AN INTRODUCTION TO THE MASLOV INDEX IN SYMPLECTIC TOPOLOGY

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The 1-dimensional Lagrangians

- **Definition** (i) Let $\mathbb{R}^2$ have the symplectic form

$$[\ ,\ ]\ :\ \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}\ ;\ ((x_1, y_1), (x_2, y_2)) \mapsto x_1 y_2 - x_2 y_1\ .$$

- (ii) A subspace $L \subset \mathbb{R}^2$ is **Lagrangian** of $(\mathbb{R}^2, [\ ,\ ])$ if

$$L \ = \ L^{\perp} \ = \ \{x \in \mathbb{R}^2 \,|\, [x, y] = 0 \text{ for all } y \in L\}\ .$$

- **Proposition** A subspace $L \subset \mathbb{R}^2$ is a Lagrangian of $(\mathbb{R}^2, [\ ,\ ])$ if and only if L is 1-dimensional,
- **Definition** (i) The **1-dimensional Lagrangian Grassmannian** $\Lambda(1)$ is the space of Lagrangians $L \subset (\mathbb{R}^2, [\ ,\ ])$, i.e. the Grassmannian of 1-dimensional subspaces $L \subset \mathbb{R}^2$.
- (ii) For $\theta \in \mathbb{R}$ let

$$L(\theta) \ = \ \{(r\cos\theta, r\sin\theta) \,|\, r \in \mathbb{R}\} \in \Lambda(1)$$

be the Lagrangian with gradient $\tan\theta$.

The topology of $\Lambda(1)$

- **Proposition** The square function

$$\det{}^2 \; : \; \Lambda(1) \to S^1 \; ; \; L(\theta) \mapsto e^{2i\theta}$$

and the square root function

$$\omega \; : \; S^1 \to \Lambda(1) \; = \; \mathbb{R}\,\mathbb{P}^1 \; ; \; e^{2i\theta} \mapsto L(\theta)$$

are inverse diffeomorphisms, and

$$\pi_1(\Lambda(1)) \; = \; \pi_1(S^1) \; = \; \mathbb{Z} \; .$$

- **Proof** Every Lagrangian L in $(\mathbb{R}^2, [\;,\;])$ is of the type $L(\theta)$, and

$$L(\theta) \; = \; L(\theta') \text{ if and only if } \theta' - \theta \; = \; k\pi \text{ for some } k \in \mathbb{Z} \; .$$

Thus there is a unique $\theta \in [0, \pi)$ such that $L = L(\theta)$. The loop $\omega : S^1 \to \Lambda(1)$ represents the generator

$$\omega = 1 \in \pi_1(\Lambda(1)) \; = \; \mathbb{Z} \; .$$

The Maslov index of a 1-dimensional Lagrangian

- **Definition** The **Maslov index** of a Lagrangian $L = L(\theta)$ in $(\mathbb{R}^2, [\ ,\ ])$ is

$$\tau(L) \ = \ \begin{cases} 1 - \dfrac{2\theta}{\pi} & \text{if } 0 < \theta < \pi \\ 0 & \text{if } \theta = 0 \end{cases}$$

- The Maslov index for Lagrangians in $\bigoplus\limits_n (\mathbb{R}^2, [\ ,\ ])$ is reduced to the special case $n = 1$ by the diagonalization of unitary matrices.
- The formula for $\tau(L)$ has featured in many guises besides the Maslov index (e.g. as assorted η-, γ-, ρ-invariants and an L^2-signature) in the papers of Arnold (1967), Neumann (1978), Atiyah (1987), Cappell, Lee and Miller (1994), Bunke (1995), Nemethi (1995), Cochran, Orr and Teichner (2003), . . .
- See http://www.maths.ed.ac.uk/~aar/maslov.htm for detailed references.

The Maslov index of a pair of 1-dimensional Lagrangians

- **Note** The function $\tau : \Lambda(1) \to \mathbb{R}$ is not continuous.
- **Examples** $\tau(L(0)) = \tau(L(\frac{\pi}{2})) = 0$, $\tau(L(\frac{\pi}{4})) = \frac{1}{2} \in \mathbb{R}$ with

$$L(0) \;=\; \mathbb{R} \oplus 0 \;,\; L(\frac{\pi}{2}) \;=\; 0 \oplus \mathbb{R} \;,\; L(\frac{\pi}{4}) \;=\; \{(x,x) \,|\, x \in \mathbb{R}\} \;.$$

- **Definition** The **Maslov index** of a pair of Lagrangians $(L_1, L_2) = (L(\theta_1), L(\theta_2))$ in $(\mathbb{R}^2, [\;,\;])$ is

$$\tau(L_1, L_2) \;=\; -\tau(L_2, L_1) \;=\; \begin{cases} 1 - \dfrac{2(\theta_2 - \theta_1)}{\pi} & \text{if } 0 \leqslant \theta_1 < \theta_2 < \pi \\ 0 & \text{if } \theta_1 = \theta_2. \end{cases}$$

- **Examples** $\tau(L) = \tau(\mathbb{R} \oplus 0, L)$, $\tau(L, L) = 0$.

