# HIGH DIMENSIONAL MANIFOLD TOPOLOGY THEN AND NOW

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Orsay 7,8,9 December 2005

- An $n$-dimensional topological manifold $M$ is a paracompact Hausdorff topological space which is locally homeomorphic to $\mathbb{R}^n$. Also called a *TOP* manifold.
  - *TOP* manifolds with boundary $(M, \partial M)$, locally $(\mathbb{R}^n_+, \mathbb{R}^{n-1})$.
- High dimensional $= n \geqslant 5$.
- Then $=$ before Kirby-Siebenmann (1970)
- Now $=$ after Kirby-Siebenmann (1970)

## Time scale

- 1905 Manifold duality (Poincaré)
- 1944 Embeddings (Whitney)
- 1952 Transversality, cobordism (Pontrjagin, Thom)
- 1952 Rochlin's theorem
- 1953 Signature theorem (Hirzebruch)
- 1956 Exotic spheres (Milnor)
- 1960 Generalized Poincaré Conjecture and $h$-cobordism theorem for *DIFF*, $n \geqslant 5$ (Smale)
- 1962–1970 Browder-Novikov-Sullivan-Wall surgery theory for *DIFF* and *PL*, $n \geqslant 5$
- 1966 Topological invariance of rational Pontrjagin classes (Novikov)
- 1968 Local contractibility of Homeo($M$) (Chernavsky)
- 1969 Stable Homeomorphism and Annulus Theorems (Kirby)
- 1970 Kirby-Siebenmann breakthrough: high-dimensional *TOP* manifolds are just like *DIFF* and *PL* manifolds, only more so!

## The triangulation of manifolds

- A triangulation $(K, f)$ of a space $M$ is a simplicial complex $K$ together with a homeomorphism

$$f \ : \ |K| \xrightarrow{\cong} M \ .$$

  - $M$ is compact if and only if $K$ is finite.
- A *DIFF* manifold $M$ can be triangulated, in an essentially unique way (Cairns, Whitehead, 1940).
- A *PL* manifold $M$ can be triangulated, by definition.
- What about *TOP* manifolds?
  - In general, still unknown.

**Are topological manifolds at least homotopy triangulable?**

- A compact *TOP* manifold $M$ is an *ANR*, and so dominated by the compact polyhedron $L = |K|$ of a finite simplicial complex $K$, with maps

$$f : M \to L \, , \; g : L \to M$$

  and a homotopy

$$gf \simeq 1 : M \to M$$

  (Borsuk, 1933).
  - $M$ has the homotopy type of the noncompact polyhedron

$$\Big(\coprod_{k=-\infty}^{\infty} L \times [k, k+1]\Big) / \{(x, k) \sim (fg(x), k+1) \,|\, x \in L, k \in \mathbb{Z}\}$$

- Does every compact *TOP* manifold $M$ have the homotopy type of a compact polyhedron?
  - Yes (K.-S., 1970)

## Are topological manifolds triangulable?

- Triangulation Conjecture
  Is every compact $n$-dimensional $TOP$ manifold $M$ triangulable?
  - Yes for $n \leqslant 3$ (Moïse, 1951)
  - No for $n = 4$ (Casson, 1985)
  - Unknown for $n \geqslant 5$.
- Is every compact $n$-dimensional $TOP$ manifold $M$ a finite $CW$ complex?
  - Yes for $n \neq 4$, since $M$ has a finite handlebody structure (K.-S., 1970)

## Homology manifolds and Poincaré duality

- A space $M$ is an $n$-dimensional homology manifold if

$$H_r(M, M - \{x\}) = \begin{cases} \mathbb{Z} & \text{if } r = n \\ 0 & \text{if } r \neq n \end{cases} (x \in M) .$$

- A compact $ANR$ $n$-dimensional homology manifold $M$ has Poincaré duality isomorphisms

$$[M] \cap - \; : \; H^{n-*}(M) \cong H_*(M)$$

with $[M] \in H_n(M)$ a fundamental class; twisted coefficients in the nonorientable case.

- An $n$-dimensional $TOP$ manifold is an $ANR$ homology manifold, and so has Poincaré duality in the compact case.
- Compact $ANR$ homology manifolds with boundary $(M, \partial M)$ have Poincaré-Lefschetz duality

$$H^{n-*}(M, \partial M) \cong H_*(M) .$$

## Are topological manifolds combinatorially triangulable?

- The polyhedron $|K|$ of a simplicial complex $K$ is an $n$-dimensional homology manifold if and only if the link of every simplex $\sigma \in K$ is a homology $S^{(n-|\sigma|-1)}$.
- An $n$-dimensional *PL* manifold is the polyhedron $M = |K|$ of a simplicial complex $K$ such that the link of every simplex $\sigma \in K$ is *PL* homeomorphic $S^{(n-|\sigma|-1)}$.
  - A *PL* manifold is a *TOP* manifold with a combinatorial triangulation.
- Combinatorial Triangulation Conjecture
  Does every compact *TOP* manifold have a *PL* manifold structure?
  - No: by the K.-S. *PL*-*TOP* analogue of the classical *DIFF*-*PL* smoothing theory, and the determination of *TOP*/*PL*.
  - There exist non-combinatorial triangulations of any triangulable *TOP* manifold $M^n$ for $n \geqslant 5$ (Edwards, Cannon, 1978)

## The Hauptvermutung: are triangulations unique?

- Hauptvermutung (Steinitz, Tietze, 1908)
  For finite simplicial complexes $K, L$ is every homeomorphism $h : |K| \cong |L|$ homotopic to a $PL$ homeomorphism?
  i.e. do $K, L$ have isomorphic subdivisions?
  - Originally stated only for manifolds.
  - No (Milnor, 1961)
    Examples of homeomorphic non-manifold compact polyhedra which are not $PL$ homeomorphic.
- Manifold Hauptvermutung Is every homeomorphism of compact $PL$ manifolds homotopic to a $PL$ homeomorphism?
  - No: by the K.-S. $PL$-$TOP$ smoothing theory.

## TOP bundle theory

- *TOP* analogues of vector bundles and *PL* bundles. Microbundles = *TOP* bundles, with classifying spaces

$$BTOP(n)\ ,\ BTOP\ =\ \varinjlim_n BTOP(n)\ .$$

(Milnor, Kister 1964)

- A *TOP* manifold $M^n$ has a *TOP* tangent bundle

$$\tau_M\ :\ M \to BTOP(n)\ .$$

- For large $k \geqslant 1$ $M \times \mathbb{R}^k$ has a *PL* structure if and only if $\tau_M : M \to BTOP$ lifts to a *PL* bundle $\widetilde{\tau}_M : M \to BPL$.

