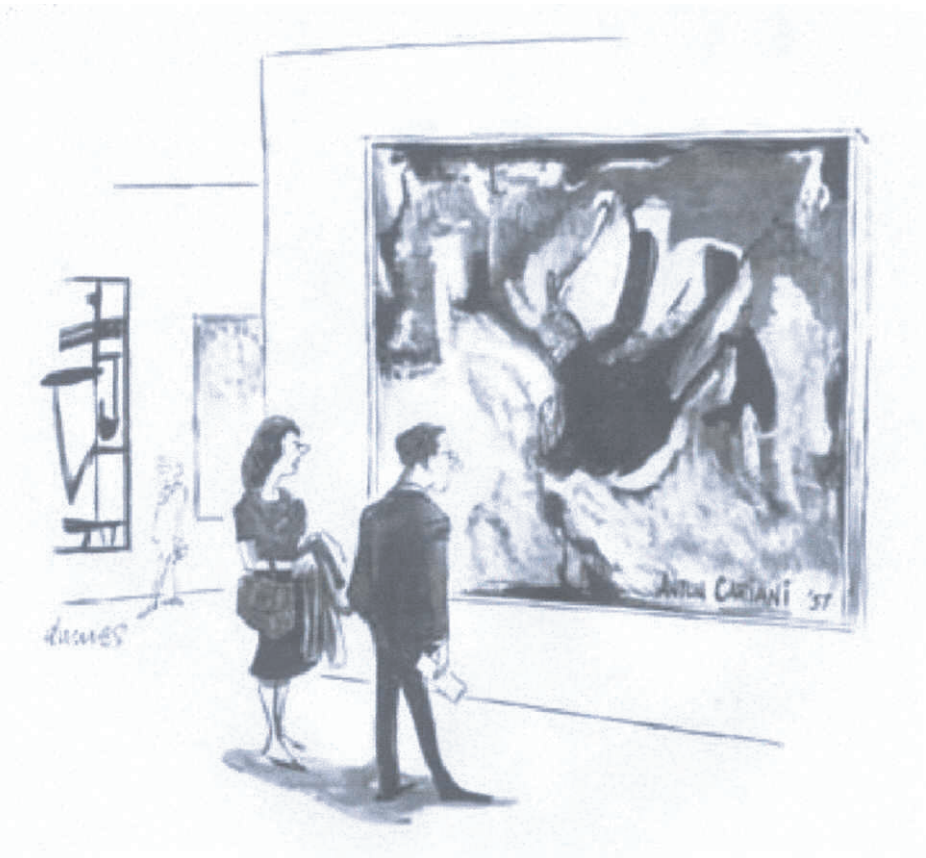
THE SIGNATURE MOD 2, 4 AND 8

Andrew Ranicki (Edinburgh)
Larry Taylor (Notre Dame)



"I notice the signatures are never abstract."

Oxford, 31st January 2005

The signature mod 2, 4 and 8 of a $4k$-dimensional Poincaré space X

- Theorem $\sigma^*(X) \equiv \chi(X) \pmod 2$ with $\sigma^*(X)$, $\chi(X) \in \mathbb{Z}$ the signature and Euler characteristic.

- Theorem $\sigma^*(X) \equiv \langle \mathcal{P}_2(v), [X] \rangle \pmod 4$ $\mathcal{P}_2 : H^{2k}(X;\mathbb{Z}_2) \to H^{4k}(X;\mathbb{Z}_4)$ Pontrjagin square, $v = v_{2k}(\nu_X) \in H^{2k}(X;\mathbb{Z}_2)$ the $2k^{\text{th}}$ Wu class of the Spivak normal fibration ν_X

  $\langle x \cup x, [X] \rangle = \langle v \cup x, [X] \rangle \in \mathbb{Z}_2$ $(x \in H^{2k}(X;\mathbb{Z}_2))$

- Theorem $\sigma^*(X) \equiv \langle \widetilde{v} \cup \widetilde{v}, [X] \rangle \pmod 8$ for any integral lift $\widetilde{v} \in H^{2k}(X)$ of v.

- To what extent are these classical results for the signature of a Poincaré space true for the 'mod 8 signature' of a 'normal space'?

Spherical fibrations

- A <u>spherical fibration</u> is a Serre fibration
$$\nu\colon S^{j-1} \to E \to X.$$

- The <u>Thom space</u> $T(\nu)$ is the mapping cone of $E \to X$. Will only consider oriented case, so have <u>Thom class</u> $U \in \widetilde{H}^j(T(\nu))$ with
$$U \cap - \colon \widetilde{H}_{*+j}(T(\nu)) \xrightarrow{\cong} H_*(X)\ ,$$
$$U \cup - \colon H^*(X) \xrightarrow{\cong} \widetilde{H}^{*+j}(T(\nu))\ .$$

- <u>Wu classes</u> $v_r(\nu) \in H^r(X;\mathbb{Z}_2)$ $(r \geqslant 0)$ characterized by dual Steenrod squares $\chi(Sq)^r(U) = U \cup v_r(\nu) \in \widetilde{H}^{r+j}(T(\nu);\mathbb{Z}_2)$.