The Maslov index of a triple of 1-dimensional Lagrangians

- **Definition** The **Maslov index** of a triple of Lagrangians

$$(L_1, L_2, L_3) = (L(\theta_1), L(\theta_2), L(\theta_3))$$

in $(\mathbb{R}^2, [\ ,\ ])$ is

$$\tau(L_1, L_2, L_3) = \tau(L_1, L_2) + \tau(L_2, L_3) + \tau(L_3, L_1) \in \{-1, 0, 1\} \subset \mathbb{R} .$$

- **Example** If $0 \leqslant \theta_1 < \theta_2 < \theta_3 < \pi$ then $\tau(L_1, L_2, L_3) = 1 \in \mathbb{Z}$.
- **Example** The Wall computation of the signature of $\mathbb{C}\,\mathbb{P}^2$ is given in terms of the Maslov index as

$$\begin{aligned}
\sigma(\mathbb{C}\,\mathbb{P}^2) &= \tau(L(0), L(\pi/4), L(\pi/2)) \\
&= \tau(L(0), L(\pi/4)) + \tau(L(\pi/4), L(\pi/2)) + \tau(L(\pi/2), L(0)) \\
&= \frac{1}{2} + \frac{1}{2} + 0 = 1 \in \mathbb{Z} \subset \mathbb{R} .
\end{aligned}$$

The Maslov index and the degree I.

- A pair of 1-dimensional Lagrangians $(L_1, L_2) = (L(\theta_1), L(\theta_2))$ determines a path in $\Lambda(1)$ from L_1 to L_2

$$\omega_{12} \ : \ I \to \Lambda(1) \ ; \ t \mapsto L((1-t)\theta_1 + t\theta_2) \ .$$

- For any $L = L(\theta) \in \Lambda(1) \backslash \{L_1, L_2\}$

$$\begin{aligned}
(\omega_{12})^{-1}(L) &= \{t \in [0,1] \,|\, L((1-t)\theta_1 + t\theta_2) = L\} \\
&= \{t \in [0,1] \,|\, (1-t)\theta_1 + t\theta_2 = \theta\} \\
&= \begin{cases} \left\{ \dfrac{\theta - \theta_1}{\theta_2 - \theta_1} \right\} & \text{if } 0 < \dfrac{\theta - \theta_1}{\theta_2 - \theta_1} < 1 \\ \emptyset & \text{otherwise .} \end{cases}
\end{aligned}$$

- The degree of a loop $\omega : S^1 \to \Lambda(1) = S^1$ is the number of elements in $\omega^{-1}(L)$ for a generic $L \in \Lambda(1)$. In the geometric applications the Maslov index counts the number of intersections of a curve in a Lagrangian manifold with the codimension 1 cycle of singular points.

The Maslov index and the degree II.

- **Proposition** A triple of Lagrangians (L_1, L_2, L_3) determines a loop in $\Lambda(1)$
$$\omega_{123} \;=\; \omega_{12}\omega_{23}\omega_{31} \;:\; S^1 \to \Lambda(1)$$
with homotopy class the Maslov index of the triple
$$\omega_{123} \;=\; \tau(L_1, L_2, L_3) \in \{-1, 0, 1\} \subset \pi_1(\Lambda(1)) \;=\; \mathbb{Z} \;.$$
- **Proof** It is sufficient to consider the special case
$$(L_1, L_2, L_3) \;=\; (L(\theta_1), L(\theta_2), L(\theta_3))$$
with $0 \leqslant \theta_1 < \theta_2 < \theta_3 < \pi$, so that
$$\begin{aligned} \det{}^2\omega_{123} \;&=\; 1 \;:\; S^1 \to S^1 \;, \\ \mathrm{degree}(\det{}^2\omega_{123}) \;&=\; 1 \;=\; \tau(L_1, L_2, L_3) \in \mathbb{Z} \end{aligned}$$