## DIFF-PL smoothing theory

- *DIFF* structures on *PL* manifolds (Cairns, Whitehead, Hirsch, Milnor, Munkres, Lashof, Mazur, . . . , 1940–1968)
  The *DIFF* structures on a compact *PL* manifold $M$ are in bijective correspondence with the lifts of $\tau_M : M \to BPL$ to a vector bundle $\widetilde{\tau}_M : M \to BO$, i.e. with $[M, PL/O]$.
- Fibration sequence of classifying spaces

$$PL/O \to BO \to BPL \to B(PL/O) \ .$$

- The difference between *DIFF* and *PL* is quantified by

$$\pi_n(PL/O) \ = \ \begin{cases} \theta_n & \text{for } n \geqslant 7 \\ 0 & \text{for } n \leqslant 6 \end{cases}$$

with $\theta_n$ the finite Kervaire-Milnor group of exotic spheres.

## PL-TOP smoothing theory

- *PL* structures on *TOP* manifolds (K.-S., 1969)
  For $n \geqslant 5$ the *PL* structures on a compact $n$-dimensional *TOP* manifold $M$ are in bijective correspondence with the lifts of $\tau_M : M \to BTOP$ to $\widetilde{\tau}_M : M \to BPL$, i.e. with $[M, TOP/PL]$.
- Fibration sequence of classifying spaces

$$TOP/PL \to BPL \to BTOP \to B(TOP/PL)$$

- The difference between *PL* and *TOP* is quantified by

$$\pi_n(TOP/PL) \; = \; \begin{cases} \mathbb{Z}_2 & \text{for } n = 3 \\ 0 & \text{for } n \neq 3 \end{cases}$$

  detected by the Rochlin signature invariant.

## Signature

- The signature $\sigma(M) \in \mathbb{Z}$ of a compact oriented $4k$-dimensional *ANR* homology manifold $M^{4k}$ with $\partial M = \emptyset$ or a homology $(4k-1)$-sphere $\Sigma$ is the signature of the Poincaré duality nonsingular symmetric intersection form

$$\phi : H^{2k}(M) \times H^{2k}(M) \to \mathbb{Z} \ ; \ (x,y) \mapsto \langle x \cup y, [M] \rangle$$

- Theorem (Hirzebruch, 1953) For a compact oriented *DIFF* manifold $M^{4k}$

$$\sigma(M) \ = \ \langle \mathcal{L}_k(M), [M] \rangle \in \mathbb{Z}$$

with $\mathcal{L}_k(M) \in H^{4k}(M;\mathbb{Q})$ a polynomial in the Pontrjagin classes $p_i(M) = p_i(\tau_M) \in H^{4i}(M)$. $\mathcal{L}_1(M) = p_1(M)/3$.

  - Signature theorem also in the *PL* category. Define $p_i(M), \mathcal{L}_i(M) \in H^{4i}(M;\mathbb{Q})$ for a *PL* manifold $M^n$ by

$$\langle \mathcal{L}_i(M), [N] \rangle \ = \ \sigma(N) \in \mathbb{Z}$$

  for compact *PL* submanifolds $N^{4i} \subset M^n \times \mathbb{R}^k$ with trivial normal *PL* bundle (Thom, 1958).

## The signature mod 8

- Theorem (Milnor, 1958–) If $M^{4k}$ is a compact oriented $4k$-dimensional *ANR* homology manifold with even intersection form

$$\phi(x,x) \equiv 0 \pmod 2 \text{ for } x \in H^{2k}(M) \qquad (*)$$

  then

$$\sigma(M) \equiv 0 \pmod 8 .$$

- For a *TOP* manifold $M^{4k}$

$$\phi(x,x) = \langle v_{2k}(\nu_M), x \cap [M] \rangle \in \mathbb{Z}_2 \text{ for } x \in H^{2k}(M)$$

  with $v_{2k}(\nu_M) \in H^{2k}(M;\mathbb{Z}_2)$ the $2k^{th}$ Wu class of the stable normal bundle $\nu_M = -\tau_M : M \to BTOP$. So condition $(*)$ is satisfied if $v_{2k}(\nu_M) = 0$.
- $(*)$ is satisfied if $M$ is almost framed, meaning that $\nu_M$ is trivial on $M - \{\text{pt.}\}$.
- For $k = 1$ spin $\iff w_2 = 0 \iff v_2 = 0 \implies (*)$.

# $\mathbf{E_8}$

- The $E_8$-form has signature 8

$$E_8 = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

- For $k \geqslant 2$ let $W^{4k}$ be the $E_8$-plumbing of 8 copies of $\tau_{S^{2k}}$, a compact $(2k-1)$-connected $4k$-dimensional framed *DIFF* manifold with $(H^{2k}(W), \phi) = (\mathbb{Z}^8, E_8)$, $\sigma(W) = 8$.
  The boundary $\partial W = \Sigma^{4k-1}$ is an exotic sphere.
- The $4k$-dimensional non-*DIFF* almost framed *PL* manifold $M^{4k} = W^{4k} \cup_{\Sigma^{4k-1}} c\Sigma$ obtained by coning $\Sigma$ has $\sigma(M) = 8$.

## Rochlin's Theorem

- Theorem (Rochlin, 1952) The signature of a compact 4-dimensional spin $PL$ manifold $M$ has $\sigma(M) \equiv 0 (\mathrm{mod}\, 16)$.
  - The Kummer surface $K^4$ has $\sigma(K) = 16$.
- Every oriented 3-dimensional $PL$ homology sphere $\Sigma$ is the boundary $\partial W$ of a 4-dimensional framed $PL$ manifold $W$. The Rochlin invariant

$$\alpha(\Sigma) \;=\; \sigma(W) \in 8\mathbb{Z}/16\mathbb{Z} = \mathbb{Z}_2$$

  accounts for the difference between $PL$ and $TOP$ manifolds!
- $\alpha(\Sigma) = 1$ for the Poincaré 3-dimensional $PL$ homology sphere $\Sigma^3 = SO(3)/A_5 = \partial W$, with $W^4 =$ the 4-dimensional framed $PL$ manifold with $\sigma(W) = 8$ obtained by the $E_8$-plumbing of 8 copies of $\tau_{S^2}$.
  - The 4-dimensional homology manifold $P^4 = W \cup_\Sigma c\Sigma$ is homotopy equivalent to a compact 4-dimensional spin $TOP$ manifold $M^4 = W \cup_\Sigma Q$ with $Q^4$ contractible, $\partial Q = \Sigma^3$, $(H^2(M), \phi) = (\mathbb{Z}^8, E_8)$, $\sigma(M) = 8$ (Freedman, 1982).

