SPECTRA AND ASSEMBLY IN ALGEBRAIC L-THEORY

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3rd December, 2012

What are spectra, assembly and algebraic L-theory doing in geometric topology?

- Answer: they are useful homotopy theoretic and algebraic tools in understanding the homotopy types of topological manifolds.
- Surgery theory thrives on these tools! Especially in dimensions $\neq 3, 4$: would be good to know how to include 3 and 4.
- Spectra = stable homotopy theory
- Assembly = passage from local to global.
- Algebraic L-theory = quadratic forms, as in the Wall obstruction groups $L_*(\mathbb{Z}[\pi])$ for surgery on manifolds with fundamental group π.

A brief history of assembly

- *Congress shall make no law ... abridging ... the right of the people to peaceable assembly*
  First Amendment to the United States Constitution, 1791
- Wall (1970) Surgery obstruction groups $L_*(\mathbb{Z}[\pi])$. Assembly modulo 2-torsion.
- Quinn (1971) Geometric L-theory assembly
  $[X, G/TOP] \to L_n(\mathbb{Z}[\pi_1(X)])$
- Ranicki (1979, 1992) Algebraic L-theory assembly
  $A : H_n(X; \mathbb{L}_\bullet) \to L_n(\mathbb{Z}[\pi_1(X)])$
- Ranicki-Weiss (1990) Chain complexes and assembly
- Weiss-Williams (1995) Assembly via stable homotopy theory
- Davis-Lück (1998) Assembly via equivariant homotopy theory
- Hambleton-Pedersen (2004) Identification of various assembly maps
- Applications of assembly to Novikov, Borel, Farrell-Jones,Baum-Connes conjectures, in many contexts besides algebraic L-theory, such as algebraic K-theory or K-theory of C^*-algebras. Lück 2010 ICM talk.

Surgery theory

- The 1960's saw a great flowering of the topology of high-dimensional manifolds, especially in dimensions > 4.
- The Browder-Novikov-Sullivan-Wall surgery theory combined with the Kirby-Siebenmann structure theory for topological manifolds provided construction methods for recognizing the homotopy types of topological manifolds among spaces with Poincaré duality.
- The **spectra**, **assembly** and L-**theory** of the title are the technical tools from homotopy theory and the algebraic theory of quadratic forms which are used to recognize topological manifolds in homotopy theory.
- Recognition only works in dimension > 4. Need much more subtle methods in dimensions $3, 4$.

Geometric Poincaré complexes

- An n-**dimensional geometric Poincaré complex** X is a finite CW complex together with a homology class $[X] \in H_n(X)$ such that there are induced Poincaré duality isomorphisms with arbitrary coefficients

$$[X] \cap - \ : \ H^*(X) \ \cong \ H_{n-*}(X) \ .$$

- An n-dimensional topological manifold M is an n-dimensional geometric Poincaré complex for $n \neq 4$, and for $n = 4$ is at least homotopy equivalent to a 4-dimensional Poincaré complex.
- Any finite CW complex homotopy equivalent to an n-dimensional topological manifold is a geometric Poincaré complex.
- When is an n-dimensional Poincaré complex X homotopy equivalent to an n-dimensional topological manifold?
- Motivational answer: for $n > 4$ if and only if the Mishchenko-R. symmetric signature $\sigma(X) \in L^n(\mathbb{Z}[\pi_1(X)])$ is in the image of the symmetric L-theory assembly map $A : H_n(X; \mathbb{L}^\bullet) \to L^n(\mathbb{Z}[\pi_1(X)])$.

