BRAIDS AND THEIR SEIFERT SURFACES

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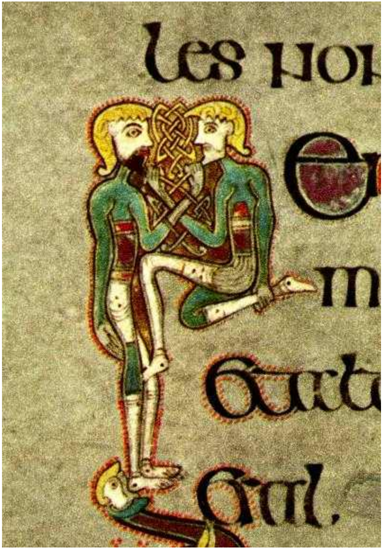
A braid in the Book of Kells



The mathematical definition of a braid

- Fix $n \geqslant 2$ and n distinct points $z_1, z_2, \dots, z_n \in D^2$.
- An n**-strand braid** β is an embedding

$$\beta \ : \ \coprod_n I \ = \ \{1,2,\dots,n\} \times I \subset D^2 \times I \ ; \ (k,t) \mapsto \beta(k,t)$$

  such that each of the composites

$$I \xrightarrow{\beta(k,-)} D^2 \times I \xrightarrow{\text{projection}} I \ \ (1 \leqslant k \leqslant n)$$

  is a homeomorphism, and

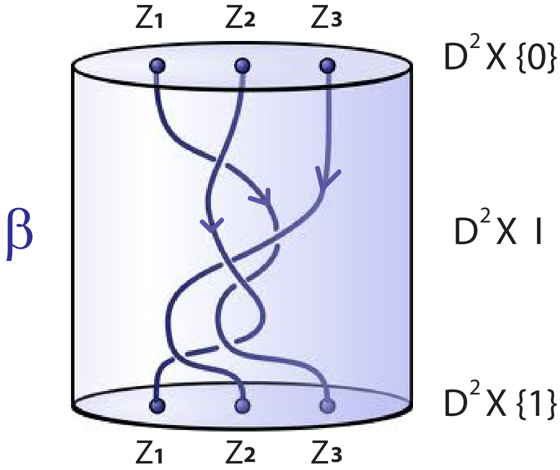
$$\beta(k,0) \ = \ (z_k,0) \ \in D^2 \times \{0\} \ , \ \beta(k,1) \ = \ (z_{\sigma(k)},1) \in D^2 \times \{1\}$$

  for a permutation $\sigma \in \Sigma_n$ of $\{1,2,\dots,n\}$.
- β defines n disjoint forward paths $t \mapsto \beta(k,t)$ in $D^2 \times I \subset \mathbb{R}^3$ from $(z_k,0)$ to $(z_{\sigma(k)},1)$, such that each section

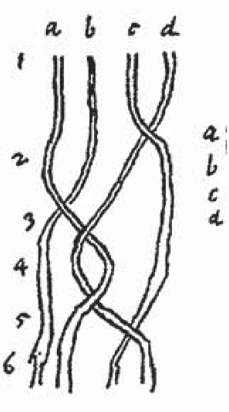
$$\beta(\{1,2,\dots,n\} \times I) \cap (D^2 \times \{t\}) \ (t \in I)$$

  consists of n points.

An example of a 3-strand braid with $\sigma = (132)$



A braid drawn by Gauss (1833)

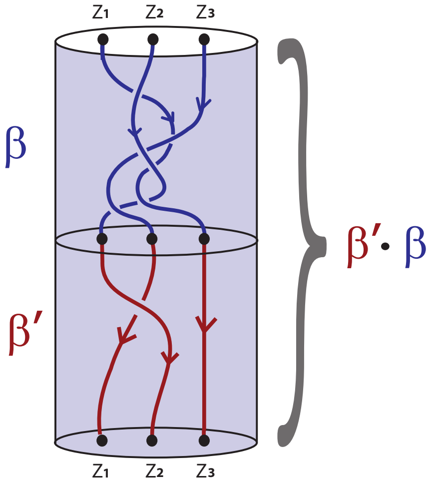
- 

- Further 19th century developments: Listing, Tait, Hurwitz.
- See Moritz Epple's history paper Orbits of asteroids, a braid, and the first link invariant, Mathematical Intelligencer, 20, 45-52 (1996)

Concatenation of braids

- The **concatenation** of n-strand braids β, β' is the n-strand braid
$$\beta'\beta \ : \ \coprod_n I \subset D^2 \times I \subset \mathbb{R}^3$$
defined by
$$(\beta'\beta)(k,t) \ = \ \begin{cases} \beta(k,2t) & \text{if } 0 \leqslant t \leqslant 1/2 \\ \beta'(k,2t-1) & \text{if } 1/2 \leqslant t \leqslant 1 \end{cases}$$
for $1 \leqslant k \leqslant n$, $t \in I$, with permutation the composite $\sigma'\sigma$.