## The topological invariance of the rational Pontrjagin classes

- Theorem (Novikov, 1965)
  If $h : M \to N$ is a homeomorphism of compact $PL$ manifolds then
  $$h^* p_i(N) \;=\; p_i(M) \in H^{4i}(M;\mathbb{Q}) \;\; (i \geqslant 1) \; .$$
- It suffices to prove the splitting theorem: for any $k \geqslant 1$ and compact $PL$ submanifold $Y^{4i} \subset N \times \mathbb{R}^k$ with $\pi_1(Y) = \{1\}$ and trivial $PL$ normal bundle the product homeomorphism
  $$h \times 1 : M \times \mathbb{R}^k \to N \times \mathbb{R}^k$$
  is proper homotopic to a $PL$ map $f : M \times \mathbb{R}^k \to N \times \mathbb{R}^k$ which is $PL$ split at $Y$, meaning that it is $PL$ transverse and $f| : X^{4i} = f^{-1}(Y) \to Y$ is also a homotopy equivalence.
- Then $\langle \mathcal{L}_i(M), [X] \rangle = \sigma(X) = \sigma(Y) = \langle \mathcal{L}_i(N), [Y] \rangle \in \mathbb{Z}$, and $h^* \mathcal{L}_i(N) = \mathcal{L}_i(M) \in H^{4i}(M;\mathbb{Q})$, so that $h^* p_i(N) = p_i(M)$.

## Splitting homotopy equivalences of manifolds

- For $CAT = DIFF$, $PL$ or $TOP$ define $CAT$ isomorphism = diffeomorphism, $PL$ homeomorphism, homeomorphism.
- A homotopy equivalence of $CAT$ manifolds $h : M \to N$ is $CAT$ split along a $CAT$ submanifold $Y \subset N$ if $h$ is homotopic to a map $f : M \to N$ $CAT$ transverse at $Y$, with the restriction

$$f| \; : \; X = f^{-1}(Y) \to Y$$

  also a homotopy equivalence of $CAT$ manifolds.
- If $h$ is homotopic to a $CAT$ isomorphism then $h$ is $CAT$ split along every $CAT$ submanifold.
- Converse: if $h$ is not $CAT$ split along one $CAT$ submanifold then $h$ is not homotopic to a $CAT$ isomorphism!

## The splitting theorem

- Theorem (Novikov 1965) Let $k \geqslant 1$, $n \geqslant 5$. If $N^n$ is a compact $n$-dimensional $PL$ manifold with $\pi_1(N) = \{1\}$, $W^{n+k}$ is a non-compact $PL$ manifold, $h : W \to N \times \mathbb{R}^k$ is a homeomorphism, then $h$ is $PL$ split along $N \times \{0\} \subset N \times \mathbb{R}^k$, with $h$ proper homotopic to a $PL$ transverse map $f$ such that $f| : X = f^{-1}(N \times \{0\}) \to N$ is a homotopy equivalence.
  - Proof: Wrap up the homeomorphism $h : W \to N \times \mathbb{R}^k$ of non-compact simply-connected $PL$ manifolds to a homeomorphism $g = \overline{h} : V \to N \times T^k$ of compact non-simply-connected $PL$ manifolds such that

    $$h \simeq \widetilde{g} \ : \ W = \widetilde{V} \to N \times \mathbb{R}^k \ .$$

    $PL$ split $g$ by $k$-fold iteration of codim. 1 $PL$ splittings along $T^0 = \{\text{pt.}\} \subset T^1 \subset T^2 \subset \cdots \subset T^k$. Lift to $PL$ splitting of $h$.
  - The $PL$ splitting needs the algebraic $K$-theory computation $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}^{k-1}]) = 0$, or Bass-Heller-Swan $Wh(\mathbb{Z}^k) = 0$. The $k$-fold iteration of the Siebenmann (1965) end obstruction (unknown to N.)

## The Stable Homeomorphism and Annulus Theorems

- A homeomorphism $h : M \to M$ is stable if

$$h = h_1 h_2 \dots h_k : M \to M$$

is the composite of homeomorphisms $h_i : M \to M$ each of which is the identity on an open subset $U_i \subset \mathbb{R}^n$.

- Stable Homeomorphism Theorem (Kirby, 1969) For $n \geqslant 5$ every orientation-preserving homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ is stable.
- Annulus Theorem (Kirby, 1969) If $n \geqslant 5$ and $h : D^n \to D^n$ is a homeomorphism such that $h(D^n) \subset D^n - S^{n-1}$ the homeomorphism

$$1 \sqcup h| \ : \ S^{n-1} \sqcup S^{n-1} \to S^{n-1} \sqcup h(S^{n-1})$$

extends to a homeomorphism

$$S^{n-1} \times [0,1] \ \cong \ \overline{D^n - h(D^n)} \ .$$

## Wrapping up and unwrapping

- Kirby's proof of the Stable Homeomorphism Theorem involves both wrapping up and unwrapping

compact non-simply-connected $T^n$

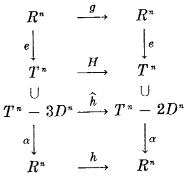
wrapping up $\uparrow$ $\quad$ $\downarrow$ unwrapping

non-compact simply-connected $\mathbb{R}^n$

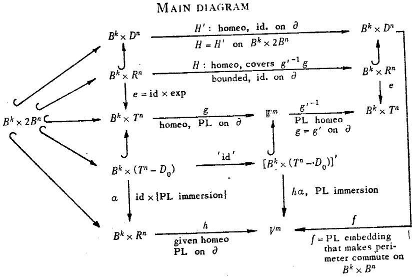
- The wrapping up passes from the homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ to a homeomorphism $\overline{h} : T^n \to T^n$ using geometric topology, via an immersion $T^n - \{\text{pt.}\} \looparrowright \mathbb{R}^n$. Also need the vanishing of the end obstruction for $\pi_1 = \{1\}$.
- $\overline{h}$ is a bounded distance from $1 : T^n \to T^n$, and hence stable.
- The unwrapping passes from $\overline{h}$ back to $h$ using the surgery theory classification of $PL$ manifolds homotopy equivalent to $T^n$ for $n \geqslant 5$ via the algebraic $L$-theory of $\mathbb{Z}[\pi_1(T^n)] = \mathbb{Z}[\mathbb{Z}^n]$.