The Euclidean structure on $\mathbb{R}^{2n}$

- The **phase space** is the $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$, with preferred basis $\{p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n\}$.
- The $2n$-dimensional phase space carries 4 additional structures.
- **Definition** The **Euclidean structure** on $\mathbb{R}^{2n}$ is the positive definite symmetric form over $\mathbb{R}$

$$(\ , \ ) \ : \ \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R} \ ; \ (v, v') \mapsto \sum_{j=1}^{n} x_j x'_j + \sum_{k=1}^{n} y_k y'_k \ ,$$

$$(v \ = \ \sum_{j=1}^{n} x_j p_j + \sum_{k=1}^{n} y_k q_k \ , \ v' \ = \ \sum_{j=1}^{n} x'_j p_j + \sum_{k=1}^{n} y'_k q_k \in \mathbb{R}^{2n}) \ .$$

- The automorphism group of $(\mathbb{R}^{2n}, (\ , \ ))$ is the **orthogonal group** $O(2n)$ of invertible $2n \times 2n$ matrices $A = (a_{jk})$ $(a_{jk} \in \mathbb{R})$ such that $A^*A = I_{2n}$ with $A^* = (a_{kj})$ the transpose.

The complex structure on $\mathbb{R}^{2n}$

- **Definition** The **complex structure** on $\mathbb{R}^{2n}$ is the linear map

$$\imath \ : \ \mathbb{R}^{2n} \to \mathbb{R}^{2n} \ ; \ \sum_{j=1}^{n} x_j p_j + \sum_{k=1}^{n} y_k q_k \mapsto \sum_{j=1}^{n} x_j p_j - \sum_{k=1}^{n} y_k q_k$$

  such that $\imath^2 = -1$. Use $\imath$ to regard $\mathbb{R}^{2n}$ as an n-dimensional complex vector space, with an isomorphism

$$\mathbb{R}^{2n} \to \mathbb{C}^n \ ; \ v \mapsto (x_1 + iy_1, x_2 + iy_2, \dots, x_n + iy_n) \ .$$

- The automorphism group of $(\mathbb{R}^{2n}, \imath) = \mathbb{C}^n$ is the **complex general linear group** $GL(n, \mathbb{C})$ of invertible $n \times n$ matrices (a_{jk}) $(a_{jk} \in \mathbb{C})$.

The symplectic structure on $\mathbb{R}^{2n}$

- **Definition** The **symplectic structure** on $\mathbb{R}^{2n}$ is the symplectic form

$$[\ ,\ ]\ :\ \mathbb{R}^{2n}\times\mathbb{R}^{2n}\to\mathbb{R}\ ;$$

$$(v,v')\mapsto[v,v']\ =\ (\imath v,v')\ =\ -[v',v]\ =\ \sum_{j=1}^{n}(x'_j y_j - x_j y'_j)\ .$$

- The automorphism group of $(\mathbb{R}^{2n},[\ ,\ ])$ is the **symplectic group** $Sp(n)$ of invertible $2n\times 2n$ matrices $A=(a_{jk})$ $(a_{jk}\in\mathbb{R})$ such that

$$A^*\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}A\ =\ \begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}\ .$$

The n-dimensional Lagrangians

- **Definition** Given a finite-dimensional real vector space V with a nonsingular symplectic form $[\ ,\ ] : V \times V \to \mathbb{R}$ let $\Lambda(V)$ be the set of Lagrangian subspaces $L \subset V$, with

$$L \;=\; L^{\perp} \;=\; \{x \in V \,|\, [x,y] = 0 \in \mathbb{R} \text{ for all } y \in L\}\ .$$

- **Terminology** $\Lambda(\mathbb{R}^{2n}) = \Lambda(n)$. The real and imaginary Lagrangians

$$\mathbb{R}^n \;=\; \{\sum_{j=1}^{n} x_j p_j \,|\, x_j \in \mathbb{R}^n\}\ ,\ \imath\mathbb{R}^n \;=\; \{\sum_{k=1}^{n} y_k q_k \,|\, y_k \in \mathbb{R}^n\} \in \Lambda(n)$$

  are complementary, with $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \imath\mathbb{R}^n$.
- **Definition** The **graph** of a symmetric form $(\mathbb{R}^n, \phi)$ is the Lagrangian

$$\Gamma_{(\mathbb{R}^n,\phi)} \;=\; \{(x,\phi(x)) \,|\, x = \sum_{j=1}^{n} x_j p_j,\ \phi(x) = \sum_{j=1}^{n}\sum_{k=1}^{n} \phi_{jk} x_j q_k\} \in \Lambda(n)$$

  complementary to $\imath\mathbb{R}^n$.
- **Proposition** Every Lagrangian complementary to $\imath\mathbb{R}^n$ is a graph.