An example of concatenation



Isotopy of braids

- Two n-strand braids

$$\beta_0\ ,\ \beta_1\ :\ \coprod_n I \subset D^2 \times I$$

are **isotopic** if there exist braids

$$\beta_s\ :\ \coprod_n I \subset D^2 \times I\ (s \in I)$$

such that the function

$$I \times \coprod_n I \to D^2 \times I\ ;\ (s,k,t) \mapsto \beta_s(k,t)$$

is continuous.

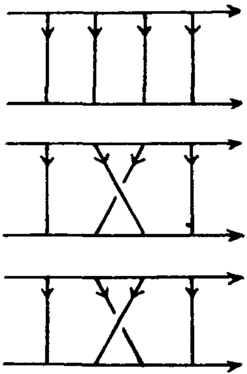
- Same permutations

$$\sigma_0\ =\ \sigma_s\ =\ \sigma_1 \in \Sigma_n\ .$$

- Braid applet

Artin

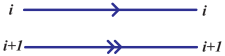
- Emil Artin founded the modern theory of braids in Theorie der Zöpfe (1925), defining the n-**strand braid group** B_n : the set of isotopy classes of n-strand braids under concatenation.
- A trivial braid, a braid with an overcrossing and a braid with an undercrossing



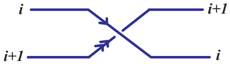
The n-strand braids $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$

- The **trivial** n-strand braid is

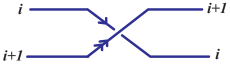
$$\sigma_0 \ : \ \coprod_n I \subset D^2 \times I \ ; \ t_i \mapsto (z_i, t_i)$$



- For $i = 1, 2, \ldots, n-1$ the **elementary** n**-strand braid** σ_i is obtained from σ_0 by introducing an overcrossing of the ith strand and the $(i+1)$th strand, with permutation $(i \ i+1) \in \Sigma_n$.



- The **elementary** n**-strand braid** σ_i^{-1} is defined in the same way but with an under crossing.



The n-strand braid group B_n

- The **concatenation** of two n-strand braids β, β' is the n-strand braid $\beta\beta'$ obtained by identifying $\beta(1_i) = \beta'(0_i)$.
- B_n is the set of isotopy classes of n-strand braids β, with composition by concatenation, and unit σ_0.
- B_n has generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and relations

$$\begin{cases} \sigma_i\sigma_j \ = \ \sigma_j\sigma_i & \text{if } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i \ = \ \sigma_j\sigma_i\sigma_j & \text{if } |i-j| \ = \ 1\ . \end{cases}$$

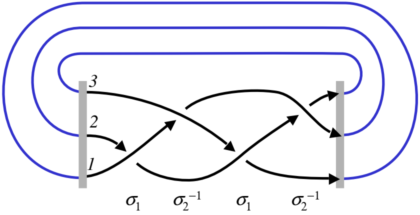
- Every n-strand braid β is represented by a word in B_n in ℓ generators, corresponding to a sequence of ℓ **crossings** in a plane projection.
- The concatenation $\beta\sigma_i$ is obtained from β by adding to the sequence a crossing of the ith strand over the $(i+1)$th strand.
- The representation theory of the braid groups much studied. Highlight: the Jones polynomial.

The closure of a braid

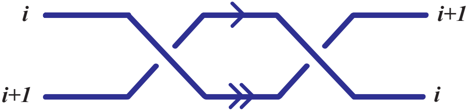
- The **closure** of an n-strand braid β is the c-component link

$$\widehat{\beta} \;=\; \beta \cup \sigma_0 \;:\; \coprod_n I \cup_\sigma \coprod_n I \;=\; \coprod_c S^1 \subset \mathbb{R}^3$$

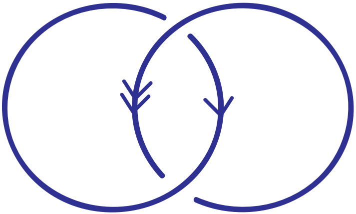
  with $c = |\{1,2,\ldots,n\}/\sigma|$ the number of cycles in $\sigma \in \Sigma_n$.
- Alexander proved in A lemma on systems of knotted curves (1923) that every link is the closure $\widehat{\beta}$ of a braid β.
- **Example** A braid representation of the figure eight knot, with 3 strands and 4 crossings



The closure of $\sigma_1\sigma_1$ is the Hopf link



The 2-strand braid $\beta = \sigma_1 \sigma_1$



The closure $\widehat{\beta}$ = Hopf link

The Seifert surfaces of a link

- A **Seifert surface** for a link

$$L \ : \ \coprod S^1 \subset \mathbb{R}^3$$

is a surface $F^2 \subset \mathbb{R}^3$ with boundary

$$\partial F \ = \ L(\coprod S^1) \subset \mathbb{R}^3 \ .$$

- Seifert in Über das Geschlecht von Knoten (1935) proved that every link L admits a Seifert surface of the type

$$F \ = \ \coprod_n D^2 \cup \coprod_\ell D^1 \times D^1 \subset \mathbb{R}^3$$

using an algorithm starting with a plane projection.