- Spherical fibrations classified by maps $\nu : X \to BSG(j)$. Stable classifying space $BSG = \varinjlim_j BSG(j)$, $\pi_*(BSG) = \pi^S_{*-1}$ with $H_*(BSG)$, $H^*(BSG)$ finite for $* \neq 0$.

Normal spaces

- Definition (Quinn, 1972)
  An n-dimensional normal space (X, ν_X, ρ_X) is a space X together with a spherical fibration $\nu_X : X \to BSG(j)$ and a map $\rho_X : S^{n+j} \to T(\nu_X)$. The fundamental class of X is the Hurewicz-Thom image
  $[X] = U \cap h(\rho_X) \in \widetilde{H}_{n+j}(T(\nu_X)) \cong H_n(X)$.

- Thom-Wu formula: for any $x \in H^{n-r}(X; \mathbb{Z}_2)$
  $[X] \cap Sq^r(x) = [X] \cap (v_r(\nu_X) \cup x) \in H_0(X; \mathbb{Z}_2)$

- Will assume that the torsion-free quotients $F^r(X) = H^r(X)/\text{torsion}$ are finitely generated, e.g. if X is finite, or $H^r(X)$ is torsion.

Poincaré spaces

- Definition An n-dimensional Poincaré space X is a finite CW complex with fundamental class $[X] \in H_n(X)$ and duality isomorphisms
$$[X] \cap - \ : \ H^{n-*}(X) \xrightarrow{\cong} H_*(X)$$

- Canonical example An oriented n-dimensional manifold is an n-dimensional Poincaré space.

- Theorem (Spivak 1965, Wall, Browder) An n-dimensional Poincaré space X is an n-dimensional normal space, with ν_X the 'Spivak normal fibration'
$$\nu_X \colon S^{j-1} \to \partial W \to W \simeq X$$
defined by a regular neighbourhood $(W, \partial W)$ of $X \subset S^{n+j}$ (j large), and
$$\rho_X \colon S^{n+j} \to W/\partial W \simeq T(\nu_X)$$
the degree 1 collapse map.

Normal maps

- A <u>normal map</u> of n-dimensional normal spaces $(f, b) : X \to Y$ is a degree 1 map $f : X \to Y$

$$f_*[X] = [Y] \in H_n(Y)$$

together with a map of normal fibrations $b : \nu_X \to \nu_Y$ s.t. $T(b)\rho_X = \rho_Y \in \pi_{n+k}(T(\nu_Y))$.

- <u>Proposition</u> (Quinn) The mapping cylinder W of a n-dimensional normal map $(f, b) : X \to Y$ defines an $(n+1)$-dimensional normal space cobordism $(W; X, Y)$.

- Basic question of surgery theory: is a Poincaré space homotopy equivalent to a manifold? Surgery obstruction to a normal map $(f, b) : X \to Y$ from a manifold X to a Poincaré space Y being bordant to a homotopy equivalence. Is a normal space bordant to a Poincaré space? Same obstruction.

The signature of a $4k$-dimensional normal space X

- Symmetric intersection pairing

  $\phi\ :\ F^{2k}(X) \times F^{2k}(X) \to \mathbb{Z}\ ;\ (x,y) \mapsto \langle x \cup y, [X] \rangle$

  Nonsingular for Poincaré X.

- The signature of X is

  $$\sigma^*(X) = \text{signature}(F^{2k}(X), \phi) \in \mathbb{Z}$$

- Warning For non-Poincaré X can have

  $$\sigma^*(X) \not\equiv \chi(X) \pmod 2$$

  Proof For any finite CW complex X with odd $\chi(X) \in \mathbb{Z}$ (e.g. $X = \{*\}$) and any $\nu_X : X \to BSG(j)$ set $\rho_X = * : S^{4k+j} \to T(\nu_X)$, so that $[X] = 0 \in H_{4k}(X)$, $\sigma^*(X) = 0 \in \mathbb{Z}$.

Normal and Poincaré cobordism (I)

- Cobordism of normal and Poincaré spaces, with groups Ω_n^N, Ω_n^P.

- Signature $\sigma^*(X) \in \mathbb{Z}$ is a Poincaré cobordism invariant, with mod 2 reduction $\chi(X) \in \mathbb{Z}_2$

- Theorem (Quinn) 'Pontrjagin-Thom' isomorphisms for normal space cobordism

  $$\Omega_n^N \xrightarrow{\cong} \pi_n(MSG) \; ; \; (X, \nu_X, \rho_X) \mapsto \nu_X \rho_X$$

  with MSG the Thom spectrum of the universal spherical fibration $1 : BSG \to BSG$.
  Proof Every normal space (X, ν_X, ρ_X) is cobordant to $(BSG, 1, \nu_X \rho_X)$ by mapping cylinder of normal map $\nu_X : X \to BSG$.

- The signature and mod 2 Euler characteristic are not normal space cobordism invariants: $F^*(BSG) = 0$ $(* \neq 0)$, $\chi(\{*\}) = 1$.

Normal and Poincaré cobordism (II)

- Theorem (Levitt-Jones-Quinn-Hausmann-Vogel, 1972-1988) For $n \geqslant 4$ there is an exact sequence

  $$\dots \to L_n(\mathbb{Z}) \to \Omega_n^P \to \Omega_n^N \to L_{n-1}(\mathbb{Z}) \to \dots$$

  with $L_*(\mathbb{Z})$ the simply-connected surgery obstruction groups.