Proto-assembly, from homotopy to homology

- A homology class $[X] \in H_n(X)$ is local in nature, depending only on the images
$$[X]_x \in H_n(X, X\backslash\{x\}) \ \ (x \in X).$$
- A map of spaces $f : X \to Y$ induces a chain map $f_* : C(X) \to C(Y)$.
- The **proto-assembly** function
$$H_0(Y^X) \to H_0(\mathrm{Hom}_{\mathbb{Z}}(C(X), C(Y))) \ ; \ f \mapsto f_*$$
sends the homotopy class of a map $f : X \to Y$ to the chain homotopy class of f_*.
- Local to global.
- **Vietoris theorem**: if $f : X \to Y$ is a surjection of reasonable spaces (e.g. simplicial complexes) with acyclic point inverses
$$H_*(f^{-1}(x)) \ \cong \ H_*(x) \ (x \in X)$$
then the proto-assembly f_* is an isomorphism in homology.
- More about this in Spiros Adams-Florou's talk tomorrow.

Proto-assembly: the diagonal map

- The diagonal map

$$\Delta : X \to X \times X \ ; \ x \mapsto (x,x)$$

sends $[X] \in H_n(X)$ to the chain homotopy class

$$\Delta[X] \in H_n(X \times X) \ = \ H_0(\mathrm{Hom}_{\mathbb{Z}}(C(X)^{n-*}, C(X)))$$

of the chain map $\Delta[X] = [X] \cap - : C(X)^{n-*} \to C(X)$. Local to global.

- If X is a closed oriented n-dimensional manifold with fundamental class $[X] \in H_n(X)$ then

$$H_r(X, X\backslash\{x\}) \ = \ \begin{cases} \mathbb{Z} & \text{for } r = n, \text{ generated by } [X]_x = 1 \\ 0 & \text{for } r \neq n \ . \end{cases}$$

- The local Poincaré duality isomorphisms

$$[X]_x \cap - \ : \ H^*(\{x\}) \ \cong \ H_{n-*}(X, X\backslash\{x\} \ (x \in X))$$

assemble to the global Poincaré duality isomorphisms

$$[X] \cap \ : \ H^*(X) \ \cong \ H_{n-*}(X) \ .$$

Suspension and loop spaces

- Only really need Ω-spectra, but suspension spectra motivational.
- The **suspension** of a pointed space X is

$$\Sigma X \;=\; S^1 \wedge X \;.$$

- The **loop space** of X is

$$\Omega X \;=\; X^{S^1} \;.$$

- Adjointness property: for any pointed X, Y

$$X^{\Sigma Y} \;=\; (\Omega X)^Y \;,\; [\Sigma Y, X] \;=\; [Y, \Omega X] \;.$$

- In particular, for $Y = S^n$ have

$$\pi_{n+1}(X) \;=\; \pi_n(\Omega X) \;.$$

Suspension spectra

- A **suspension spectrum** is a sequence of pointed spaces and maps

$$\mathbb{E} \;=\; \{E_k, \Sigma E_k \to E_{k+1} \,|\, k \geqslant 0\}$$

- The **homotopy groups** of $\mathbb{E}$ are defined by

$$\pi_n(\mathbb{E}) \;=\; \varinjlim_k \pi_{n+k}(E_k)\ .$$

- **Example** The homology groups of a space X are the homotopy groups of the Eilenberg-MacLane suspension spectrum $\mathbb{H}(X)$

$$H_n(X) \;=\; \pi_n(\mathbb{H}(X))\ ,\ \mathbb{H}(X)_k \;=\; X_+ \wedge K(\mathbb{Z}, k)$$

with $X_+ = X \sqcup \{+\}$.

- Hard to see the local nature of $H_*(X)$.

The Pontrjagin-Thom transversality construction

- Given an oriented k-plane bundle $\eta : X \to BSO(k)$ let $T(\eta)$ be the Thom space.
- **Pontrjagin-Thom construction**: Every map $\rho : N^{n+k} \to T(\eta)$ from an oriented $(n+k)$-dimensional manifold N is homotopic to a map transverse regular at the zero section $X \subset T(\eta)$. The inverse image is an oriented n-dimensional submanifold

$$M^n \;=\; \rho^{-1}(X) \subset N \;.$$

- The normal bundle of $M \subset N$ is the pullback oriented k-plane bundle

$$\nu_{M\subset N} \;=\; f^*\eta \;:\; M \to X \to BSO(k)$$

of η along the restriction $f = \rho| : M \to X$.

