CUTTING AND PASTING MANIFOLDS FROM THE ALGEBRAIC POINT OF VIEW

ANDREW RANICKI (Edinburgh)

http://www.maths.ed.ac.uk/~aar

Cutting and Pasting

- Cut a closed n-dimensional manifold M

$$M = M_1 \cup M_2$$

with M_1, M_2 manifolds with boundary

$$\partial M_1 = \partial M_2$$

- Paste together M_1 and M_2 using an isomorphism

$$h : \partial M_1 \to \partial M_2$$

to obtain a new closed n-dimensional manifold

$$M' = M_1 \cup_h M_2$$

- What are the invariants of manifolds which do not change under cutting and pasting?

Schneiden und Kleben

- Jänich (1968) characterized signature and Euler characteristic as cut and paste invariants.

- Karras, Kreck, Neumann and Ossa (1973) defined SK-groups, universal groups of cut and paste invariants.

- Applications to index of elliptic operators.

- Some recent applications of cut and paste methods to higher signatures, L^2-cohomology

  – Leichtnam, Lott, Lück, Weinberger, ...

The bordism SK-groups

- $\Omega_n(X)$ = bordism of maps from closed n-dimensional manifolds

$$f : M^n \to X$$

- Definition $\overline{SK}_n(X) = \Omega_n(X)/\sim$ with

$$M_1 \cup_g M_2 \sim M_1 \cup_h M_2$$

for any isomorphisms $g, h : \partial M_1 \to \partial M_2$

Twisted doubles

- A closed n-dimensional manifold M is a <u>twisted double</u> if

$$M \;=\; N \cup_h N$$

for n-dimensional manifold with boundary $(N, \partial N)$ and automorphism $h : \partial N \to \partial N$.

- <u>Lemma</u> A map $f : M \to X$ from a closed n-dimensional manifold M represents 0 in $\overline{SK}_n(X)$ if and only if $f : M \to X$ is bordant to a twisted double.

- <u>Proof</u> Whitehead identity in bordism

$$M_1 \cup_g M_2 + M_2 \cup_h M_3 \;=\; M_1 \cup_{hg} M_3 \in \Omega_n(X)$$

with $M_3 = M_1$.

Main result

- The identification of $\overline{SK}_n(X)$ for $n \geq 6$ with the image of the assembly map in the asymmetric L-theory of $\mathbb{Z}[\pi_1(X)]$.

- Geometric realization of algebraic result:

  – A symmetric Poincaré complex is an algebraic twisted double if and only if it is null-cobordant as an asymmetric Poincaré complex.

- Identification almost proved in *High dimensional knot theory* (Springer, 1998)

Symmetric L-theory (I.)

- A = ring with involution

- An $\underline{n\text{-dimensional symmetric Poincaré}}$ $\underline{\text{complex}}$ (C, ϕ) is an n-dimensional f.g. free A-module chain complex

$$C : \cdots \to 0 \to C_n \to \cdots \to C_1 \to C_0$$

with a chain equivalence

$$\phi : C^{n-*} = \mathrm{Hom}_A(C, A)_{*-n} \to C$$

such that $\phi \simeq \phi^*$, and higher symmetry conditions.

- Cobordism of symmetric Poincaré complexes

- $L^n(A)$ = cobordism group (Mishchenko, R.)

Symmetric L-theory (II.)

- Symmetric L-groups = Wall quadratic L-groups modulo 2-primary torsion

$$L^n(A) \otimes \mathbb{Z}[1/2] \ \cong \ L_n(A) \otimes \mathbb{Z}[1/2]$$

- $L^{4*}(\mathbb{Z}) \ = \ \mathbb{Z}$ (signature)

- The <u>symmetric signature</u> of an n-dimensional manifold M

$$\sigma^*(M) \ = \ (C(\widetilde{M}), \phi) \in L^n(\mathbb{Z}[\pi_1(M)])$$

- Symmetric signature map on bordism

$$\sigma^* \ : \ \Omega_*(X) \to L^*(\mathbb{Z}[\pi_1(X)])$$

Asymmetric L-theory (I.)

- An <u>n-dimensional asymmetric Poincaré complex</u> (C,ϕ) is an n-dimensional f.g. free A-module chain complex

$$C \ : \ \cdots \to 0 \to C_n \to \cdots \to C_0$$

with a chain equivalence

$$\phi \ : \ C^{n-*} \ = \ \mathrm{Hom}_A(C,A)_{*-n} \to C$$

(no symmetry condition)

- <u>Cobordism</u> of asymmetric Poincaré complexes

- $LAsy^n(A) =$ cobordism group

- Forgetful maps $L^n(A) \to LAsy^n(A)$

Asymmetric L-theory (II.)