- Theorem (Brumfiel and Morgan, 1976) The signature and the mod-8-Hirzebruch number define surjections

  $$\sigma^* : \Omega_{4k}^P \to \mathbb{Z} \; ; \; X \mapsto \sigma^*(X) \; ,$$
  $$\widehat{\sigma}^* : \Omega_{4k}^N \to \mathbb{Z}_8 \; ; \; X \mapsto \langle \nu_X^*(\ell_{4k}), [X] \rangle$$

  with $\ell_{4k} \in H^{4k}(BSG; \mathbb{Z}_8)$ the mod 8 ℓ-class. σ^* and $\widehat{\sigma}^*$ are isomorphisms for $k = 1$. The forgetful maps $\Omega_{4k}^P \to \Omega_{4k}^N$ ($k \geqslant 1$) are surjections, since $L_{4k-1}(\mathbb{Z}) = 0$.

The mod 8 signature of a $4k$-dimensional normal space X

- Definition The mod 8 signature is the Brumfiel-Morgan mod 8 Hirzebruch number

$$\widehat{\sigma}^*(X) \;=\; \langle \nu_X^*(\ell_{4k}),[X]\rangle \in \mathbb{Z}_8 \ .$$

- The mod 8 signature of a Poincaré X is the signature mod 8, $\widehat{\sigma}^*(X) = [\sigma^*(X)] \in \mathbb{Z}_8$.

- Every X is normal cobordant to a Poincaré space Y, with $\widehat{\sigma}^*(X) = [\sigma^*(Y)] \in \mathbb{Z}_8$.

- Warning For non-Poincaré X can have mod 8 signature $\neq$ signature mod 8

$$\widehat{\sigma}^*(X) \neq [\sigma^*(X)] \in \mathbb{Z}_8 \ .$$

Proof Take $\nu_X = 1 : X = BSG(j) \to BSG(j)$. Every $d \neq 0 \in \mathbb{Z}_8$ is realized as $d = \widehat{\sigma}^*(X)$ for some $\rho_X : S^{4k+j} \to X$, but $\sigma^*(X) = 0 \in \mathbb{Z}$.

Homological formulae for the mod 2 and 4 signatures of normal spaces

- Theorem 1 (R.-T.) The mod 4 reduction of the mod 8 signature of a $4k$-dimensional normal space X is

  $$[\widehat{\sigma}^*(X)] = \langle \mathcal{P}_2(v_{2k}(\nu_X)), [X]\rangle \in \mathbb{Z}_4$$

  with $\mathcal{P}_2 : H^{2k}(X;\mathbb{Z}_2) \to H^{4k}(X;\mathbb{Z}_4)$ the Pontrjagin square.

  (True for Poincaré X: Morita (1971), Brumfiel-Morgan (1974))

- Corollary (R.-T.) The mod 2 reduction of the mod 8 signature of a $4k$-dimensional normal space X is

  $$[\widehat{\sigma}^*(X)] = \langle v_{2k}(\nu_X) \cup v_{2k}(\nu_X), [X]\rangle \in \mathbb{Z}_2$$

Homological formulae for the mod 8 signature of certain normal spaces (I)

Theorem 2 (R.-T.) Let X be a $4k$-dimensional normal space. Suppose that

$$v_{2k}(\nu_X) \in \ker(\delta_4 : H^{2k}(X;\mathbb{Z}_2) \to H^{2k+1}(X;\mathbb{Z}_2))$$
$$= \mathrm{im}(H^{2k}(X;\mathbb{Z}_4) \to H^{2k}(X;\mathbb{Z}_2)),$$

with δ_4 = the Bockstein for

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$$

For any lift $v \in H^{2k}(X;\mathbb{Z}_4)$ of $v_{2k}(\nu_X) \in H^{2k}(X;\mathbb{Z}_2)$

$$\widehat{\sigma}^*(X) \;=\; \langle \mathcal{P}_4(v), [X] \rangle \in \mathbb{Z}_8$$

with $\mathcal{P}_4 : H^{2k}(X;\mathbb{Z}_4) \to H^{4k}(X;\mathbb{Z}_8)$ the Pontrjagin square.

Homological formulae for the mod 8 signature of certain normal spaces (II)

Corollary (R.-T.) Suppose that

$$v_{2k}(\nu_X) \in \ker(\delta_\infty : H^{2k}(X;\mathbb{Z}_2) \to H^{2k+1}(X)) = \mathrm{im}(H^{2k}(X) \to H^{2k}(X;\mathbb{Z}_2))$$

with δ_∞ = the Bockstein for

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}_2 \to 0 \ .$$

For any lift $v \in H^{2k}(X)$ of $v_{2k}(\nu_X) \in H^{2k}(X;\mathbb{Z}_2)$

$$\langle x \cup x, [X] \rangle = \langle v \cup x, [X] \rangle \in \mathbb{Z}_2 \ (x \in H^{2k}(X))$$

and

$$\widehat{\sigma}^*(X) \ = \ \langle v \cup v, [X] \rangle \in \mathbb{Z}_8 \ .$$

(True for Poincaré X: Hirzebruch and Hopf (1958), van der Blij (1959))

Strategy of proofs (I)

- Use the chain complex theory of algebraic surgery to interpret the mod 8 signature $\widehat{\sigma}^*(X) \in \mathbb{Z}_8$ as the cobordism class of the 'algebraic normal complex' $(C(X), \phi, \gamma, \chi)$ of X, computing it as a 'characteristic number' of the 'algebraic normal structure' (ϕ, γ, χ).