The Pontrjagin-Thom assembly in bordism theory

- The Thom space $MSO(k) = T(1_k)$ of the universal k-plane bundle $1_k : BSO(k) \to BSO(k)$ is the kth space of the **universal Thom suspension spectrum**

$$MSO \;=\; \{MSO(k) \,|\, \Sigma MSO(k) \to MSO(k+1)\}\ .$$

- Let $\Omega_n^{SO}(X)$ be the bordism groups of closed oriented n-dimensional manifolds M^n with a map $M \to X$
- The Pontrjagin-Thom isomorphism

$$\pi_n(X_+ \wedge MSO) \to \Omega_n^{SO}(X)\ ;$$
$$(\rho : S^{n+k} \to X_+ \wedge MSO(k)) \mapsto (\rho| : M^n = \rho^{-1}(X \times BSO(k)) \to X)$$

will serve as a model for the algebraic L-theory assembly map A, but it is hard to see it as local to global. The Pontrjagin-Thom construction is too analytic to translate into algebra directly. Also, A is not in general an isomorphism.

Ω-spectra

- An Ω-**spectrum** is a sequence of pointed spaces and homotopy equivalences
$$\mathbb{F} \ = \ \{F_k, F_k \simeq \Omega F_{k-1} \,|\, k \in \mathbb{Z}\}$$
so that there are homotopy equivalences
$$F_0 \ \simeq \ \Omega F_{-1} \ \simeq \ \dots \ \simeq \ \Omega^k F_{-k} \ .$$
- The homotopy groups of F are defined by
$$\pi_n(\mathbb{F}) \ = \ \pi_n(F_0) \ = \ \dots \ = \ \pi_{n+k}(F_{-k}) \ .$$
- There is no essential difference between the homotopy theoretic properties of the suspension spectra and Ω-spectra.
- A suspension spectrum $\mathbb{E} = \{E_k, \Sigma E_k \to E_{k+1}\}$ determines an Ω-spectrum $\Omega^\infty \mathbb{E} = \mathbb{F}$ with the same homotopy groups
$$\mathbb{F} \ = \ \{F_k \simeq \Omega F_{k+1}\} \ , \ F_k \ = \ \varinjlim_j \Omega^j E_{j-k} \ , \ \pi_n(\mathbb{F}) \ = \ \pi_n(\mathbb{E}) \ .$$

Homotopy invariant functors

- The homotopy groups of a covariant functor

$$F \ : \ \{\text{topological spaces}\} \to \{\Omega\text{-spectra}\} \ ; \ X \mapsto F(X)$$

are written

$$F_n(X) \ = \ \pi_n(F(X)) \ (n \in \mathbb{Z}) \ .$$

- F is **homotopy invariant** if for a homotopy equivalence $X \to Y$, or equivalently there are induced isomorphisms

$$F_*(X) \ \xrightarrow{\cong} \ F_*(Y) \ .$$

- The relative homotopy groups of a pair $(Y, X \subseteq Y)$

$$F_n(Y, X) = \pi_n(F(Y)/F(X))$$

fit into the usual exact sequence

$$\dots \to F_n(X) \to F_n(Y) \to F_n(Y, X) \to F_{n-1}(X) \to \dots \ .$$

Generalized homology theories

- The functor

$$F \;:\; \{\text{topological spaces}\} \to \{\Omega\text{-spectra}\} \;;\; X \mapsto F(X)$$

is **excisive** if for $X = X_1 \cup_Y X_2$ the inclusion $(X_1, Y) \subset (X, X_2)$ induces excision isomorphisms

$$F_n(X_1, Y) \xrightarrow{\cong} F_n(X, X_2)$$

and there is defined a Mayer-Vietoris exact sequence

$$\dots \to F_n(Y) \to F_n(X_1) \oplus F_n(X_2) \to F_n(X) \to F_{n-1}(Y) \to \dots\;.$$

- F is a **generalized homology functor** if it is both homotopy invariant and excisive.
- The homotopy groups $F_*(X) = \pi_*(F(X))$ are called **generalized homology groups**.