- A link L has many projections, and many Seifert surfaces.

The algorithm for a Seifert surface

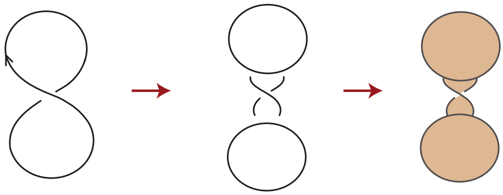
- For any link $L : \coprod S^1 \subset \mathbb{R}^3$ there exists a linear map $P : \mathbb{R}^3 \to \mathbb{R}^2$ (many in fact) such that the image of the composite $PL : \coprod S^1 \to \mathbb{R}^2$ is a collection of oriented curves with ℓ transverse double points labelled as over/underpasses. This is a **plane projection** of L.
- Given L and a plane projection traverse the curves, switching each intersection according to over/underpasses, giving n "Seifert circles". Construct a Seifert surface with n 0-handles and ℓ 1-handles

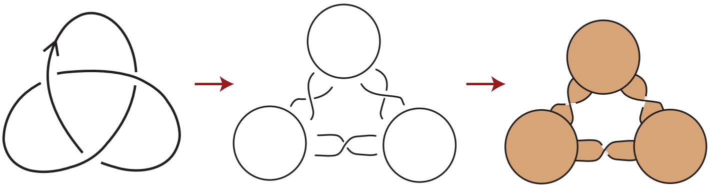
$$F \ = \ \coprod_n D^2 \cup \coprod_\ell D^1 \times D^1 \subset \mathbb{R}^3$$

with

$$\partial F \ = \ L(\coprod S^1) \subset \mathbb{R}^3 \ .$$

Examples of Seifert's algorithm for knots

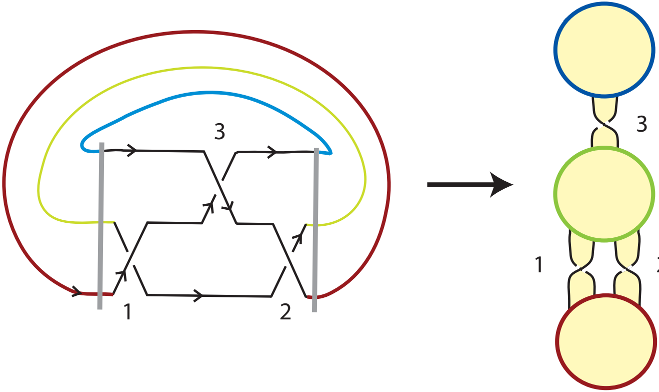
- 

- 

The canonical Seifert surface F_β of a braid

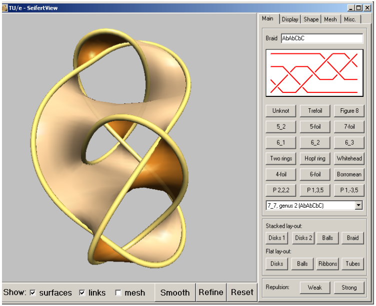
- An n-strand braid β with ℓ crossings is represented by a word in B_n of length ℓ in the generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$, so that $\beta = \beta_1\beta_2 \ldots \beta_\ell$ is the concatenation of ℓ elementary braids.
- Stallings in Constructions of fibred knots and links (1978) observed that the closure $\widehat{\beta}$ has a canonical projection with n Seifert circles and ℓ intersections, and hence a canonical Seifert surface with n 0-handles and ℓ 1-handles
$$F_\beta \;=\; \coprod_n D^2 \cup \coprod_\ell D^1 \times D^1 \subset \mathbb{R}^3 \;.$$
- **Lemma** F_β is homotopy equivalent to the CW complex
$$X_\beta \;=\; \coprod_{i=1}^{n} e_i^0 \cup \coprod_{j=1}^{\ell} e_j^1$$
with $\partial e_j^1 = e_i^0 \cup e_{i+1}^0$ if jth crossing is between strands $i, i+1$
$$\begin{aligned} H_1(F_\beta) \;=\; H_1(X_\beta) \;&=\; \ker(d : C_1(X_\beta) \to C_0(X_\beta)) \\ &=\; \ker(d : \mathbb{Z}^\ell \to \mathbb{Z}^n) \;=\; \mathbb{Z}^m \;. \end{aligned}$$