- $\phi = \{\phi_s | s \geqslant 0\}$ consists of the chain map

$$\phi_0 = [X] \cap - : C(X)^{4k-*} \to C(X)$$

  and the chain homotopies $\phi_{s+1} : \phi_s \simeq T\phi_s$, which determine the evaluation of the Steenrod and Pontrjagin squares on the fundamental class $[X] \in H_{4k}(X)$.

- γ is the 'chain bundle' of $\nu_X : X \to BSG(j)$, determined by Wu classes $v_*(\nu_X) \in H^*(X; \mathbb{Z}_2)$. χ is determined by $\rho_X \in \pi_{4k+j}(T(\nu_X))$.

Strategy of proofs (II)

- ϕ and γ are essentially homological in nature, but χ is more subtle: difference between $\rho_X \in \pi_{4k+j}(T(\nu_X))$ and the Hurewicz-Thom image $U \cap h(\rho_X) = [X] \in H_{4k}(X)$.

- It turns out that the mod 4 reduction $[\widehat{\sigma}^*(X)] \in \mathbb{Z}_4$ is determined by ϕ and γ, and hence by $\mathcal{P}_2 : H^{2k}(X;\mathbb{Z}_2) \to H^{4k}(X;\mathbb{Z}_4)$ as in Theorem 1.

- The mod 8 signature $\widehat{\sigma}^*(X) \in \mathbb{Z}_8$ is in general determined by ϕ, γ and also χ. However, if the Bockstein hypothesis of Theorem 2 is satisfied then $\widehat{\sigma}^*(X)$ is determined only by ϕ, γ, and hence by $\mathcal{P}_4 : H^{2k}(X;\mathbb{Z}_4) \to H^{4k}(X;\mathbb{Z}_8)$ as in the conclusion.

The L-groups of $\mathbb{Z}$

- Exact sequence relating the quadratic, symmetric Poincaré and normal cobordism groups of chain complexes with duality

$$\dots \to L_n(\mathbb{Z}) \to L^n(\mathbb{Z}) \to \widehat{L}^n(\mathbb{Z}) \to L_{n-1}(\mathbb{Z}) \to \dots$$

n mod 4	$L_n(\mathbb{Z})$	$L^n(\mathbb{Z})$	$\widehat{L}^n(\mathbb{Z})$
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_8$
1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
2	$\mathbb{Z}_2$	0	0
3	0	0	$\mathbb{Z}_2$

- The signature $\sigma^* : \Omega^P_{4k} \to \mathbb{Z}$ and the mod 8 signature $\widehat{\sigma}^* : \Omega^N_{4k} \to \mathbb{Z}_8$ extend to a natural transformation of exact sequences

$$\begin{array}{ccccccc}
L_n(\mathbb{Z}) & \longrightarrow & \Omega^P_n & \longrightarrow & \Omega^N_n & \longrightarrow & L_{n-1}(\mathbb{Z}) \\
\| & & \downarrow \sigma^* & & \downarrow \widehat{\sigma}^* & & \| \\
L_n(\mathbb{Z}) & \longrightarrow & L^n(\mathbb{Z}) & \longrightarrow & \widehat{L}^n(\mathbb{Z}) & \longrightarrow & L_{n-1}(\mathbb{Z})
\end{array}$$

The symmetric and normal signatures from the chain complex point of view

- The signature $\sigma^*(X) \in L^{4k}(\mathbb{Z}) = \mathbb{Z}$ of a Poincaré space X is the algebraic cobordism invariant of the 'symmetric Poincaré structure' on $C(X)$ given by the Poincaré duality chain equivalence

$$[X] \cap - : C(X)^{n-*} \xrightarrow{\simeq} C(X)$$

- The mod 8 signature $\widehat{\sigma}^*(X) \in \widehat{L}^{4k}(\mathbb{Z}) = \mathbb{Z}_8$ of a normal space X is the algebraic cobordism invariant of the 'algebraic normal structure' on $C(X)$, given by the chain map

$$[X] \cap - : C(X)^{n-*} \to C(X)$$

  and the 'chain bundle' properties $C(X)$ inherits from $\nu_X : X \to BSG(j)$ and $\rho_X : S^{4k+j} \to T(\nu_X)$.

Steenrod and Pontrjagin squares

- Natural Alexander-Whitney-Steenrod diagonal chain approximation for any space X

  $\Delta_s \colon C(X)_r \to (C(X) \otimes C(X))_{r+s} \ (s \geqslant 0)$

  such that up to signs

  $d\Delta_s + \Delta_s d + (1-T)\Delta_{s-1} = 0 \ (\Delta_{-1} = 0)$

  $T(x \otimes y) = y \otimes x$, $\Delta_0 \colon C(X) \to C(X) \otimes C(X)$ chain map, $\Delta_1 \colon \Delta_0 \simeq T\Delta_0$ chain homotopy,...