Generalized homology functors and Ω-spectra I.

- **Theorem** (G.W. Whitehead 1962) An Ω-spectrum

$$\mathbb{F} \;=\; \{F_k, \Omega F_k \simeq F_{k-1} \mid k \in \mathbb{Z}\}$$

determines a generalized homology functor

$$F \;=\; H(?;\mathbb{F}) \;:\; \{\text{topological spaces}\} \to \{\Omega\text{-spectra}\}\;;$$
$$X \mapsto F(X) \;=\; H(X;\mathbb{F}) \;=\; X_+ \wedge \mathbb{F}\;.$$

- The generalized homology groups are

$$H_n(X;\mathbb{F}) \;=\; F_n(X) \;=\; \varinjlim_k \pi_{n+k}(X_+ \wedge F_{-k})\;.$$

- Moreover, every generalized homology theory arises in this way.

Generalized homology functors and Ω-spectra II.

- **Theorem** (Weiss-Williams 1995) For every homotopy invariant functor

$$F \;:\; \{\text{topological spaces}\} \to \{\Omega\text{-spectra}\}$$

there is an assembly natural transformation $A : F^{\%} \to F$. with

$$F^{\%} \;=\; H(?; F(*)) \;:\; \{\text{topological spaces}\} \to \{\Omega\text{-spectra}\}$$

the $F(*)$-coefficient generalized homology functor..

- $F^{\%}$ is the best approximation to a generalized homology theory with a natural transformation to F.
- The algebraic L-spectrum $F(X) = \mathbb{L}(\mathbb{Z}[\pi_1(X)])$ does give the algebraic L-theory assembly A, but very abstractly.

Bordism is a generalized homology theory

- **Theorem** (Thom 1954, Atiyah 1960) The functor

$$\Omega_*^{SO} \ : \ \{\text{topological spaces}\} \to \{\mathbb{Z}\text{-graded abelian groups}\} \ ;$$
$$X \mapsto \Omega_*^{SO}(X)$$

  is a generalized homology theory, i.e. satisfies the Eilenberg-Steenrod axioms other than dimension.
- **Example**: The Mayer-Vietoris exact sequence for a union $X = X_1 \cup_Y X_2$ with $Y \times \mathbb{R} \subset X$

$$\dots \to \Omega_n^{SO}(Y) \to \Omega_n^{SO}(X_1) \oplus \Omega_n^{SO}(X_2) \to \Omega_n^{SO}(X) \to \Omega_{n-1}^{SO}(Y) \to \dots$$

  is proved by codimension 1 transversality, with

$$\partial \ : \ \Omega_n^{SO}(X_1 \cup_Y X_2) \to \Omega_{n-1}^{SO}(Y) \ ;$$
$$(f : M^n \to X_1 \cup_Y X_2) \mapsto (f| : N^{n-1} = f^{-1}(Y) \to Y) \ .$$

The generalized homology functor of bordism.

- Want to construct a generalized homology functor

$$\Omega^{SO} \ : \ \{\text{topological spaces}\} \to \{\Omega\text{-spectra}\}$$

  such that

$$\pi_*(\Omega^{SO}(X)) \ = \ \Omega^{SO}_*(X)\ .$$

- The Ω-spectrum $\Omega^\infty MSO$ of the Thom suspension spectrum MSO to construct a generalized homology functor

$$\Omega^{SO} \ : \ \{\text{topological spaces}\} \to \{\Omega\text{-spectra}\}$$

  such that $\pi_*(\Omega^{SO}(X)) = \Omega^{SO}_*(X)$.
- However, this procedure does not adapt gracefully to algebraic L-theory.