# The original wrapping up/unwrapping diagrams

- From Kirby's 1969 Annals paper




- From the Kirby-Siebenmann 1969 AMS Bulletin paper



# TOP/PL

- Theorem (K.-S., 1969) Fibration sequence

$$TOP/PL \simeq K(\mathbb{Z}_2, 3) \longrightarrow BPL \longrightarrow BTOP \xrightarrow{\kappa} B(TOP/PL) \simeq K(\mathbb{Z}_2, 4)$$

- The Pontrjagin classes $p_k(\eta) \in H^{4k}(X;\mathbb{Q})$ for $TOP$ bundles $\eta : X \to BTOP$ are defined by pullback from universal classes

$$p_k \in H^{4k}(BTOP;\mathbb{Q}) = H^{4k}(BPL;\mathbb{Q}) \ .$$

- The $\mathcal{L}$-genus of a $TOP$ manifold $M^n$ is defined by $\mathcal{L}_k(M) = \mathcal{L}_k(\tau_M) \in H^{4k}(M;\mathbb{Q})$, and for $n = 4k$

$$\sigma(M) = \langle \mathcal{L}_k(M), [M] \rangle \in \mathbb{Z} \ .$$

- Bundles over $S^4$ classified by $p_1 \in 2\mathbb{Z} \subset H^4(S^4;\mathbb{Q}) = \mathbb{Q}$ and $\kappa \in H^4(S^4;\mathbb{Z}_2) = \mathbb{Z}_2$, with isomorphisms

$$\pi_4(BPL) \xrightarrow{\cong} \mathbb{Z} \ ; \ \widetilde{\eta} \mapsto p_1(\widetilde{\eta})/2 \ ,$$

$$\pi_4(BTOP) \xrightarrow{\cong} \mathbb{Z} \oplus \mathbb{Z}_2 \ ; \ \eta \mapsto (p_1(\eta)/2, \kappa(\eta)) \ .$$

## TOP bundles over $\mathbf{S}^4$

- A *TOP* bundle $\eta : S^4 \to BTOP$ has a *PL* lift $\widetilde{\eta} : S^4 \to BPL$ if and only if

$$\kappa(\eta) \;=\; 0 \in H^4(S^4;\mathbb{Z}_2) = \mathbb{Z}_2 \;.$$

- A *TOP* bundle $\eta : S^4 \to BTOP$ is fibre homotopy trivial if and only if $J(\eta) = 0 \in \pi_4(BG) = \pi_3^S = \mathbb{Z}_{24}$ or equivalently

$$p_1(\eta)/2 \;\equiv\; 12\kappa(\eta) \pmod{24} \;.$$

- A fibre homotopy trivial *TOP* bundle $\eta : S^4 \to BTOP$ has a *PL* lift $\widetilde{\eta} : S^4 \to BPL$ if and only if $p_1(\eta) \equiv 0 (\bmod 48)$.
- The Poincaré homology sphere $\Sigma^3$ is used to construct a non-*PL* homeomorphism $h : \mathbb{R}^n \times S^3 \to \mathbb{R}^n \times S^3$ $(n \geqslant 4)$ with $ph = p : \mathbb{R}^n \times S^3 \to S^3$. The $TOP(n)$-bundle $\eta : S^4 \to BTOP(n)$ with clutching function $h$ is fibre homotopy trivial but does not have a *PL* lift, with

$$p_1(\eta) = 24 \in \mathbb{Z} \;,\; \kappa(\eta) = 1 \in \mathbb{Z}_2 \;.$$

## PL structures on TOP manifolds

- The *PL* structure obstruction of a compact $n$-dimensional *TOP* manifold $M$

  $$\kappa(M) \in [M, B(TOP/PL)] \;=\; H^4(M;\mathbb{Z}_2)$$

  is the *PL* lifting obstruction of the stable tangent bundle $\tau_M$

  $$\kappa(M) : M \xrightarrow{\tau_M} BTOP \xrightarrow{\kappa} B(TOP/PL) \simeq K(\mathbb{Z}_2, 4) \ .$$

  For $n \geqslant 5$ $\kappa(M) = 0$ if and only if $M$ has a *PL* structure. (K.-S. 1969)
- If $n \geqslant 5$ and $\kappa(M) = 0$ the *PL* structures on $M$ are in bijective correspondence with $[M, TOP/PL] = H^3(M;\mathbb{Z}_2)$.
- For each $n \geqslant 4$ there exist compact $n$-dimensional *TOP* manifolds $M$ with $\kappa(M) \neq 0$. Such $M$ do not have a *PL* structure, and are counterexamples to the Combinatorial Triangulation Conjecture.
  - All known counterexamples for $n \geqslant 5$ can be triangulated.

## The triangulation obstruction

- Rochlin invariant map $\alpha$ fits into short exact sequence

$$0 \longrightarrow \ker(\alpha) \longrightarrow \theta_3^H \xrightarrow{\alpha} \mathbb{Z}_2 \longrightarrow 0$$

  with $\theta_3^H$ the cobordism group of oriented 3-dimensional *PL* homology spheres.
  - $\ker(\alpha)$ is infinitely generated (Fintushel-Stern 1990, using Donaldson, 1982).
- (Galewski-Stern, Matumoto, 1976)
  The triangulation obstruction of a compact *n*-dimensional *TOP* manifold *M* is

$$\delta\kappa(M) \in H^5(M; \ker(\alpha))$$

  with $\delta : H^4(M; \mathbb{Z}_2) \to H^5(M; \ker(\alpha))$ the Bockstein. For $n \geqslant 5$ *M* can be triangulated if and only if $\delta\kappa(M) = 0$.
  - Still unknown if $\delta\kappa(M)$ can be non-zero for $M^n$ with $n \geqslant 5$!
  - $M^4$ with $\kappa(M) \neq 0$ cannot be triangulated (Casson, 1985). E.g. the 4-dim. Freedman $E_8$-manifold cannot be triangulated.

# The handle straightening obstruction

- A homeomorphism $h : M \to N$ of compact $n$-dimensional $PL$ manifolds has a handle straightening obstruction

$$\kappa(h) \ = \ \tau_M - h^*\tau_N \in [M, TOP/PL] \ = \ H^3(M;\mathbb{Z}_2) \ .$$

  For $n \geqslant 5$ $\kappa(h) = 0$ if and only if $h$ is isotopic to a $PL$ homeomorphism (K.-S., 1969).

- The mapping cylinder of $h$ is a $TOP$ manifold $W$ with a $PL$ structure on boundary $\partial W = M \cup N$, such that $W$ is homeomorphic to $M \times [0,1]$. The handle straightening obstruction is the rel $\partial$ $PL$ structure obstruction

$$\kappa(h) \ = \ \kappa_\partial(W) \in H^4(W, \partial W;\mathbb{Z}_2) = H^3(M;\mathbb{Z}_2).$$

- For each $n \geqslant 5$ every element $\kappa \in H^3(M;\mathbb{Z}_2)$ is $\kappa = \kappa(h)$ for a homeomorphism $h : M \to N$.