- Steenrod squares for any $k \geqslant r \geqslant 0$

  $Sq^r \colon H^k(X;\mathbb{Z}_2) \to H^{k+r}(X;\mathbb{Z}_2); x \mapsto (x \otimes x)\Delta_{k-r}$

  For $k = r$ $Sq^k(x) = x \cup x \in H^{2k}(X;\mathbb{Z}_2)$.

- Pontrjagin squares for any $j \geqslant 1$, $k \geqslant 0$

  $\mathcal{P}_{2j} \colon H^k(X;\mathbb{Z}_{2j}) \to H^{2k}(X;\mathbb{Z}_{4j})$;

  $x \mapsto (x \otimes x)(\Delta_0 + d\Delta_1)$

  Reduction mod $\mathbb{Z}_{2j}$: $\mathcal{P}_{2j}(x) \mapsto x \cup x \in H^{2k}(X;\mathbb{Z}_{2j})$

The symmetric Q-groups

- Let A be a commutative ring. Given a f.g. free A-module chain complex C let $T \in \mathbb{Z}_2$ act on $C \otimes_A C$ by $T(x \otimes y) = \pm y \otimes x$.
  The <u>symmetric Q-groups</u> of C are

  $$\begin{aligned} Q^n(C) &= H^n(\mathbb{Z}_2; C \otimes_A C) \\ &= H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_A C)) \end{aligned}$$

  with W=free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$,

  $d = 1+(-1)^s T : W_s = \mathbb{Z}[\mathbb{Z}_2] \to W_{s-1} = \mathbb{Z}[\mathbb{Z}_2]$

- An element $\phi \in Q^n(C)$ is represented by

  $\phi = \{\phi_s : C^r = \mathrm{Hom}_A(C_r, A) \to C_{n-r+s} \,|\, s \geqslant 0\}$

  such that up to signs

  $$d\phi_s + \phi_s d^* + \phi_{s-1} + \phi^*_{s-1} = 0 \quad (\phi_{-1} = 0)$$

  A chain map $f : C \to D$ induces morphisms

  $f^{\%} : Q^n(C) \to Q^n(D); \phi \mapsto (f \otimes f)\phi = f\phi f^*$.

Symmetric complexes

- An <u>n-dimensional symmetric complex</u> (C, ϕ) over A is an n-dimensional f.g. free A-module chain complex C together with an element $\phi \in Q^n(C)$.

- A symmetric complex (C, ϕ) is <u>Poincaré</u> if $\phi_0 : C^{n-*} \to C$ is a chain equivalence.

- <u>Example</u> For

$$C : \cdots \to 0 \to C_{2k} \to 0 \to \ldots$$

an element $\phi \in Q^{4k}(C)$ is a symmetric form $\phi_0 : C^{2k} \times C^{2k} \to A$. In this case Poincaré complex = nonsingular form.

- $L^n(\mathbb{Z})$ = cobordism group of n-dimensional symmetric Poincaré complexes over $\mathbb{Z}$. For $n = 4k$ isomorphism

$$L^{4k}(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}; (C, \phi) \mapsto \text{signature}(F^{2k}(C), \phi_0)$$

The symmetric complex of a normal space

- The symmetric construction on a space X

$$\Delta\colon H_n(X) \to Q^n(C(X))$$

is induced by the Alexander-Whitney-Steenrod diagonal chain approximation.

- An n-dimensional normal space X determines an n-dimensional symmetric complex $(C(X),\phi)$ over $\mathbb{Z}$, with

$$\phi = \Delta[X] \in Q^n(C(X))$$

such that

$$\phi_0 = [X] \cap - \colon C(X)^{n-*} \to C(X)\ .$$

- X is Poincaré if and only if $(C(X),\phi)$ is a symmetric Poincaré complex, i.e. ϕ_0 is a chain equivalence.

Universal examples for the Steenrod squares

- For any $r \geqslant 0$ the $\mathbb{Z}_2$-module chain complex

$$B : \dots \to 0 \to B_r = \mathbb{Z}_2 \to 0 \to \dots$$

has $Q^n(B) \xrightarrow{\cong} \mathbb{Z}_2 \; ; \; \phi \mapsto \phi_{n-2r} \quad (n \geqslant 2r)$

- For any space X and any element

$y \in H^r(X;\mathbb{Z}_2) = H_0(\mathrm{Hom}_{\mathbb{Z}_2}(C(X;\mathbb{Z}_2), B))$

the composite

$$H_n(X;\mathbb{Z}_2) \xrightarrow{\Delta} Q^n(C(X;\mathbb{Z}_2)) \xrightarrow{y\%} Q^n(B) \cong \mathbb{Z}_2$$

is given by $x \mapsto \langle Sq^{n-r}(y), x\rangle$.