The hermitian structure on $\mathbb{R}^{2n}$

- **Definition** The **hermitian inner product** on $\mathbb{R}^{2n}$ is defined by

$$\langle\ ,\ \rangle\ :\ \mathbb{R}^{2n}\times\mathbb{R}^{2n}\to\mathbb{C}\ ;$$

$$(v,v')\mapsto\langle v,v'\rangle\ =\ (v,v')+i[v,v']\ =\ \sum_{j=1}^{n}(x_j+iy_j)(x_j'-iy_j')\ .$$

or equivalently by

$$\langle\ ,\ \rangle\ :\ \mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}\ ;\ (z,z')\mapsto\langle z,z'\rangle\ =\ \sum_{j=1}^{n}z_j\overline{z}_j'\ .$$

- The automorphism group of $(\mathbb{C}^n,\langle\ ,\ \rangle)$ is the **unitary group** $U(n)$ of invertible $n\times n$ matrices $A=(a_{jk})$ $(a_{jk}\in\mathbb{C})$ such that $AA^*=I_n$, with $A^*=(\overline{a}_{kj})$ the conjugate transpose.

The general linear, orthogonal and unitary groups

- **Proposition** (Arnold, 1967) (i) The automorphism groups of $\mathbb{R}^{2n}$ with respect to the various structures are related by

$$O(2n) \cap GL(n,\mathbb{C}) \;=\; GL(n,\mathbb{C}) \cap Sp(n) \;=\; Sp(n) \cap O(2n) \;=\; U(n)\ .$$

- (ii) The determinant map $\det : U(n) \to S^1$ is the projection of a fibre bundle

$$SU(n) \to U(n) \to S^1\ .$$

- (iii) Every $A \in U(n)$ sends the standard Lagrangian $\imath\mathbb{R}^n$ of $(\mathbb{R}^{2n}, [\ ,\ ])$ to a Lagrangian $A(\imath\mathbb{R}^n)$. The unitary matrix $A = (a_{jk})$ is such that $A(\imath\mathbb{R}^n) = \imath\mathbb{R}^n$ if and only if each $a_{jk} \in \mathbb{R} \subset \mathbb{C}$, with

$$O(n) \;=\; \{A \in U(n) \,|\, A(\imath\mathbb{R}^n) = \imath\mathbb{R}^n\} \subset U(n)\ .$$

The Lagrangian Grassmannian $\Lambda(n)$ I.

- $\Lambda(n)$ is the space of all Lagrangians $L \subset (\mathbb{R}^{2n}, [\ ,\ ])$.
- **Proposition** (Arnold, 1967) The function

$$U(n)/O(n) \to \Lambda(n)\ ;\ A \mapsto A(\imath\mathbb{R}^n)$$

  is a diffeomorphism.
- $\Lambda(n)$ is a compact manifold of dimension

$$\dim \Lambda(n) \;=\; \dim U(n) - \dim O(n) \;=\; n^2 - \frac{n(n-1)}{2} \;=\; \frac{n(n+1)}{2}\ .$$

  The graphs $\{\Gamma_{(\mathbb{R}^n,\phi)} \,|\, \phi^* = \phi \in M_n(\mathbb{R})\} \subset \Lambda(n)$ define a chart at $\mathbb{R}^n \in \Lambda(n)$.
- **Example** (Arnold and Givental, 1985)

$$\begin{aligned}\Lambda(2)^3 \;&=\; \{[x,y,z,u,v] \in \mathbb{R}\,\mathbb{P}^4 \,|\, x^2+y^2+z^2 = u^2+v^2\} \\ &=\; S^2 \times S^1/\{(x,y) \sim (-x,-y)\}\ .\end{aligned}$$

The Lagrangian Grassmannian $\Lambda(n)$ II.