**TOP transversality**

- Theorem (K.-S. 1970, Rourke-Sanderson 1970, Marin, 1977) Let $(X, Y \subset X)$ be a pair of spaces such that $Y$ has a $TOP$ $k$-bundle neighbourhood

$$\nu_{Y \subset X} \ : \ Y \to BTOP(k) \ .$$

For $n - k \neq 4$, every map $f : M \to X$ from a compact $n$-dimensional $TOP$ manifold $M$ is homotopic to a map $g : M \to X$ which is $TOP$ transverse at $Y \subset X$, meaning that

$$N^{n-k} \ = \ f^{-1}(Y) \subset M^n$$

is a codimension $k$ $TOP$ submanifold with normal $TOP$ $k$-bundle

$$\nu_{N \subset M} \ = \ f^* \nu_{Y \subset X} \ : \ N \to BTOP(k)$$

  - Also for $n - k = 4$ (Quinn, 1988)
- $TOP$ analogue of Sard-Thom transversality for $DIFF$ and $PL$, but much harder to prove.

## TOP handlebodies

- Theorem (K.-S. 1970)
  For $n \geqslant 6$ every compact $n$-dimensional $TOP$ manifold $M^n$ has a handlebody decomposition
  $$M = \bigcup h^0 \cup \bigcup h^1 \cup \cdots \cup \bigcup h^n$$
  with every $i$-handle $h^i = D^i \times D^{n-i}$ attached to lower handles at
  $$\partial_+ h^i = S^{i-1} \times D^{n-i} \subset h^i .$$
  - In particular, $M$ is a finite $CW$ complex.
- $TOP$ analogue of handlebody decomposition for $DIFF$ and $PL$, but much harder to prove.
- There is also a $TOP$ analogue of Morse theory for $DIFF$ and $PL$.

**The TOP h- and s-cobordism theorems**

- An *h-cobordism* is a cobordism $(W; M, N)$ such that the inclusions $M \hookrightarrow W$, $N \hookrightarrow W$ are homotopy equivalences.
- *TOP h- and s-cobordism theorems* (K.-S. 1970). For $n \geqslant 5$ an $(n+1)$-dimensional *TOP h*-cobordism $(W^{n+1}; M, N)$ is homeomorphic to $M \times ([0,1]; \{0\}, \{1\})$ rel $M$ if and only if it is an *s*-cobordism, i.e. the Whitehead torsion is

$$\tau(M \simeq W) \ = \ 0 \in Wh(\pi_1(M)) \ .$$

  - If $\tau = 0$ the composite homotopy equivalence

$$M \xrightarrow{\simeq} W \xrightarrow{\simeq} N$$

    is homotopic to a homeomorphism.
- Generalization of the *DIFF* and *PL* cases originally due to Smale, 1962 and Barden-Mazur-Stallings, 1964.

## Why are TOP manifolds harder than DIFF and PL manifolds?

- For *CAT* = *DIFF* and *PL* the structure theory of *CAT* manifolds can be developed working entirely in *CAT* to obtain transversality and handlebody decompositions.
  - Need $n \geqslant 5$ for Whitney trick for removing double points.
  - But do not need sophisticated algebraic computation beyond

    $$Wh(1) = 0$$

    required for the combinatorial invariance of Whitehead torsion.
- The high-dimensional *TOP* manifold structure theory cannot be developed just in the *TOP* category!
  - The *TOP* theory also needs the *PL* surgery classification of the homotopy types of the tori $T^n$ for $n \geqslant 5$ which depends on the Bass-Heller-Swan (1964) computation

    $$Wh(\mathbb{Z}^n) = 0$$

    or some controlled *K*- or *L*-theory analogue.

## Why are TOP manifolds easier than DIFF and PL manifolds?

- Topological manifolds bear the simplest possible relation to their underlying homotopy types. This is a broad statement worth testing.

  L.C.Siebenmann (Nice ICM article, 1970)

- (R., 1992) The homotopy types of high-dimensional *TOP* manifolds are in one-one correspondence with the homotopy types of Poincaré duality spaces with some additional chain level quadratic structure.
- Homeomorphisms correspond to homotopy equivalences preserving the additional structure.

## Poincaré duality spaces

- An *n-dimensional Poincaré duality space* $X$ is a space with the simple homotopy type of a finite $CW$ complex, and a fundamental class $[X] \in H_n(X)$ such that cap product defines a simple chain equivalence

$$[X] \cap - \ : \ C(X)^{n-*} \xrightarrow{\simeq} C(X)$$

  inducing duality isomorphisms $[X] \cap - : H^{n-*}(X) \cong H_*(X)$ with arbitrary $\mathbb{Z}[\pi_1(X)]$-module coefficients.
- A compact $n$-dimensional $TOP$ manifold is an $n$-dimensional Poincaré space (K.-S., 1970).
- Any space homotopy equivalent to a Poincaré duality space is again a Poincaré duality space.
- There exist $n$-dimensional Poincaré duality spaces which are not homotopy equivalent to compact $n$-dimensional $TOP$ manifolds (Gitler-Stasheff, 1965 and Wall, 1967 for $PL$, K.-S. 1970 for $TOP$)

## The CAT manifold structure set

- Let $CAT = DIFF$, $PL$ or $TOP$.
- The *CAT* structure set $\mathbb{S}^{CAT}(X)$ of an $n$-dimensional Poincaré duality space $X$ is the set of equivalence classes of pairs $(M, f)$ with $M$ a compact $n$-dimensional $CAT$ manifold and $f : M \to X$ a homotopy equivalence, with
  - $(M, f) \sim (M', f')$ if there exists a $CAT$ isomorphism $h : M \to M'$ with a homotopy $f \simeq f'h : M \to X$.
- Fundamental problem of surgery theory:
  decide if $\mathbb{S}^{CAT}(X)$ is non-empty, and if so compute it by algebraic topology.
- This can be done for $n \geqslant 5$, allowing the systematic construction and classification of $CAT$ manifolds and homotopy equivalences using algebra.

## The Spivak normal fibration

- A spherical fibration $\eta$ over a space $X$ is a fibration
$$S^{k-1} \to S(\eta) \to X$$
e.g. the sphere bundle of a $k$-plane vector or $TOP$ bundle.
- Classifying spaces $BG(k)$, $BG = \varinjlim_k BG(k)$ with homotopy groups the stable homotopy groups of spheres
$$\pi_n(BG) \ = \ \pi^S_{n-1} \ = \ \varinjlim_k \pi_{n+k-1}(S^k)$$
- The Spivak normal fibration $\nu_X : X \to BG$ of an $n$-dimensional Poincaré duality space $X$ is
$$S^{k-1} \to S(\nu_X) = \partial W \to W \simeq X$$
$(W, \partial W)$ regular neigbhd. of embedding $X \subset S^{n+k}$ ($k$ large).
- If $M$ is a $CAT$ manifold the Spivak normal fibration $\nu_M : M \to BG$ lifts to the $BCAT$ stable normal bundle $\nu_M^{CAT} : M \to BCAT$ of an embedding $M \subset S^{n+k}$ ($k$ large).