- For $4k$-dimensional normal space X, $x = [X] \in H_{4k}(X;\mathbb{Z}_2)$, $y = v_{2k}(\nu_X) \in H^{2k}(X;\mathbb{Z}_2)$ obtain

$$\langle Sq^{2k}(y), x\rangle = \langle v_{2k}(\nu_X) \cup v_{2k}(\nu_X), [X]\rangle \in \mathbb{Z}_2$$

Universal examples for the Pontrjagin squares

- For any $j \geqslant 1$ the $\mathbb{Z}_{4j}$-module chain complex concentrated in dimensions $2k, 2k+1$

  $B = B(2j, 2k) : \cdots \to 0 \to \mathbb{Z}_2 \xrightarrow{2j} \mathbb{Z}_{4j} \to 0 \to \ldots$
  has $H_{2k}(B) = \mathbb{Z}_{2j}$, $H_{2k+1}(B) = 0$ and

  $$Q^{4k}(B) \xrightarrow{\cong} \mathbb{Z}_{4j} \ ; \ \phi \mapsto \phi_0 + d\phi_1 \ .$$

- For any space X and any element

  $y \in H^{2k}(X; \mathbb{Z}_{2j}) = H_0(\mathrm{Hom}_{\mathbb{Z}_{4j}}(C(X; \mathbb{Z}_{4j}), B))$
  the composite

  $H_{4k}(X; \mathbb{Z}_{4j}) \xrightarrow{\Delta} Q^{4k}(C(X; \mathbb{Z}_{4j})) \xrightarrow{y_\%} Q^{4k}(B) \cong \mathbb{Z}_{4j}$
  is given by $x \mapsto \langle \mathcal{P}_{2j}(y), x \rangle$.

- For $4k$-dimensional normal space X, $x = [X] \in H_{4k}(X; \mathbb{Z}_4)$, $y = v_{2k}(\nu_X) \in H^{2k}(X; \mathbb{Z}_2)$ obtain $\langle \mathcal{P}_2(v_{2k}(\nu_X)), [X] \rangle \in \mathbb{Z}_4$.

The hyperquadratic Q-groups

- The hyperquadratic Q-groups of a f.g. free A-module chain complex C

$$\widehat{Q}^n(C) \;=\; H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}, C \otimes_A C))$$

  $\widehat{W}$ = complete free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$, for all $s \in \mathbb{Z}$

  $d = 1 + (-1)^s T\colon \widehat{W}_s = \mathbb{Z}[\mathbb{Z}_2] \to \widehat{W}_{s-1} = \mathbb{Z}[\mathbb{Z}_2]$

- An element $\theta \in \widehat{Q}^n(C)$ is represented by

  $\theta = \{\theta_s : C^r = \mathrm{Hom}_A(C_r, A) \to C_{n-r+s} \,|\, s \in \mathbb{Z}\}$

  such that $d\theta_s + \theta_s d^* + \theta_{s-1} + \theta^*_{s-1} = 0$ ($\pm$)

- The symmetric construction on any S-dual of X is the hyperquadratic construction

$$\widehat{\Delta}\colon H^*(X) \to \widehat{Q}^*(C(X)^{-*})$$

  which generalizes dual Steenrod squares $\chi(Sq)^r$.

Chain bundles

- A chain bundle (C,γ) is a $\mathbb{Z}$-module chain complex C together with an element $\gamma \in \widehat{Q}^0(C^{-*})$. The Wu classes of (C,γ) are $v_{2k}(\gamma)\colon H_{2k}(C) \to \mathbb{Z}_2; x \mapsto \gamma_{-4k}(x)(x)$.

- Theorem (R., 1978) A spherical fibration $\nu : X \to BSG(j)$ determines a chain bundle $(C(X), \gamma(\nu) = \widehat{\Delta}U)$ with

$$\widehat{\Delta}\colon \widetilde{H}^j(T(\nu)) \to \widehat{Q}^j(\widetilde{C}(T(\nu))^{-*}) = \widehat{Q}^0(C(X)^{-*})$$

$$v_{2*}(\gamma) = v_{2*}(\nu) \in \widehat{Q}^0(C(X)^{-*}) = H^{2*}(X;\mathbb{Z}_2)$$

- Theorem (Weiss, 1985) The chain bundle (B,β) with $B\colon \ldots \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z}$ and $v_{2k}(\beta) : H_{2k}(B) \xrightarrow{\cong} \mathbb{Z}_2$ is universal: for any $\mathbb{Z}$-module chain complex C isomorphism

$$H_0(\mathrm{Hom}_{\mathbb{Z}}(C,B)) = H^{2*}(C;\mathbb{Z}_2) \xrightarrow{\cong} \widehat{Q}^0(C^{-*});$$

$$v = v_{2*}(\gamma) \mapsto (v^*)^{\%}(\beta) = \gamma\ .$$

The twisted quadratic Q-groups

- Definition (Weiss) The twisted quadratic Q-groups $Q_*(C,\gamma)$ fit into exact sequence

$$\ldots \to \widehat{Q}^{n+1}(C) \to Q_n(C,\gamma) \to Q^n(C) \xrightarrow{J_\gamma} \widehat{Q}^n(C) \to \ldots$$

with $Q_n(C,\gamma) \to Q^n(C); (\phi,\chi) \mapsto \phi$,

$J_\gamma : Q^n(C) \to \widehat{Q}^n(C); \phi \mapsto \{\phi_s - \phi_0^*\gamma_{s-n}\phi_0 | s \in \mathbb{Z}\}$