- In view of the fibration

$$\Lambda(n) \;=\; U(n)/O(n) \to BO(n) \to BU(n)$$

  $\Lambda(n)$ classifies real n-plane bundles β with a trivialisation $\delta\beta : \mathbb{C} \otimes \beta \cong \epsilon^n$ of the complex n-plane bundle $\mathbb{C} \otimes \beta$.
- The canonical real n-plane bundle η over $\Lambda(n)$ is

$$E(\eta) \;=\; \{(L, \ell) \,|\, L \in \Lambda(n),\; \ell \in L\}\ .$$

  The complex n-plane bundle $\mathbb{C} \otimes \eta$

$$E(\mathbb{C} \otimes \eta) \;=\; \{(L, \ell_{\mathbb{C}}) \,|\, L \in \Lambda(n),\; \ell_{\mathbb{C}} \in \mathbb{C} \otimes_{\mathbb{R}} L\}$$

  is equipped with the canonical trivialisation $\delta\eta : \mathbb{C} \otimes \eta \cong \epsilon^n$ defined by

$$\begin{aligned}
&\delta\eta \;:\; E(\mathbb{C} \otimes \eta) \xrightarrow{\;\cong\;} E(\epsilon^n) \;=\; \Lambda(n) \times \mathbb{C}^n\ ;\\
&(L, \ell_{\mathbb{C}}) \mapsto (L, (p, q)) \text{ if } \ell_{\mathbb{C}} = (p, q) \in \mathbb{C} \otimes_{\mathbb{R}} L = L \oplus iL = \mathbb{C}^n\ .
\end{aligned}$$

The fundamental group $\pi_1(\Lambda(n))$ I.

- **Theorem** (Arnold, 1967) The square of the determinant function

$$\det{}^2 \ : \ \Lambda(n) \to S^1 \ ; \ L = A(\imath\mathbb{R}^n) \mapsto \det(A)^2$$

induces an isomorphism

$$\det{}^2 \ : \ \pi_1(\Lambda(n)) \xrightarrow{\cong} \pi_1(S^1) \ = \ \mathbb{Z} \ .$$

- **Proof** By the homotopy exact sequence of the commutative diagram of fibre bundles

$$\begin{tikzcd}
SO(n) \arrow[r] \arrow[d] & O(n) \arrow[r, "\det"] \arrow[d] & O(1) = S^0 \arrow[d] \\
SU(n) \arrow[r] \arrow[d] & U(n) \arrow[r, "\det"] \arrow[d] & U(1) = S^1 \arrow[d, "z \mapsto z^2"] \\
S\Lambda(n) \arrow[r] & \Lambda(n) \arrow[r, "\det^2"] & \Lambda(1) = S^1
\end{tikzcd}$$

The fundamental group $\pi_1(\Lambda(n)) = \mathbb{Z}$ II.

- **Corollary 1** The function

$$\pi_1(\Lambda(n)) \to \mathbb{Z}\ ;\ (\omega : S^1 \to \Lambda(n)) \mapsto \text{degree}(\ S^1 \xrightarrow{\omega} \Lambda(n) \xrightarrow{\det^2} S^1\ )$$

  is an isomorphism.

- **Corollary 2** The cohomology class

$$\alpha \in H^1(\Lambda(n)) \ = \ \text{Hom}(\pi_1(\Lambda(n)), \mathbb{Z})$$

  characterized by

$$\alpha(\omega) \ = \ \text{degree}(\ S^1 \xrightarrow{\omega} \Lambda(n) \xrightarrow{\det^2} S^1\ ) \in \mathbb{Z}$$

  is a generator $\alpha \in H^1(\Lambda(n)) = \mathbb{Z}$.

$\pi_1(\Lambda(1)) = \mathbb{Z}$ in terms of line bundles

- **Example** The universal real line bundle $\eta : S^1 \to BO(1) = \mathbb{R}\mathbb{P}^\infty$ is the infinite Möbius band over S^1

$$E(\eta) \ = \ \mathbb{R} \times [0,\pi]/\{(x,0) \sim (-x,\pi)\} \to S^1 \ = \ [0,\pi]/(0 \sim \pi) \ .$$

The complexification $\mathbb{C} \otimes \eta : S^1 \to BU(1) = \mathbb{C}\mathbb{P}^\infty$ is the trivial complex line bundle over S^1

$$E(\mathbb{C} \otimes \eta) \ = \ \mathbb{C} \times [0,\pi]/\{(z,0) \sim (-z,\pi)\} \to S^1 \ = \ [0,\pi]/(0 \sim \pi) \ .$$

The canonical trivialisation $\delta\eta : \mathbb{C} \otimes \eta \cong \epsilon$ is defined by

$$\delta\eta \ : \ E(\mathbb{C} \otimes \eta) \xrightarrow{\cong}$$
$$E(\epsilon) \ = \ \mathbb{C} \times [0,\pi]/\{(z,0) \sim (z,\pi)\} \ = \ \mathbb{C} \times S^1 \ ; \ [z,\theta] \mapsto [ze^{i\theta},\theta] \ .$$

- The canonical pair $(\delta\eta, \eta)$ represents

$$(\delta\eta,\eta) \ = \ 1 \in \pi_1(\Lambda(1)) \ = \ \pi_1(U(1)/O(1)) \ = \ \mathbb{Z} \ ,$$

corresponding to the loop $\omega : S^1 \to \Lambda(1); e^{2i\theta} \mapsto L(\theta)$.