## Surgery obstruction theory

- Wall (1969) defined the algebraic $L$-groups $L_n(A)$ of a ring with involution $A$. Abelian Grothendieck-Witt groups of quadratic forms on based f.g. free $A$-modules and their automorphisms. 4-periodic: $L_n(A) = L_{n+4}(A)$.
- Let $CAT = DIFF$, $PL$ or $TOP$. A $CAT$ normal map $f : M \to X$ from a compact $n$-dimensional $CAT$ manifold $M$ to an $n$-dimensional Poincaré duality space $X$ has $f_*[M] = [X] \in H_n(X)$ and $\nu_M \simeq f^*\nu_X^{CAT} : M \to BCAT$ for a $CAT$ lift $\nu_X^{CAT} : X \to BCAT$ of $\nu_X : X \to BG$.
- The surgery obstruction of a $CAT$ normal map $f$

$$\sigma_*(f) \in L_n(\mathbb{Z}[\pi_1(X)])$$

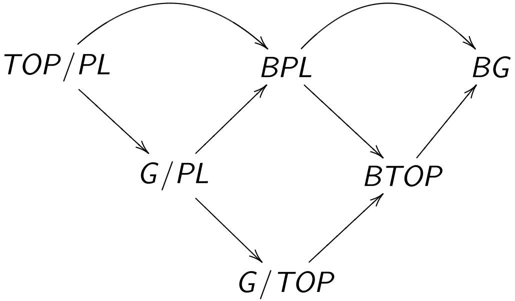
  is such that for $n \geqslant 5$ $\sigma_*(f) = 0$ if and only if $f$ is $CAT$ normal bordant to a homotopy equivalence.
    - Same obstruction groups in each $CAT$.
    - Also a rel $\partial$ version, with homotopy equivalences on the boundaries.

## The surgery theory construction of homotopy equivalences of manifolds from quadratic forms

- Theorem (Wall, 1969, for $CAT = DIFF, PL$, after K.-S. also for $TOP$).
  For an $n$-dimensional $CAT$ manifold $M$ with $n \geqslant 5$ every element $x \in L_{n+1}(\mathbb{Z}[\pi_1(M)])$ is realized as the rel $\partial$ surgery obstruction $x = \sigma_*(f)$ of a $CAT$ normal map
  $$(f;1,h) : (W;M,N) \to M \times ([0,1];\{0\},\{1\})$$
  with $h : N \to M$ a homotopy equivalence.
- Build $W$ by attaching middle-dimensional handles to $M \times I$
  $$W^{n+1} = \begin{cases} M \times [0,1] \cup \bigcup h^i & \text{if } n+1 = 2i \\ M \times [0,1] \cup \bigcup h^i \cup \bigcup h^{i+1} & \text{if } n+1 = 2i+1 \end{cases}$$
  using $x$ to determine the intersections and self-intersections.
- Interesting quadratic forms $x$ lead to interesting homotopy equivalences $h : N \to M$ of $CAT$ manifolds!

## G/PL and G/TOP

- The classifying spaces $BPL$, $BTOP$, $BG$ for $PL$, $TOP$ bundles and spherical fibrations fit into a braid of fibrations




- $G/CAT$ classifies $CAT$ bundles with fibre homotopy trivialization.
- If $X$ is a Poincaré duality space with $CAT$ lift of $\nu_X$ then $[X, G/CAT]$ = the set of cobordism classes of $CAT$ normal maps $f : M \to X$. Abelian group $\pi_n(G/CAT)$ for $X = S^n$.

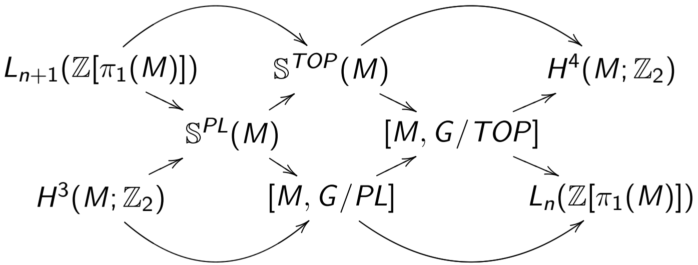
## The surgery exact sequence

Theorem (B.-N.-S.-W. for $CAT = DIFF$, $PL$, K.-S. for $TOP$)
Let $X$ be an $n$-dimensional Poincaré duality space, $n \geqslant 5$.

- $X$ is homotopy equivalent to a compact $n$-dimensional $CAT$ manifold if and only if there exists a lift of $\nu_X : X \to BG$ to $\widetilde{\nu}_X : X \to BCAT$ for which the corresponding $CAT$ normal map $f : M \to X$ with $\nu_M^{CAT} = f^*\widetilde{\nu}_X : M \to BCAT$ has surgery obstruction
$$\sigma_*(f) \;=\; 0 \in L_n(\mathbb{Z}[\pi_1(X)]) \; .$$
- If $X$ is a $CAT$ manifold the structure set $\mathbb{S}^{CAT}(X)$ fits into the $CAT$ surgery exact sequence of pointed sets
$$\cdots \to L_{n+1}(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}^{CAT}(X) \to [X, G/CAT] \to L_n(\mathbb{Z}[\pi_1(X)]) \; .$$

## The Manifold Hauptvermutung from the surgery point of view

- The *TOP* and *PL* surgery exact sequences of a compact $n$-dimensional *PL* manifold $M$ ($n \geqslant 5$) interlock in a braid of exact sequences of abelian groups




- A homeomorphism $h : M \to N$ is homotopic to a *PL* homeomorphism if and only if $\kappa(h) \in \ker(H^3(M;\mathbb{Z}_2) \to \mathbb{S}^{PL}(M))$.
- $[\kappa(h)] \in [M, G/PL]$ is the Hauptvermutung obstruction of Casson and Sullivan (1966-7) - complete for $\pi_1(M) = \{1\}$.

## Why is the TOP surgery exact sequence better than the DIFF and PL sequences?

- Because it has an algebraic model (R., 1992)!
- For 'any space' $X$ can define the algebraic surgery exact sequence of cobordism groups of quadratic Poincaré complexes

$$\dots \longrightarrow L_{n+1}(\mathbb{Z}[\pi_1(X)]) \longrightarrow \mathbb{S}_{n+1}(X) \longrightarrow$$

$$H_n(X;\mathbb{L}(\mathbb{Z})) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \longrightarrow \mathbb{S}_n(X) \longrightarrow \dots$$

  with $\mathbb{L}(\mathbb{Z})$ a 1-connective spectrum of quadratic forms over $\mathbb{Z}$, and $A$ the assembly map from the local generalized $\mathbb{L}(\mathbb{Z})$-coefficient homology of $X$ to the global $L$-theory of $\mathbb{Z}[\pi_1(X)]$.
- $\pi_*(\mathbb{L}(\mathbb{Z})) = L_*(\mathbb{Z})$ and $\mathbb{S}_*(\{\text{pt.}\}) = 0$.