- An element $(\phi,\chi) \in Q_n(C,\gamma)$ is represented by collections of $\mathbb{Z}$-module morphisms

$$\phi = \{\phi_s : C^r \to C_{n-r+s} \,|\, r,s \geqslant 0\}$$
$$\chi = \{\chi_s : C^r \to C_{n-r+s+1} \,|\, r \geqslant 0, s \in \mathbb{Z}\}$$

such that up to signs

$$d\phi_s + \phi_s d^* + \phi_{s-1} + \phi_{s-1}^* = 0 \quad (\phi_{-1} = 0)$$
$$\phi_s - \phi_0^*\gamma_{s-n}\phi_0 = d\chi_s + \chi_s d^* + \chi_{s-1} + \chi_{s-1}^*.$$

Nonlinear addition by

$$(\phi,\chi)+(\phi',\chi')=(\{\phi_s+\phi'_s\},\{\chi_s+\chi'_s+\phi_0^*\gamma_{s-n}\phi'_0\})$$

The algebraic normal complex of a normal space

- An n-dimensional normal space (X,ν_X,ρ_X) determines an n-dimensional algebraic normal complex $(C(X),\phi,\gamma,\chi)$ over $\mathbb{Z}$, with $\phi=\Delta[X]$, $\gamma=\gamma(\nu_X)$, such that

  $$\phi_0 = [X]\cap - : H^*(X)\to H_{n-*}(X)$$
  $$\phi_{n-2r}(x)(x) = \langle Sq^r(x),[X]\rangle \in \mathbb{Z}_2$$
  $$\gamma_{-2r}(y)(y) = \langle v_r(\nu_X),y\rangle \in \mathbb{Z}_2$$
  $$[X]\cap Sq^r(x) = [X]\cap(v_r(\nu_X)\cup x)\in H_0(X;\mathbb{Z}_2)$$

  for any $x\in H^{n-r}(X;\mathbb{Z}_2)$, $y\in H_r(X;\mathbb{Z}_2)$.

- The element $(\phi,\chi)\in Q_n(C(X),\gamma)$ is the 'algebraic normal invariant' of (X,ν_X,ρ_X).

Certain exact sequences

- For $\nu : X \to BSG(j)$, $(C,\gamma) = (C(X),\gamma(\nu))$ the Alexander-Whitney-Steenrod diagonal chain approximation extends to a natural transformation from the certain exact sequence of Whitehead

$$\begin{array}{ccccccc}
\Gamma_{n+j} & \longrightarrow & \pi_{n+j}(T(\nu)) & \xrightarrow{h} & \widetilde{H}_{n+j}(T(\nu)) & \to & \Gamma_{n+j-1} \\
\downarrow & & \downarrow & & \downarrow \Delta U & & \downarrow \\
\widehat{Q}^{n+1}(C) & \longrightarrow & Q_n(C,\gamma) & \longrightarrow & Q^n(C) & \xrightarrow{J_\gamma} & \widehat{Q}^n(C)
\end{array}$$

with h the Hurewicz map and
$\Delta U : \widetilde{H}_{n+j}(T(\nu)) \cong H_n(X) \xrightarrow{\Delta} Q^n(C)$

- The mod 8 signature $\widehat{\sigma}^*(X) \in \mathbb{Z}_8$ of a $4k$-dimensional normal space X is the evaluation on $\rho_X \in \pi_{4k+j}(T(\nu_X))$ of composite

$$\pi_{4k+j}(T(\nu_X)) \to Q_{4k}(C,\gamma) \xrightarrow{v_\%} Q_{4k}(B,\beta) = \mathbb{Z}_8$$

with $v : C = C(X) \to B$ classifying $\gamma = \gamma(\nu_X) = v_{2*}(\nu_X) \in \widehat{Q}^0(C^{-*}) = H^{2*}(X;\mathbb{Z}_2)$.

The proof of Theorem 1

The $\mathbb{Z}$-module chain complex concentrated in dimensions $2k, 2k+1$

$$B : \cdots \to 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0 \to \ldots$$

is an integral lift of the universal example $B(2,2k)$ for $\mathcal{P}_2 : H^{2k}(X;\mathbb{Z}_2) \to H^{4k}(X;\mathbb{Z}_4)$. Let $v = v_{2k}(\nu_X) \in H_0(\mathrm{Hom}_{\mathbb{Z}}(C(X),B)) = H^{2k}(X;\mathbb{Z}_2)$. The commutative diagram

$$\begin{array}{ccc}
\pi_{4k+j}(T(\nu_X)) & \longrightarrow & \widetilde{H}_{4k+j}(T(\nu_X)) \\
\big\downarrow & & \big\downarrow \Delta U \\
Q_{4k}(C(X),\gamma(\nu_X)) & \longrightarrow & Q^{4k}(C(X)) \\
v\% \big\downarrow & & \big\downarrow v\% \\
Q_{4k}(B,\beta) = \mathbb{Z}_8 & \longrightarrow & Q^{4k}(B) = \mathbb{Z}_4
\end{array}$$

sends $\rho_X \in \pi_{4k+j}(T(\nu_X))$ to

$$[\widehat{\sigma}^*(X)] = \langle \mathcal{P}_2(v_{2k}(\nu_X)),[X]\rangle \in \mathbb{Z}_4 \ .$$