The Maslov index for n-dimensional Lagrangians

- Unitary matrices can be diagonalized. For every $A \in U(n)$ there exists $B \in U(n)$ such that
$$BAB^{-1} \;=\; D(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})$$
is the diagonal matrix, with $e^{i\theta_j} \in S^1$ the eigenvalues, i.e. the roots of the characteristic polynomial
$$\mathrm{ch}_z(A) \;=\; \det(zI_n - A) \;=\; \prod_{j=1}^{n}(z - e^{i\theta_j}) \in \mathbb{C}[z] \;.$$
- **Definition** The **Maslov index** of $L \in \Lambda(n)$ is
$$\tau(L) \;=\; \sum_{j=1}^{n}(1 - 2\theta_j/\pi) \in \mathbb{R}$$
with $\theta_1, \theta_2, \ldots, \theta_n \in [0, \pi)$ such that $\pm e^{i\theta_1}, \pm e^{i\theta_2}, \ldots, \pm e^{i\theta_n}$ are the eigenvalues of any $A \in U(n)$ such that $A(\imath\mathbb{R}^n) = L$.
- There are similar definitions of $\tau(L, L') \in \mathbb{R}$ and $\tau(L, L', L'') \in \mathbb{Z}$.

The Maslov cycle

- Corollary 2 above shows that the Maslov index $\alpha \in H^1(\Lambda(n))$ is the pullback along $\det^2 : \Lambda(n) \to S^1$ of the standard form $\mathrm{d}\theta$ on S^1.
- **Definition** The Poincaré dual of α is the **Maslov cycle**

$$\mathfrak{m} \ = \ \{L \in \Lambda(n) \,|\, L \cap \imath\mathbb{R}^n \neq \{0\}\} \ .$$

- In his 1967 paper Arnold constructs this cycle following ideas that generalise the standard constructions on Schubert varieties in the Grassmannians of linear subspaces of Euclidean spaces, and proves that

$$[\mathfrak{m}] \in H_{(n+2)(n-1)/2}(\Lambda(n))$$

is the Poincaré dual of $\alpha \in H^1(\Lambda(n))$.

Lagrangian submanifolds of $\mathbb{R}^{2n}$ I.

- An n-dimensional submanifold $M^n \subset \mathbb{R}^{2n}$ is **Lagrangian** if for each $x \in M$ the tangent n-plane

$$\mathrm{T}_x(M) \subset \mathrm{T}_x(\mathbb{R}^{2n}) \;=\; \mathbb{R}^{2n}$$

  is a Lagrangian subspace of $(\mathbb{R}^{2n}, [\ ,\ ])$.
- The tangent bundle $\mathrm{T}_M : M \to BO(n)$ is the pullback of the canonical real n-plane bundle over $\Lambda(n)$

$$E(\eta) \;=\; \{(\lambda, \ell) \,|\, \lambda \in \Lambda(n),\ \ell \in \lambda\}$$

  along the classifying map $\zeta : M \to \Lambda(n); x \mapsto \mathrm{T}_x(M)$, with $\mathrm{T}_M : M \xrightarrow{\zeta} \Lambda(n) \xrightarrow{\eta} BO(n)$. The complex n-plane bundle $\mathbb{C} \otimes \mathrm{T}_M = \mathrm{T}_{\mathbb{C}^n}|_M$ has a canonical trivialisation.
- **Definition** The **Maslov index** of a Lagrangian submanifold $M^n \subset \mathbb{R}^{2n}$ is the pullback of the generator $\alpha \in H^1(\Lambda(n))$

$$\tau(M) \;=\; \zeta^*(\alpha) \in H^1(M)\ .$$

Lagrangian submanifolds of $\mathbb{R}^{2n}$ II.