An example of the canonical Seifert surface F_β for the closure $\widehat{\beta}$ of a braid β



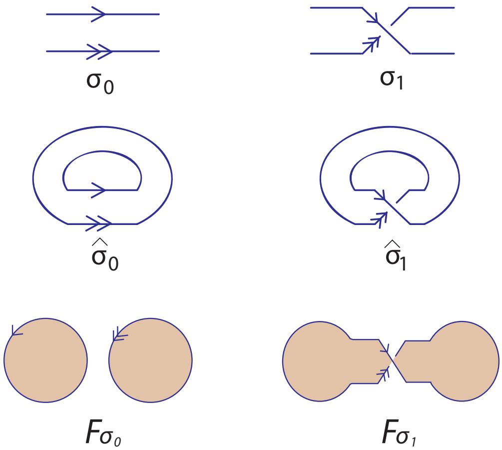
SeifertView

- Arjeh Cohen and Jack van Wijk wrote a programme SeifertView (2005) and a paper The visualization of Seifert surfaces (2006) for drawing the canonical Seifert surfaces F_β of the closures $\widehat{\beta}$ of braids β.
- A screenshot

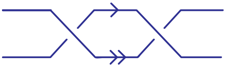


- Try the SeifertView rollercoaster!

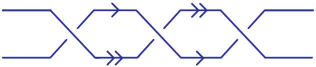
More braids β and canonical Seifert surfaces F_β I.



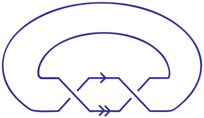
More braids β and canonical Seifert surfaces F_β II.



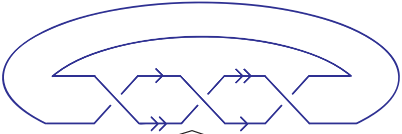
$\sigma_1\sigma_1$



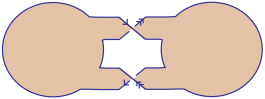
$\sigma_1\sigma_1\sigma_1$



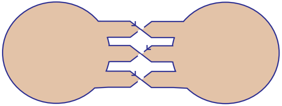
$\widehat{\sigma_1\sigma_1}$



$\widehat{\sigma_1\sigma_1\sigma_1}$



$F_{\sigma_1\sigma_1}$



$F_{\sigma_1\sigma_1\sigma_1}$

Duality and matrices

- The **dual** of an abelian group A is the abelian group
$$A^* \;=\; \mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Z})\ .$$
- The **dual** of a morphism $f : A \to B$ of abelian groups is the morphism
$$f^* \;:\; B^* \to A^*\ ;\ (g : B \to \mathbb{Z}) \mapsto (gf : A \to B \to \mathbb{Z})\ .$$
- If A is f.g. free with basis $\{a_1, a_2, \dots, a_m\}$ then A^* is f.g. free with **dual** basis $\{a_1^*, a_2^*, \dots, a_m^*\}$ such that $a_j^*(a_k) = \delta_{jk}$.
- A morphism $f : A \to B$ of f.g. free abelian groups with bases $\{a_1, a_2, \dots, a_m\}$, $\{b_1, b_2, \dots, b_n\}$ has the $n \times m$ matrix (f_{jk}) with
$$f(a_k) \;=\; \sum_{j=1}^{n} f_{jk} b_j \in B \quad (1 \leqslant k \leqslant m)\ .$$
- The dual morphism f^* has the **transpose** $m \times n$ matrix
$$(f_{jk})^* \;=\; (f_{kj}^*)\ ,\ f_{kj}^* \;=\; f_{jk}\ .$$

The Seifert form of a surface $F \subset \mathbb{R}^3$

- The **intersection form** of a surface with boundary $(F, \partial F)$ is the symplectic bilinear form
$$\Phi \ = \ -\Phi^* \ : \ H_1(F) \to H_1(F)^* \ = \ H^1(F) \ = \ H_1(F, \partial F)$$
defined by intersection numbers, with an exact sequence
$$0 \to H^0(F) \to H_1(\partial F) \to H_1(F) \xrightarrow{\Phi} H^1(F) \to H_0(\partial F) \to H_0(F) \to 0$$
- An embedding $F \subset \mathbb{R}^3$ determines a **Seifert matrix** $\Psi = (\Psi_{jk})$: given cycles $b_1, b_2, \dots, b_m : S^1 \subset F$ representing a basis $\{b_1, b_2, \dots, b_m\} \subset H_1(F) = \mathbb{Z}^m$
$$\Psi_{jk} \ = \ \text{linking number}(b_j^+, b_k^- : S^1 \subset \mathbb{R}^3) \in \mathbb{Z}$$
with $b_j^+, b_k^- : S^1 \subset \mathbb{R}^3$ the cycles $b_j, b_k : S^1 \subset F$ pushed off from $\partial F \subset F \subset \mathbb{R}^3$ in opposite directions.
- The **Seifert form** $\Psi : H_1(F) \to H_1(F)^*$ is independent of the choice of basis, and such that
$$\Phi \ = \ \Psi - \Psi^* \ : \ H_1(F) \to H_1(F)^* \ .$$

The canonical Seifert matrix Ψ_β of a braid I.