## Quadratic Poincaré complexes

- An $n$-dimensional quadratic Poincaré complex $C$ over a ring with involution $A$ is an $A$-module chain complex $C$ with a chain equivalence $\psi : C^{n-*} = \mathrm{Hom}_A(C, A)_{*-n} \simeq C$.
- $L_n(A)$ is the cobordism group of $n$-dimensional quadratic Poincaré complexes $C$ of based f.g. free $A$-modules with Whitehead torsion $\tau(\psi) = 0 \in \widetilde{K}_1(A)$.
- $H_n(X; \mathbb{L}(\mathbb{Z}))$ is the cobordism group of 'sheaves' $C$ over $X$ of $n$-dimensional quadratic Poincaré complexes over $\mathbb{Z}$, with Verdier-type duality. Assembly $A(C) = q_! p^! C$ with $p : \widetilde{X} \to X$ the universal cover projection, $q : \widetilde{X} \to \{\text{pt.}\}$.
- $\mathbb{S}_{n+1}(X)$ is the cobordism group of sheaves $C$ over $X$ of $n$-dimensional quadratic Poincaré complexes over $\mathbb{Z}$ with the assembly $A(C)$ a contractible quadratic Poincaré complex over $\mathbb{Z}[\pi_1(X)]$.

**The total surgery obstruction of a Poincaré duality space**

- An $n$-dim. P. duality space $X$ has a total surgery obstruction

$$s(X) = C \in \mathbb{S}_n(X)$$

  such that for $n \geqslant 5$ $s(X) = 0$ if and only if $X$ is homotopy equivalent to a compact $n$-dimensional $TOP$ manifold.
- The stalks $C(x)$ $(x \in X)$ of $C$ are quadratic Poincaré complexes over $\mathbb{Z}$ measuring the failure of $X$ to be an $n$-dimensional homology manifold, with exact sequences

$$\cdots \to H_r(C(x)) \to H^{n-r}(\{x\}) \to H_r(X, X - \{x\}) \to \dots \ .$$

  $s(X) = 0$ if and only if stalks are coherently null-cobordant.
- For $n \geqslant 5$ the difference between the homotopy types of $n$-dimensional $TOP$ manifolds and Poincaré duality spaces is measured by the failure of the functor

$$\{\text{spaces}\} \to \{\mathbb{Z}_4\text{-graded abelian groups}\} \ ; \ X \mapsto L_*(\mathbb{Z}[\pi_1(X)])$$

  to be a generalized homology theory.

## The total surgery obstruction of a homotopy equivalence of manifolds

- A homotopy equivalence $f : N \to M$ of compact $n$-dimensional $TOP$ manifolds has a total surgery obstruction
$$s(f) \;=\; C \in \mathbb{S}_{n+1}(M)$$
such that for $n \geqslant 5$ $s(f) = 0$ if and only if $f$ is homotopic to a homeomorphism.
- The stalks $C(x)$ $(x \in M)$ of $C$ are quadratic Poincaré complexes over $\mathbb{Z}$ measuring the failure of $f$ to have acyclic point inverses $f^{-1}(x)$, with exact sequences
$$\dots \to H_r(C(x)) \to H_r(f^{-1}\{x\}) \to H_r(\{x\}) \to \dots \;.$$
$s(f) = 0$ if and only if the stalks are coherently null-cobordant.
- Theorem (R., 1992) The $TOP$ surgery exact sequence is isomorphic to the algebraic surgery exact sequence. Bijection
$$\mathbb{S}^{TOP}(M) \xrightarrow{\cong} \mathbb{S}_{n+1}(M)\;;\;(N,f) \mapsto s(f)\;.$$

## Manifolds with boundary

- For an $n$-dimensional $CAT$ manifold with boundary $(X, \partial X)$ let $\mathbb{S}^{CAT}(X, \partial X)$ be the structure set of homotopy equivalences $h : (M, \partial M) \to (X, \partial X)$ with $(M, \partial M)$ a $CAT$ manifold with boundary and $\partial h : \partial M \to \partial X$ a $CAT$ isomorphism.
- For $n \geqslant 5$ rel $\partial$ surgery exact sequence

$$\begin{aligned} \cdots \to L_{n+k+1}(\mathbb{Z}[\pi_1(X)]) &\to \mathbb{S}^{CAT}(X \times D^k, \partial(X \times D^k)) \\ &\to [X \times D^k, \partial; G/CAT, *] \to L_{n+k}(\mathbb{Z}[\pi_1(X)]) \to \\ \cdots \to L_{n+1}(\mathbb{Z}[\pi_1(X)]) &\to \mathbb{S}^{CAT}(X, \partial X) \to [X, \partial X; G/CAT, *] \\ &\to L_n(\mathbb{Z}[\pi_1(X)]) \ . \end{aligned}$$

- $TOP$ case isomorphic to algebraic surgery exact sequence. Bijections $\mathbb{S}^{TOP}(X \times D^k, \partial(X \times D^k)) \ \cong \ \mathbb{S}_{n+k+1}(X)$ $(k \geqslant 0)$.

## The algebraic L-groups of Z

- (Kervaire-Milnor, 1963) The $L$-groups of $\mathbb{Z}$ are given by

$$L_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ (signature } \sigma/8) \\ 0 \\ \mathbb{Z}_2 \text{ (Arf invariant)} \\ 0 \end{cases} \text{for } n \equiv \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \pmod 4$$

- Define the *PL L-groups* of $\mathbb{Z}$ by

$$\widetilde{L}_n(\mathbb{Z}) = \begin{cases} L_n(\mathbb{Z}) & \text{for } n \neq 4 \\ \{\sigma \in L_4(\mathbb{Z}) | \sigma \equiv 0 (\bmod 16)\} & \text{for } n = 4 \end{cases}$$

as in Rochlin's theorem, with

$$L_4(\mathbb{Z})/\widetilde{L}_4(\mathbb{Z}) = \mathbb{Z}_2 .$$

## Spheres

- Generalized Poincaré Conjecture For $n \geqslant 4$ a compact $n$-dimensional $TOP$ manifold $M^n$ homotopy equivalent to $S^n$ is homeomorphic to $S^n$.
  - For $n \geqslant 5$: Smale (1960, *DIFF*), Stallings (1961, *PL*), Newman (1962, *TOP*).
  - For $n = 4$: Freedman (1982, *TOP*).
- For $n + k \geqslant 4$

$$\mathbb{S}^{TOP}(S^n \times D^k, \partial) \;=\; \mathbb{S}_{n+k+1}(S^n) \;=\; 0 \;.$$