- Given a Lagrangian submanifold $M^n \subset \mathbb{R}^{2n}$ let

$$\pi \ = \ \text{proj}| \ : \ M^n \to \mathbb{R}^n \ ; \ (p,q) \mapsto p$$

be the restriction of the projection $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \imath\mathbb{R}^n \to \mathbb{R}^n$.
The differentials of π are the differentiable maps

$$\mathrm{d}\pi_x \ : \ \mathrm{T}_x(M) \to \mathrm{T}_{\pi(x)}(\mathbb{R}^n) \ = \ \mathbb{R}^n \, ; \, \ell \mapsto p \text{ if } \ell = (p,q) \in \mathrm{T}_x(M) \subset \mathbb{R}^{2n}$$

with $\ker(\mathrm{d}\pi_x) = \mathrm{T}_x(M) \cap \imath\mathbb{R}^n \subseteq \imath\mathbb{R}^n$.
- If π is a local diffeomorphism then each $\mathrm{d}\pi_x$ is an isomorphism of real vector spaces, with kernel $\mathrm{T}_x(M) \cap \imath\mathbb{R}^n = \{0\}$.
- **Definition** The **Maslov cycle** of a Lagrangian submanifold $M \subset \mathbb{R}^{2n}$ is

$$\mathfrak{m} \ = \ \{x \in M \,|\, \mathrm{T}_x(M) \cap \imath\mathbb{R}^n \neq \{0\}\} \ .$$

- **Theorem** (Arnold, 1967) Generically, $\mathfrak{m} \subset M$ is a union of open submanifolds of codimension $\geqslant 1$. The homology class $[\mathfrak{m}] \in H_{n-1}(M)$ is the Poincaré dual of the Maslov index class $\tau(M) \in H^1(M)$, measuring the failure of $\pi : M \to \mathbb{R}^n$ to be a local diffeomorphism.

Symplectic manifolds

- **Definition** A manifold N is **symplectic** if it admits a 2-form ω which is closed and non-degenerate.
- **Examples** $S^2, \mathbb{R}^{2n} = \mathbb{C}^n = \mathrm{T}^*\mathbb{R}^n$; the cotangent bundle T^*B of any manifold B with a **canonical** symplectic form.
- **Definition** A submanifold $M \subset N$ of a symplectic manifold is **Lagrangian** if $\omega|_M = 0$ and each $\mathrm{T}_x M \subset \mathrm{T}_x N$ $(x \in M)$ is a Lagrangian subspace.
- **Remark** A symplectic manifold N is necessarily even dimensional (because of the non-degeneracy of ω) and the dimension of a Lagrangian submanifold M is necessarily half the dimension of the ambient manifold.
- **Examples** If N is two dimensional, any one dimensional submanifold is Lagrangian (*why?*). If $N = \mathrm{T}^*B$ for some manifold B with the canonical symplectic form, then each cotangent space $\mathrm{T}^*_b B$ is a Lagrangian submanifold.

The $\Lambda(n)$-affine structure of $\Lambda(V)$

- Let V be a $2n$-dimensional real vector space with nonsingular symplectic form $[\ ,\ ] : V \times V \to \mathbb{R}$ and a complex structure $\imath : V \to V$ such that $V \times V \to \mathbb{R}; (v, w) \mapsto [\imath v, w]$ is an inner product. Let $U(V)$ be the group of automorphisms $A : (V, [\ ,\ ], \imath) \to (V, [\ ,\ ], \imath)$.
- For $L \in \Lambda(V)$ let $O(L) = \{A \in U(V) \,|\, A(L) = L\}$.
- **Proposition** (Guillemin and Sternberg) (i) The function

$$U(V)/O(L) \to \Lambda(V)\ ;\ A \mapsto A(L)$$

  is a diffeomorphism.
- (ii) There exists an isomorphism $f_L : (\mathbb{R}^{2n}, [\ ,\ ], \imath) \cong (V, [\ ,\ ], \imath)$ such that $f_L(\imath\mathbb{R}^n) = L$.
- (iii) For any f_L as in (ii) the diffeomorphism

$$f_L\ :\ \Lambda(n) = U(n)/O(n) \to \Lambda(V) = U(V)/O(L)\ ;\ \lambda \mapsto f_L(\lambda)$$

  only depends on L, and not on the choice of f_L.
- Thus $\Lambda(V)$ has a $\Lambda(n)$-affine structure: for any $L_1, L_2 \in \Lambda(V)$ there is defined a difference element $(L_1, L_2) \in \Lambda(n)$, with $(L_1, L_1) = \imath\mathbb{R}^n \subset \mathbb{R}^{2n}$

An application of the Maslov index I.