- A **Seifert matrix** for a link $L : \coprod_c S^1 \subset \mathbb{R}^3$ is a Seifert matrix Ψ of a Seifert surface $F \subset \mathbb{R}^3$.
- The **canonical Seifert matrix** Ψ_β of a braid β is the Seifert $m \times m$ matrix of the canonical Seifert surface F_β for the closure $\widehat{\beta} : S^1 \subset \mathbb{R}^3$, with $m = \operatorname{rank} H_1(F_\beta)$.
- **Example 1** For the elementary braid $\beta = \sigma_1$ with closure $\widehat{\beta}$ the trivial knot the canonical Seifert surface F_β is homotopy equivalent to

$$X_\beta \;=\; e^0 \cup e^0 \cup e^1 \;=\; I \;.$$

Thus $H_1(F_\beta) = 0$ and the canonical Seifert 0×0 matrix is

$$\Psi_\beta \;=\; (0) \;.$$

The canonical Seifert matrix Ψ_β of a braid II.

- **Example 2** For the braid $\beta = \sigma_1\sigma_1$ with closure $\widehat{\beta}$ the Hopf link the canonical Seifert surface F_β is homotopy equivalent to

$$X_\beta \;=\; e^0 \cup e^0 \cup e^1 \cup e^1 \;=\; S^1 \;.$$

  Thus $H_1(F_\beta) = \mathbb{Z}$ and the canonical Seifert 1×1 matrix is

$$\Psi_\beta \;=\; (1) \;.$$

- **Example 3** For $\beta = \sigma_1^{-1}\sigma_1^{-1}$

$$\Psi_\beta \;=\; (-1) \;.$$

- **Problem** For any n-strand braids β, β' what is the relation between the canonical Seifert matrices $\Psi_\beta, \Psi_{\beta'}, \Psi_{\beta\beta'}$?

An algorithm for the canonical Seifert matrix Ψ_β

- In 2007 Julia Collins computed the canonical Seifert matrix Ψ_β of a braid β, with a programme Seifert Matrix Computation and a paper An algorithm for computing the Seifert matrix of a link from a braid representation
- For a sequence $x_1, x_2, \ldots, x_\ell$ with $x_i \in \{\pm 1, \pm 2, \ldots, \pm(n-1)\}$ let
$$\epsilon(i) \;=\; \mathrm{sign}(x_i) \in \{-1, 1\}\;,\;\sigma(x_i) \;=\; \sigma_{|x_i|}^{\epsilon(i)} \in B_n\;.$$
- Define the braid with n strands and ℓ crossings
$$[x_1, x_2, \ldots, x_\ell] \;=\; \sigma(x_1)\sigma(x_2)\ldots\sigma(x_\ell) \in B_n\;.$$
- The algorithm uses a basis for the homology $H_1(F_\beta) = \mathbb{Z}^m$ with one basis element for each pair of adjacent crossings on the same strands, i.e. between each x_i and x_j where $|x_i| = |x_j|$ and $|x_k| \neq |x_i|$ for all $i < k < j$.
- The entries in the canonical Seifert matrix Ψ_β are either 0,+1 or -1.

Braids and signatures

- The Tristram-Levine ω-**signature** of a link $L : \coprod S^1 \subset \mathbb{R}^3$ is defined for $\omega \neq 1 \in \mathbb{C}$ by

$$\sigma_\omega(L) \ = \ \text{signature}((1-\omega)\Psi + (1-\overline{\omega})\Psi^*) \in \mathbb{Z}$$

  for any Seifert matrix Ψ. Independent of choice of Ψ.
- Gambaudo and Ghys (2005) and Bourrigan (2013) used the Burau-Squier hermitian representation of B_n to express the non-additivity

$$\sigma_\omega(\widehat{\beta\beta}') - \sigma_\omega(\widehat{\beta}) - \sigma_\omega(\widehat{\beta}') \in \mathbb{Z}$$

  in terms of the Wall-Maslov-Mayer formula for the nonadditivity of signature.
- Proofs rather complicated, for lack of an explicit formula for the canonical Seifert matrix Ψ_β of the closure $\widehat{\beta}$ of a braid β. Could get such a formula from an expression for the canonical Seifert matrix of a concatenation $\Psi_{\beta\beta'}$ in terms of $\Psi_\beta, \Psi_{\beta'}$. Rather tricky, because of the nonadditivity of rank $H_1(F_\beta)$.

