 & \lambda_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix})$$
$$= \;(\ker(\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} : \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \oplus \mathbb{R}), \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}) \;=\; (\mathbb{R}, 1)$$

  so that
$$\sigma(N) \;=\; \sigma(\mathbb{R}, 1) \;=\; 1 \in \mathbb{Z}\,.$$