- For any n-dimensional manifold B the tangent bundle of the cotangent bundle
$$E \;=\; \mathrm{T}(\mathrm{T}^*B) \to \mathrm{T}^*B$$
has each fibre a $2n$-dimensional symplectic vector space. The symplectic form is given in canonical (or Darboux) coordinates as
$$\omega \;=\; \mathrm{d}p \wedge \mathrm{d}q \;\; (q \in B,\, p \in \mathrm{T}^*_q B)\ .$$
- The **bundle of Lagrangian planes** $\Pi : \Lambda(E) \to \mathrm{T}^*B$ is the fibre bundle with fibres the affine $\Lambda(n)$-sets $\Pi^{-1}(x) = \Lambda(\mathrm{T}_x\mathrm{T}^*B)$ of Lagrangians in the $2n$-dimensional symplectic vector space $\mathrm{T}_x\mathrm{T}^*B$.

An application of the Maslov index II.

- The projection $\pi : \mathrm{T}^*B \to B$ induces the

$$\textbf{vertical subbundle } V \;=\; \ker \mathrm{d}\pi \subset \mathrm{T}(\mathrm{T}^*B) \ ,$$

  whose fibres are Lagrangian subspaces.
- The map

$$s \ : \ \mathrm{T}^*B \to \Lambda(E)\,;\ x = (p,q) \mapsto V_x = \ker(\mathrm{d}\pi_x : \mathrm{T}_x(\mathrm{T}^*B) \to \mathrm{T}_{\pi(x)}(B))$$

  is a section of the bundle of Lagrangian planes.
- For any other section $r : \mathrm{T}^*B \to \Lambda(E)$ use the $\Lambda(n)$-affine structures of the fibres $\Pi^{-1}(x)$ to define a difference map

$$\zeta \ = \ (r,s) \ : \ \mathrm{T}^*B \to \Lambda(n) \ ; \ x \mapsto (r(x), s(x))$$

An application of the Maslov index III.

- Let M be a Lagrangian submanifold of T^*B, $\dim M = \dim B = n$.
- If $\pi|_M : M \to B$ is a local diffeomorphism then each $\mathrm{d}(\pi|_M)_x : \mathrm{T}_x(M) \to \mathrm{T}_{\pi(x)}(B)$ is an isomorphism of vector bundles, with kernel $\mathrm{T}_x(M) \cap V_x = \{0\}$.
- The Maslov index measures the failure of $\pi|_M : M \to B$ to be a local diffeomorphism, in general.
- We let

$$E_M \;=\; \mathrm{T}M \subset \mathrm{T}(\mathrm{T}^*B)\;,\; \Lambda(E_M) \;=\; \Pi^{-1}(M) \subset \Lambda(E)$$

so that $\Pi| : \Lambda(E_M) \to M$ is a fibre bundle with the fibre over $x \in M$ the $\Lambda(n)$-affine set $(\Pi|)^{-1}(x) = \Lambda(\mathrm{T}_x M)$.

An application of the Maslov index IV.

- The maps $r, s|_M : M \to \Lambda(E_M)$ defined by

$$r(x) \;=\; \mathrm{T}_x M \;,\; s|_M(x) \;=\; V_x \subset \mathrm{T}_x(\mathrm{T}^*B)$$

  are sections of $\Pi|$. The difference map $\zeta = (r, s|_M) : M \to \Lambda(n)$ classifies the real n-plane bundle over M with complex trivialisation in which the fibre over $x \in M$ is the Lagrangian $(\mathrm{T}_x M, V_x) \in \Lambda(n)$.
- **Definition** The **Maslov index class** is the pullback along ζ of the generator $\alpha = 1 \in H^1(\Lambda(n)) = \mathbb{Z}$

$$\alpha_M \;=\; \zeta^* \alpha \in H^1(M) \;.$$

- **Proposition** The homology class $[\mathfrak{m}] \in H_{n-1}(M)$ of the Maslov cycle

$$\mathfrak{m} \;=\; \{x \in M \,|\, \mathrm{T}_x M \cap V_x \neq \{0\}\} \;,$$

  is the Poincaré dual of the Maslov index class $\alpha_M \in H^1(M)$.
- If the class $[\mathfrak{m}] = \alpha_M \in H_{n-1}(M) = H^1(M)$ is non-zero then the projection $\pi|_M : M \to B$ fails to be a local diffeomorphism.

An application of the Maslov index V.

- (Butler, 2007) This formulation of the Maslov index can be used to prove that if Σ is an n-dimensional manifold which is topologically T^n, and which admits a geometrically semi-simple convex Hamiltonian, then Σ has the standard differentiable structure.

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