Surgery on manifolds

- An r-**surgery** on an m-dimensional manifold M uses an embedding

$$S^r \times D^{m-r} \subset M \ (-1 \leqslant r \leqslant m)$$

  to create a new m-dimensional manifold, the **effect**

$$M' \ = \ \mathrm{cl.}(M \backslash S^r \times D^{m-r}) \cup D^{r+1} \times S^{m-r-1}$$

- The **trace** of the r-surgery is the $(m+1)$-dimensional cobordism $(W; M, M')$ with

$$W \ = \ (M \times I) \cup D^{r+1} \times D^{m-r}$$

  obtained from $M \times I$ by attaching an $(r+1)$-handle at $S^r \times D^{m-r} \subset M \times \{1\}$.
- **Theorem** (Thom, Milnor, 1961) Every $(m+1)$-dimensional cobordism is a union of traces of successive surgeries.
- For surgery on manifolds with boundary $(M, \partial M)$ require $S^r \times D^{m-r} \subset M \backslash \partial M$.

Surgery on 1-manifolds

- A 1-dimensional manifold is a disjoint union of circles

$$M = \coprod_n S^1 .$$

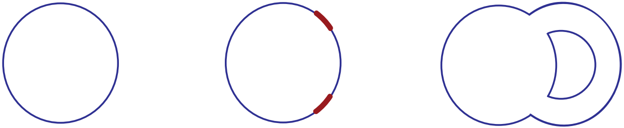
- The effect of a (-1)-surgery on M is to add another circle

$$M' = M \sqcup S^1 = \coprod_{n+1} S^1 .$$

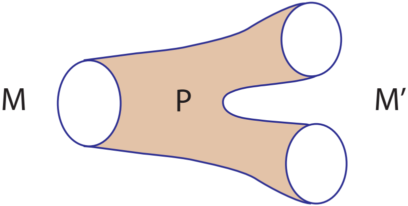
- The effect of a 0-surgery using an embedding $S^0 \times D^1 \subset M$ is

$$M' = \begin{cases} \coprod_{n+1} S^1 & \text{if } S^0 \subset M \text{ in same component of } M \\ \coprod_{n-1} S^1 & \text{if } S^0 \subset M \text{ in different components of } M . \end{cases}$$

The pair of pants



$M = S^1$ $\qquad$ $S^0 \mathrm{x}\, D^1 \subset M = S^1$ $\qquad$ $M' = \overline{M \setminus S^0 \mathrm{x}\, D^1} \cup D^1 \mathrm{x}\, S^0$



$P = M \mathrm{x}\, I \cup D^1 \mathrm{x}\, D^1 = \text{trace}$

Generalized intersection matrices

- Given an n-strand braid β with ℓ crossings, let $C = C(X_\beta)$ be the cellular $\mathbb{Z}$-module chain complex of $X_\beta \simeq F_\beta$ with

$$d = \begin{pmatrix} 1 & \vdots \\ -1 & \vdots \\ 0 & \vdots \\ \vdots & \vdots \end{pmatrix} : C_1 = \mathbb{Z}^\ell = \mathbb{Z}[e^1_1, \dots, e^1_\ell]$$

$$\to C_0 = \mathbb{Z}^n = \mathbb{Z}[e^0_1, \dots, e^0_n] ; \ e^1_j \mapsto e^0_i - e^0_{i+1} .$$

- A **generalized intersection matrix** for β is an $\ell \times \ell$ matrix ϕ_β such that

$$d^* d = \phi_\beta + \phi^*_\beta \ : C_1 \to C^1$$

and which induces the intersection form

$$\Phi_\beta = [\phi_\beta] : H_1(F_\beta) = H_1(C) = \ker(d)$$

$$\to H_1(F_\beta, \partial F_\beta) = H^1(C) = \operatorname{coker}(d^*) .$$

The canonical generalized intersection matrix ϕ_β I.