Assembly via simplicial complexes

- There is a direct construction of an assembly map

$$A \;:\; H(X;\mathbb{F}(*)) \to \mathbb{F}(X)$$

  for any functor $\mathbb{F}$, with $X = |K|$ the polyhedron of a simplicial complex K.
- Will concentrate on the bordism functors $\mathbb{F}$ in various contexts: manifolds, geometric Poincaré complexes, algebraic Poincaré complexes.
- Method also works for arbitrary $\mathbb{F}$ - see Chapter 6 of Algebraic L-theory and topological manifolds (CUP, 1992)
- The key idea is to construct $H(X;\mathbb{F}) = H(K;\mathbb{F})$ as an Ω-spectrum of Kan Δ-sets which keeps track of one piece of $\mathbb{F}$ for each simplex $\sigma \in K$, and these pieces fit together according to the simplicial structure of K. The assembly map $A : H(K;\mathbb{F}) \to \mathbb{F}(X) = \mathbb{F}(K)$ forgets the K-local structure.

Nerves

- Let X be a topological space with a covering

$$X = \bigcup_{v \in V} X(v)$$

  by subspaces $X(v) \subseteq X$, some of which may be empty.
- The **nerve** of the cover is the simplicial complex K with vertex set

$$K^{(0)} = \{v \in V \,|\, X(v) \neq \emptyset\} .$$

  The vertices $v_0, v_1, \ldots, v_n \in K^{(0)}$ span an n-simplex of K if

$$X(v_0) \cap X(v_1) \cap \cdots \cap X(v_n) \neq \emptyset .$$

Dissections

- Let K be a simplicial complex. A K-**dissection** of a space X is a covering
$$X = \bigcup_{\sigma \in K} X(\sigma)$$
by subspaces $X(\sigma) \subseteq X$, some of which may be empty, such that
$$X(\sigma) \cap X(\tau) = \begin{cases} X(\sigma\tau) & \text{if } \sigma\tau \in K \\ \emptyset & \text{otherwise.} \end{cases}$$

- The nerve of the cover is the subcomplex
$$\{\sigma \in K \mid X(\sigma) \neq \emptyset\} \subseteq K .$$

The barycentric subdivision

- The **barycentric subdivision** K' is the simplicial complex with vertices the barycentres $\widehat{\sigma}$ of the simplexes $\sigma \in K$. There is one n-simplex $(\widehat{\sigma}_0\widehat{\sigma}_1 \dots \widehat{\sigma}_n) \in X'$ for each flag of simplexes
$$\sigma_0 < \sigma_1 < \cdots < \sigma_n \in K \ .$$
- Same polyhedron
$$|K'| \ = \ |K| \ = \ \coprod_{\sigma \in K} \Delta^{\dim \sigma}/ \sim \ .$$
- Poincaré used the dual cells $D(\sigma, K) \subseteq K'$ $(\sigma \in K)$ to prove Poincaré duality $H^{n-*}(K) \cong H_*(K)$ for an n-dimensional combinatorial homology manifold K.
- The assembly $A : H(K; \mathbb{F}(*)) \to \mathbb{F}(K)$ will also use dual cells, in the first instance to just describe $H(K; \mathbb{F}(*))$.

Dual cells

- The **dual cell** of a simplex $\sigma \in K$ is the contractible subcomplex

$$D(\sigma, K) \;=\; \{(\widehat{\sigma}_0\widehat{\sigma}_1 \ldots \widehat{\sigma}_n) \,|\, \sigma \leqslant \sigma_0 < \sigma_1 < \cdots < \sigma_n \in K\} \subseteq K' \ .$$

- The **boundary** of $D(\sigma, K)$ is the subcomplex

$$\begin{aligned} \partial D(\sigma, K) \;&=\; \bigcup_{\tau > \sigma} D(\tau, K) \\ &=\; \{(\widehat{\sigma}_0\widehat{\sigma}_1 \ldots \widehat{\sigma}_n) \,|\, \sigma < \sigma_0 < \sigma_1 < \cdots < \sigma_n \in K\} \subset D(\sigma, K) \ . \end{aligned}$$