- 2-periodic

$$LAsy^n(A) \cong LAsy^{n+2}(A)$$

- Odd-dimensional asymmetric L-groups vanish

$$LAsy^{2*+1}(A) = 0$$

- Even-dimensional asymmetric L-groups are large, e.g.

$$LAsy^0(\mathbb{Z}) = \bigoplus_{\infty} \mathbb{Z} \oplus \bigoplus_{\infty} \mathbb{Z}_2 \oplus \bigoplus_{\infty} \mathbb{Z}_4$$

- Asymmetric signature of n-dimensional manifold M

$$Asy\sigma^*(M) = (C(\widetilde{M}), \phi) \in LAsy^n(\mathbb{Z}[\pi_1(M)])$$

Algebraic twisted doubles

- Theorem An n-dimensional symmetric Poincaré complex (C,ϕ) over A is an algebraic twisted symmetric Poincaré double if and only if

$$(C,\phi) \ = \ 0 \in LAsy^n(A)$$

- Proof Chapter 30 of *High dimensional knot theory*

- Example If $M = N \cup_h N$ is a twisted double manifold then $C(M) \to C(N,\partial N)$ determines an asymmetric Poincaré null-cobordism of the symmetric Poincaré complex of M, so that

$$\sigma^*(M) \in \ker(L^n(\mathbb{Z}[\pi_1(M)]) \to LAsy^n(\mathbb{Z}[\pi_1(M)]))$$

Recognizing twisted doubles

- Theorem For $n \geq 6$ an n-dimensional manifold M is a twisted double if and only if

$$Asy([M]_{\mathbb{L}}) \ = \ 0 \in LAsy^n(\mathbb{Z}[\pi_1(M)])$$

- Proof The asymmetric signature is the Quinn (1979) obstruction to the existence of open book structure on M

$$M \ = \ T(h : F \to F) \cup \partial F \times D^2$$

- $(h, \text{id.}) \ = \$ rel ∂ automorphism of $(n-1)$-dimensional manifold with boundary $(F, \partial F)$.

- For $n \geq 6$ open book if and only if twisted double.

Assembly

- Assembly map in symmetric L-theory

  $$A : H_n(X;\mathbb{L}(\mathbb{Z})) \to L^n(\mathbb{Z}[\pi_1(X)])$$

  for any space X, with $\pi_*(\mathbb{L}(\mathbb{Z})) = L^*(\mathbb{Z})$.

- Every n-dimensional manifold M has an L-theory orientation

  $$[M]_{\mathbb{L}} \in H_n(M;\mathbb{L}(\mathbb{Z}))$$

  with $A([M]_{\mathbb{L}}) = \sigma^*(M) \in L^n(\mathbb{Z}[\pi_1(M)])$

- Symmetric signature factors through assembly

  $$\sigma^* : \Omega_n(X) \to H_n(X;\mathbb{L}(\mathbb{Z})) \xrightarrow{A} L^n(\mathbb{Z}[\pi_1(X)])$$

- Assembly map in asymmetric L-theory

  $$\begin{aligned} Asy : H_n(X;\mathbb{L}(\mathbb{Z})) &\xrightarrow{A} L^n(\mathbb{Z}[\pi_1(X)]) \\ &\to LAsy^n(\mathbb{Z}[\pi_1(X)]) \end{aligned}$$

The identification of the bordism SK-groups

- <u>Corollary</u> For any space X and $n \geq 6$ the asymmetric signature defines an isomorphism

  $\overline{SK}_n(X) \cong$
  $\mathrm{im}(Asy : H_n(X; \mathbb{L}(\mathbb{Z})) \to LAsy^n(\mathbb{Z}[\pi_1(X)]))$

- <u>Proof</u> Theorem gives that

  $\overline{SK}_n(X) \cong \mathrm{im}(Asy\, \sigma^* : \Omega_n(X) \to LAsy^n(\mathbb{Z}[\pi_1(X$

  with

  $\sigma^* : \Omega_*(X) \to H_*(X; \mathbb{L}(\mathbb{Z}))\ ;\ (f : M \to X) \mapsto f_*[$

- Computation of homotopy type of $\mathbb{L}(\mathbb{Z})$ (Taylor and Williams, 1979) shows that σ^* is onto, so

  $\mathrm{im}(Asy) = \mathrm{im}(Asy\, \sigma^*) \subseteq LAsy^n(\mathbb{Z}[\pi_1(X)])$