- **Definition** The **canonical generalized intersection** 1×1 **matrices** for the elementary n-strand braids σ_i, σ_i^{-1} are

$$\phi_{\sigma_i} \;=\; \phi_{\sigma_i^{-1}} \;=\; (1) \;.$$

- Let β, β' be n-strand braids with ℓ, ℓ' crossings and chain complexes

$$d : C_1 = \mathbb{Z}^\ell \to C_0 = \mathbb{Z}^n \;,\; d' : C_1' = \mathbb{Z}^{\ell'} \to C_0' = \mathbb{Z}^n \;.$$

- **Lemma** The concatenation n-strand braid $\beta'' = \beta\beta'$ with $(\ell + \ell')$ crossings has chain complex

$$d'' = (d \;\; d') : C_1'' = \mathbb{Z}^\ell \oplus \mathbb{Z}^{\ell'} \to C_0'' = \mathbb{Z}^n$$

- **Definition** The **concatenation** of generalized intersection matrices $\phi_\beta, \phi_{\beta'}$ for β, β' is the generalized intersection matrix for β''

$$\phi_{\beta''} \;=\; \phi_\beta \phi_{\beta'} \;=\; \begin{pmatrix} \phi_\beta & d^* d' \\ 0 & \phi_{\beta'} \end{pmatrix} \;.$$

The canonical generalized intersection matrix ϕ_β II.

- **Proposition** An n-strand braid $\beta = \beta_1\beta_2\dots\beta_\ell$ with ℓ crossings has the canonical generalized intersection matrix

$$\phi_\beta \ = \ \phi_{\beta_1}\phi_{\beta_2}\dots\phi_{\beta_\ell} \ : \ C_1 \ = \ \mathbb{Z}^\ell \to C^1 \ = \ \mathbb{Z}^\ell \ .$$

- The generalized intersection matrix ϕ_β encodes the sequence of ℓ 0-surgeries on $\coprod\limits_n S^1$ determined by β with combined trace $(\mathrm{cl.}(F_\beta\backslash \coprod\limits_n D^2); \coprod\limits_n S^1, \partial F_\beta)$.
- The algebraic theory of surgery (A.R., 1980) expresses the chain complex of $\partial F_\beta = \widehat{\beta}(\coprod\limits_n S^1) \subset \mathbb{R}^3$ up to chain equivalence as

$$d' \ = \ \begin{pmatrix} \phi_\beta & -d^* \\ d & 0 \end{pmatrix} \ : \ C'_1 \ = \ C_1 \oplus C^0 \to C'_0 \ = \ C^1 \oplus C_0 \ .$$

- **Proposition**

$$\text{no. of components of } \widehat{\beta} \ = \ \operatorname{rank} H_0(C') \ .$$

How many components does the Hopf link have?

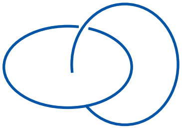
- **Example** The canonical Seifert surface F_β of the closure $\widehat{\beta}$ of the 2-strand braid $\beta = \sigma_1\sigma_1$ has chain complex

$$d = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} : C_1 = \mathbb{Z} \oplus \mathbb{Z} \to C_0 = \mathbb{Z} \oplus \mathbb{Z}$$

and generalized intersection matrix

$$\phi_\beta = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} : C_1 = \mathbb{Z} \oplus \mathbb{Z} \to C^1 = \mathbb{Z} \oplus \mathbb{Z} .$$

The 4×4 matrix $d' = \begin{pmatrix} \phi_\beta & -d^* \\ d & 0 \end{pmatrix}$ has rank 2, so the Hopf link $\widehat{\beta}$ has 4-2=2 components.



Surgery on submanifolds

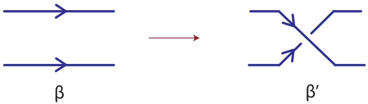
- An **ambient** r-**surgery** on a codimension q submanifold $M^m \subset N^{m+q}$ is an r-surgery on $S^r \times D^{m-r} \subset M$ with a codimension q embedding of the trace

$$(W; M, M') \subset N \times (I; \{0\}, \{1\}) .$$

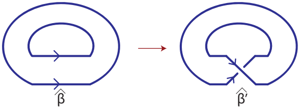
- **Key idea 1** The closure $\widehat{\beta} : \coprod S^1 \subset \mathbb{R}^3$ of an n-strand braid β with ℓ crossings is the effect of n ambient (-1)-surgeries on the codimension 2 submanifold $\emptyset \subset \mathbb{R}^3$ (i.e. $\coprod\limits_n S^1$) followed by ℓ ambient 0-surgeries.
- **Key idea 2** The canonical Seifert surface $F_\beta \subset \mathbb{R}^3$ is the union of the traces of n ambient (-1)-surgeries on the codimension 1 submanifold $\emptyset \subset \mathbb{R}^3$ (i.e. $\coprod\limits_n D^2$) followed by ℓ ambient 0-surgeries.
- **Problems** What are the algebraic effects of the corresponding chain level algebraic surgeries?