- The dual cells constitute a K-dissection of $|K|$ with nerve K

$$|K| \;=\; \bigcup_{\sigma \in K} D(\sigma, K)$$

such that

$$D(\sigma, K) \cap D(\tau, K) \;=\; \begin{cases} D(\sigma\tau, K) & \text{if } \sigma\tau \in K \\ \emptyset & \text{otherwise} \ . \end{cases}$$

Inverse images of the dual cells

- Given a simplicial complex K and a map $f : M \to |K'|$ write the inverse images of the dual cells and their boundaries as

$$(M(\sigma), \partial M(\sigma)) \ = \ f^{-1}(D(\sigma, K), \partial D(\sigma, K)) \subset M$$

(which may be empty).

- Properties:

$$\partial M(\sigma) \ = \ \bigcup_{\tau > \sigma} M(\tau) \ ,$$

$$M(\sigma) \cap M(\tau) \ = \ \begin{cases} M(\sigma\tau) & \text{if } \sigma\tau \in K \\ \emptyset & \text{otherwise} \ . \end{cases}$$

- The nerve of the cover of M

$$M \ = \ \bigcup_{\sigma \in K} M(\sigma)$$

is the subcomplex

$$\{\sigma | \, M(\sigma) \neq \emptyset\} \subseteq K \ .$$

Manifold transversality from the simplicial complex point of view

- **Theorem** (Marshall Cohen, 1967)
  Let M be an n-dimensional PL manifold M and K a simplicial complex. A simplicial map $f : M \to K'$ is automatically transverse at the dual cells $D(\sigma, K) \subset K'$, with the inverse images codimension k submanifolds with boundary
  $$(M(\sigma)^{n-k}, \partial M(\sigma)) \;=\; f^{-1}(D(\sigma, K), \partial D(\sigma, K)) \subset M$$
  (which may be empty), where $k = \dim(\sigma)$.
- Converse: given a simplicial complex K and a space M with a K-dissection $\{M(\sigma) \,|\, \sigma \in K\}$ there is defined a map $f : M \to |K'|$ such that
  $$M(\sigma) \;=\; f^{-1}D(\sigma, K) \; (\sigma \in K)$$
  If each $(M(\sigma), \partial M(\sigma))$ is an $(n - \dim(\sigma))$-dimensional PL manifold with boundary then M is an n-dimensional PL manifold.
- There are also versions for $CAT = O$, TOP.

Assembly via Δ-sets

- Chapters 11,12 of Algebraic L-theory and topological manifolds use the Rourke-Sanderson 1971 theory of Kan Δ-sets to construct the assembly $A : H(K;\mathbb{F}) \to \mathbb{F}(K') \simeq F(K)$ for a homotopy invariant functor $\mathbb{F} : \{\text{simplicial complexes}\} \to \{\Omega\text{-spectra of Kan } \Delta\text{-sets}\}$.
- The construction uses an abstract version of the theorem of Marshall Cohen: a simplex $x \in H(K;\mathbb{F})$ is a compatible collection of simplices

$$x(\sigma) \in \mathbb{F}(D(\sigma, K)) \ \ (\sigma \in K) \ .$$

The Kan extension condition is used to form the union

$$A(x) \ = \ \bigcup_{\sigma \in K} x(\sigma) \ .$$

- Model: a simplicial map $f : M^n \to K'$ is a compatible collection

$$f| \ : \ M(\sigma)^{n-\dim\sigma} \ = \ f^{-1}D(\sigma, K) \to D(\sigma, K) \ (\sigma \in K)$$

with $M = \bigcup_{\sigma \in K} M(\sigma)$.

Δ-sets I.