The proof of Theorem 2

The $\mathbb{Z}$-module chain complex concentrated in dimensions $2k, 2k+1$

$$B : \dots \to 0 \to \mathbb{Z} \xrightarrow{4} \mathbb{Z} \to 0 \to \dots$$

is an integral lift of the universal example $B(4, 2k)$ for $\mathcal{P}_4 : H^{2k}(X;\mathbb{Z}_4) \to H^{4k}(X;\mathbb{Z}_8)$. If

$$\delta_4(v_{2k}(\nu_X)) = 0 \in H^{2k+1}(X;\mathbb{Z}_2)$$

then $v_{2k}(\nu_X) \in H^{2k}(X;\mathbb{Z}_2)$ can be lifted to $v \in H^{2k}(X;\mathbb{Z}_4) = H_0(\mathrm{Hom}_{\mathbb{Z}}(C(X), B))$. The commutative diagram

$$\begin{array}{ccc}
\pi_{4k+j}(T(\nu_X)) & \xrightarrow{h} & \widetilde{H}_{4k+j}(T(\nu_X)) \\
\downarrow & & \downarrow \Delta U \\
Q_{4k}(C(X), \gamma(\nu_X)) & \longrightarrow & Q^{4k}(C(X)) \\
v_\% \downarrow & & \downarrow v^\% \\
Q_{4k}(B, \beta) & \xrightarrow{\cong} & Q^{4k}(B) = \mathbb{Z}_8
\end{array}$$

sends $\rho_X \in \pi_{4k+j}(T(\nu_X))$ to

$$\widehat{\sigma}^*(X) = \langle \mathcal{P}_4(v), [X] \rangle \in \mathbb{Z}_8 \ .$$

Here be dragons!

The mod 2^{m+2} signature for $m \geqslant 1$

- Let X be a $4k$-dimensional normal space s.t. $v_{2k}(\nu_X) \in \ker(\delta_{2^{m+1}}) = \mathrm{im}(H^{2k}(X;\mathbb{Z}_{2^{m+1}})) \subseteq H^{2k}(X;\mathbb{Z}_2)$. For any lift $v \in H^{2k}(X;\mathbb{Z}_{2^{m+1}})$ of $v_{2k}(\nu_X)$ define the <u>mod 2^{m+2} signature</u>

  $$\widehat{\sigma}^*(X,v) = \langle \mathcal{P}_{2^{m+1}}(v),[X]\rangle \in \mathbb{Z}_{2^{m+2}} .$$

  For $m = 1$ agrees with previous definition of mod 8 signature by Theorem 2.

- <u>Theorem $m+1$</u> For a $4k$-dimensional Poincaré X and any lift $v \in H^{2k}(X;\mathbb{Z}_{2^{m+1}})$ of $v_{2k}(\nu_X) \in \mathrm{im}(H^{2k}(X)) \subseteq H^{2k}(X;\mathbb{Z}_2)$

  $$[\sigma^*(X)] = \widehat{\sigma}^*(X,v) = \langle \mathcal{P}_{2^{m+1}}(v),[X]\rangle \in \mathbb{Z}_{2^{m+2}} .$$

- <u>Proof</u> The integral lift of $B(2^{m+1},2k)$ with $d = 2^{m+1} : B_{2k+1} = \mathbb{Z} \to B_{2k} = \mathbb{Z}$ has

  $$Q_{4k}(B,\beta) = Q^{4k}(B) = \mathbb{Z}_{2^{m+2}} .$$

Wu orientations for Poincaré cobordism?

- For any $m \geqslant 1$ let $\Omega^N_{4k}\langle\delta_{2^{m+1}}\rangle$ be the cobordism group of $4k$-dimensional normal spaces X such that

  $v_{2k}(\nu_X) \in \ker(\delta_{2^{m+1}}) = \mathrm{im}(H^{2k}(X;\mathbb{Z}_{2^{m+1}}))$

  $\subseteq H^{2k}(X;\mathbb{Z}_2)$ with a lift $v \in H^{2k}(X;\mathbb{Z}_{2^{m+1}})$.
  The mod 2^{m+2} signature is a morphism

  $\widehat{\sigma}^* : \Omega^N_{4k}\langle\delta_{2^{m+1}}\rangle \to \mathbb{Z}_{2^{m+2}}; (X,v) \mapsto \langle \mathcal{P}_{2^{m+1}}(v), [X]\rangle$

- Conjecture There exist Wu-orientation maps

  $$\Omega^P_{4k} \to \Omega^N_{4k}\langle\delta_{2^{m+1}}\rangle; X \mapsto (X,v)$$

  to fit into commutative diagram such that

  $$\begin{array}{ccc} \Omega^P_{4k} & \xrightarrow{\sigma^*} & \mathbb{Z} \\ \downarrow & & \downarrow \\ \Omega^N_{4k}\langle\delta_{2^{m+1}}\rangle & \xrightarrow{\widehat{\sigma}^*} & \mathbb{Z}_{2^{m+2}} \end{array}$$