Surgery on braids

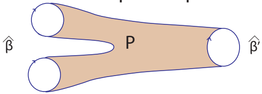
- The effect of a 1-surgery on a 2-strand braid $\beta : I \sqcup I \subset D^2 \times I$ with $S^0 \times D^1 \subset I \sqcup I$ in different components is the 2-strand braid $\beta' = \beta\sigma_1 : I \sqcup I \subset D^2 \times I$



- Corresponding 1-surgery on the closure $\widehat{\beta}$ of β with effect the closure $\widehat{\beta}'$ of β'



With trace the pair of pants:



Generalized Seifert matrices

- Define the $n \times n$ matrix

$$\chi = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 1 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- A **generalized Seifert matrix** for an n-strand braid β with ℓ crossings is an $\ell \times \ell$ matrix ψ_β such that

$$\phi_\beta + d^*\chi d = \psi_\beta - \psi_\beta^* : C_1 = \mathbb{Z}^\ell \to C^1 = \mathbb{Z}^\ell$$

and $\psi_\beta : C_1 \to C^1$ induces the Seifert form

$$\Psi_\beta = [\psi_\beta] : H_1(F_\beta) = H_1(C) = \ker(d)$$
$$\to H_1(F_\beta, \partial F_\beta) = H^1(C) = \operatorname{coker}(d^*) .$$

- Motivated by the algebraic surgery properties of the Pontrjagin-Thom map $S^3 \to \Sigma(F_\beta/\partial F_\beta)$ of $F_\beta \subset \mathbb{R}^3$.

The canonical generalized Seifert matrix ψ_β I.

- **Definition** The **canonical generalized Seifert** 1×1 **matrices** for the elementary n-strand braids σ_i, σ_i^{-1} are
$$\psi_{\sigma_i} ~=~ (1)~,~\psi_{\sigma_i^{-1}} ~=~ (-1)~.$$
- Let β, β' be n-strand braids with ℓ, ℓ' crossings and chain complexes
$$d : C_1 = \mathbb{Z}^\ell \to C_0 = \mathbb{Z}^n~,~d' : C_1' = \mathbb{Z}^{\ell'} \to C_0' = \mathbb{Z}^n~.$$
As before, the concatenation n-strand braid $\beta'' = \beta\beta'$ has
$$d'' = (d~~d') : C_1'' = \mathbb{Z}^\ell \oplus \mathbb{Z}^{\ell'} \to C_0'' = \mathbb{Z}^n$$
and a canonical generalized intersection matrix $\phi_{\beta''} = \phi_\beta \phi_{\beta'}$.
- **Definition** The **concatenation** of generalized Seifert matrices $\psi_\beta, \psi_{\beta'}$ for β, β' is the generalized Seifert matrix for β''
$$\psi_{\beta''} ~=~ \psi_\beta \psi_{\beta'} ~=~ \begin{pmatrix} \psi_\beta & -d^*\chi^*d' \\ 0 & \psi_{\beta'} \end{pmatrix}~.$$

The canonical generalized Seifert matrix ψ_β II.

- **Lemma** Concatenation is associative.
- **Proposition** An n-strand braid with ℓ crossings $\beta = \beta_1\beta_2\ldots\beta_\ell$ has the canonical generalized Seifert matrix

$$\psi_\beta \ = \ \psi_{\beta_1}\psi_{\beta_2}\ldots\psi_{\beta_\ell} \ : \ C_1 \ = \ \mathbb{Z}^\ell \to C^1 \ = \ \mathbb{Z}^\ell \ .$$

- The generalized Seifert matrix ψ_β encodes the sequence of ℓ ambient 1-surgeries on $\coprod_n S^1 \subset \mathbb{R}^3$ determined by β with combined trace $(\mathrm{cl.}(F_\beta \backslash \coprod_n D^2); \coprod_n S^1, \partial F_\beta) \subset \mathbb{R}^3$.
- Maciej Borodzik extended Julia Collins' algorithm to construct an $\ell \times \ell$ matrix inducing the Seifert form directly from the braid, but it is not clear if this is the canonical generalized Seifert matrix.

What is the Seifert form of the trefoil knot?

- **Example** The 2-strand braid $\beta = \sigma_1\sigma_1\sigma_1$ with 3 crossings has closure $\widehat{\beta}$ the trefoil knot. The chain complex for β is

$$d = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix} : C_1 = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \to C_0 = \mathbb{Z}\oplus\mathbb{Z}$$

  so $H_1(C) = \mathbb{Z}\oplus\mathbb{Z}$ with basis $b_1 = (1,0,-1)$, $b_2 = (0,1,-1)$.
- The canonical generalized Seifert matrix is

$$\psi_\beta = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} : C_1 = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \to C^1 = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$$

  The Seifert matrix of the trefoil knot with respect to b_1, b_2 is

$$[\psi_\beta] = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} : H_1(C) = \mathbb{Z}\oplus\mathbb{Z} \to H^1(C) = \mathbb{Z}\oplus\mathbb{Z}\,.$$

Braids in the Book of Durrow