- A **Δ-set** K is a sequence $K^{(n)}$ $(n \geqslant 0)$ of sets, together with face maps

$$\partial_i \ : \ K^{(n)} \to K^{(n-1)} \ \ (0 \leqslant i \leqslant n)$$

  such that $\partial_i \partial_j = \partial_{j-1} \partial_i$ $((i < j)$.
- **Example** An ordered simplicial complex K determines a Δ-set K, with $K^{(n)}$ the set of n-simplexes.
- A Δ-set K is **Kan** if every Δ-map

$$\Lambda_{i,n} \ = \ \Delta^n \backslash \{n\text{-face} \cup i\text{th } (n-1)\text{-face}\} \to K$$

  extends to a Δ-map $\Delta^n \to K$.
- The homotopy theory of Kan Δ-sets is essentially the same as the homotopy theory of simplicial complexes.

Δ-sets II.

- A Δ-set K is **pointed** if there is a base simplex $\emptyset \in K^{(n)}$ in each dimension $n \geqslant 0$.
- The **homotopy groups** of a Kan pointed Δ-set K are

$$\pi_n(K) \;=\; \{x \in K^{(n)} \,|\, \partial_i x = \emptyset \text{ for } 0 \leqslant i \leqslant n\}/\sim \;\; .$$

- The **loop space** of a Kan pointed Δ-set K is the Kan pointed Δ-set ΩK with

$$(\Omega K)^{(n)} \;=\; \{x \in K^{(n+1)} \,|\, \partial_{n+1} x = \emptyset, \partial_0 \partial_1 \dots \partial_n x = \emptyset\} \;,$$

  such that

$$\pi_n(\Omega K) \;=\; \pi_{n+1}(K) \;.$$

The Ω-spectrum $\Omega^{CAT}(K)$

- Let CAT be one of the categories O, PL, TOP.
- An $(n+k)$-**dimensional** CAT **manifold** k-**ad** M is an $(n+k)$-dimensional CAT manifold M with transverse codimension 0 submanifolds $\partial_0 M, \partial_1 M, \dots, \partial_k M \subset \partial M$ such that

$$\bigcap_{j=0}^{k} \partial_j M = \emptyset \ , \ \bigcup_{j=0}^{k} \partial_j M = \partial M \ .$$

- Examples: 0-ad = closed manifold, 1-ad = cobordism.
- Let $\Omega^{CAT}(K)$ be the Ω-spectrum with

$$\Omega^{CAT}(K)_n^{(k)} = \{(n+k)\text{-dimensional } CAT \text{ manifold } k\text{-ads } M, \text{ with a map } f : M \to |K|\} \ .$$

  Base points the empty manifold k-ads $\emptyset$.
- The functor

$$\Omega^{CAT} : \{\text{simplicial complexes}\} \to \{\Omega\text{-spectra}\} \ ; \ K \mapsto \Omega^{CAT}(K)$$

  is homotopy invariant, with $\pi_n(\Omega^{CAT}(K)) = \Omega_n^{CAT}(K)$.

The Ω-spectrum $H(K;\Omega^{CAT})$

- $H(K;\Omega^{CAT})$ is the subspectrum of $\Omega^{CAT}(K)$ in which $f : M \to |K|$ is required to be CAT transverse at the dual cells $D(\sigma, K) \subset |K|$ ($\sigma \in K$).
- The assembly map is the inclusion

$$A^{CAT} \ : \ H(K;\Omega^{CAT}) \to \Omega^{CAT}(K) \ .$$

- CAT transversality $= A^{CAT}$ is a homotopy equivalence.
- Apart from transversality, everything works just as well in the category of geometric Poincaré complexes, with assembly the inclusion

$$A^{P} \ : \ H(K;\Omega^{P}) \to \Omega^{P}(K) \ .$$

- **Theorem** For $n \geqslant 5$ an n-dimensional geometric Poincaré complex K is homotopy equivalent to a compact n-dimensional topological manifold if and only if

$$(1 : K \to K) \in \mathrm{im}(A^{P} : H_n(K;\Omega^{P}) \to \Omega^{P}_n(K)) \ .$$