- $\pi_n(G/PL) = \widetilde{L}_n(\mathbb{Z})$ (Sullivan, 1967)
- $\pi_n(G/TOP) = L_n(\mathbb{Z})$ (K.-S., 1970), so

$$\mathbb{L}_0(\mathbb{Z}) \simeq G/TOP \;.$$

## Simply-connected surgery theory

- Theorem (K.-S., 1970) For $n \geqslant 5$ a simply-connected $n$-dimensional Poincaré duality space $X$ is homotopy equivalent to a compact $n$-dimensional $TOP$ manifold if and only if the Spivak normal fibration $\nu_X : X \to BG$ lifts to a $TOP$ bundle $\widetilde{\nu}_X : X \to BTOP$.
  - $TOP$ version of original $DIFF$ theorem of Browder, 1962.
  - Also true for $n = 4$ by Freedman, 1982.
- Corollary For $n \geqslant 5$ a homotopy equivalence of simply-connected compact $n$-dimensional $TOP$ manifolds $h : M \to N$ is homotopic to a homeomorphism if and only if a canonical homotopy

$$g \ : \ h^*\nu_N \simeq \nu_M \ : \ M \to BG$$

lifts to a homotopy

$$\widetilde{g} \ : \ h^*\nu_N^{TOP} \simeq \nu_M^{TOP} \ : \ M \to BTOP \ .$$

## Products of spheres

- For $m, n \geqslant 2$, $m+n \geqslant 5$

$$\begin{aligned} \mathbb{S}^{PL}(S^m \times S^n) &= \widetilde{L}_m(\mathbb{Z}) \oplus \widetilde{L}_n(\mathbb{Z}) \\ \mathbb{S}^{TOP}(S^m \times S^n) &= \mathbb{S}_{m+n+1}(S^m \times S^n) = L_m(\mathbb{Z}) \oplus L_n(\mathbb{Z}) \end{aligned}$$

- For $CAT = PL$ and $TOP$ there exist homotopy equivalences $M^{m+n} \simeq S^m \times S^n$ of $CAT$ manifolds which are not $CAT$ split, and so not homotopic to $CAT$ isomorphisms. For $CAT = PL$ these are counterexamples to the Manifold Hauptvermutung.
- There exist compact $TOP$ manifolds $M^{m+4}$ which are homotopy equivalent to $S^m \times S^4$, but do not have a $PL$ structure. Counterexamples to Combinatorial Triangulation Conjecture.

## TOP/PL and homotopy structures

- A map $h : S^k \to TOP(n)/PL(n)$ is represented by a homeomorphism $h : \mathbb{R}^n \times D^k \to \mathbb{R}^n \times D^k$ such that $ph = p : \mathbb{R}^n \times D^k \to D^k$ and which is a $PL$ homeomorphism on $\mathbb{R}^n \times S^{k-1}$. For $n + k \geqslant 6$ can wrap up $h$ to a homeomorphism $\overline{h} : M^{n+k} \to T^n \times D^k$ with $(M, \partial M)$ $PL$, such that $\partial\overline{h} : \partial M \to T^n \times S^{k-1}$ is a $PL$ homeomorphism.
- Theorem (K.-S., 1970) For $1 \leqslant k < n$, $n \geqslant 5$ the wrapping up

$$\pi_k(TOP(n)/PL(n)) \to \mathbb{S}^{PL}(T^n \times D^k, \partial) \; ; \; h \mapsto \overline{h}$$

is injective with image the subset

$$\mathbb{S}^{PL}_*(T^n \times D^k, \partial) \subseteq \mathbb{S}^{PL}(T^n \times D^k, \partial)$$

invariant under transfers for finite covers $T^n \to T^n$, and

$$\pi_k(TOP(n)/PL(n)) \cong \pi_k(TOP/PL) \; .$$

- Key: approximate homeomorphism $h : \mathbb{R}^n \times D^k \to \mathbb{R}^n \times D^k$ by homotopy equivalence $\overline{h} : M^{n+k} \to T^n \times D^k$.

**The algebraic L-groups of polynomial extensions**

- Theorem (Wall, Shaneson 1969 geometrically for $A = \mathbb{Z}[\pi]$, Novikov, R. 1970– algebraically)
  For any ring with involution $A$

$$L_m(A[z, z^{-1}]) \;=\; L_m(A) \oplus L^h_{m-1}(A)$$

  with $L^h_*$ defined just like $L_*$ but ignoring Whitehead torsion.
- Inductive computation

$$L_m(\mathbb{Z}[\mathbb{Z}^n]) \;=\; \sum_{i=0}^{n} \binom{n}{i} L_{m-i}(\mathbb{Z})$$

  for any $n \geqslant 1$, using

$$\begin{aligned} \mathbb{Z}[\mathbb{Z}^n] \;&=\; \mathbb{Z}[\mathbb{Z}^{n-1}][z_n, z_n^{-1}] \\ &=\; \mathbb{Z}[z_1, z_1^{-1}, z_2, z_2^{-1}, \dots, z_n, z_n^{-1}] \end{aligned}$$

  and the Bass-Heller-Swan computation $Wh(\mathbb{Z}^n) = 0$.

## Tori

- Theorem (Wall, Hsiang, Shaneson 1969)

$$[T^n \times D^k, \partial; G/PL, *] \;=\; \sum_{i=0}^{n-1} \binom{n}{i} \widetilde{L}_{n+k-i}(\mathbb{Z})$$
$$\subset L_{n+k}(\mathbb{Z}[\mathbb{Z}^n]) \;=\; \sum_{i=0}^{n} \binom{n}{i} L_{n+k-i}(\mathbb{Z}) \;\; (n, k \geqslant 0) \; ,$$
$$\mathbb{S}^{PL}(T^n \times D^k, \partial) = H^{3-k}(T^n; \mathbb{Z}_2) = \binom{n}{n+k-3} \mathbb{Z}_2 \; (n + k \geqslant 5)$$

- Corollary (K.-S., 1970) For $k < n$ and $n \geqslant 5$

$$\pi_k(TOP/PL) \;=\; \mathbb{S}_*^{PL}(T^n \times D^k, \partial) \;=\; \begin{cases} \mathbb{Z}_2 & \text{if } k = 3 \\ 0 & \text{if } k \neq 3 \end{cases}$$

  so that $TOP/PL \simeq K(\mathbb{Z}_2, 3)$.
- Need $\mathbb{S}_*^{PL}(T^n \times D^k, \partial) = 0$ $(k \neq 3)$ for handle straightening.
- $\mathbb{S}^{TOP}(T^n \times D^k, \partial) = \mathbb{S}_{n+k+1}(T^n) = 0$ for $n + k \geqslant 5$.

## A counterexample to the Manifold Hauptvermutung from the surgery theory point of view

- The morphism

$$L_{n+1}(\mathbb{Z}[\mathbb{Z}^n]) = [T^n \times D^1, \partial; G/TOP, *] \\ \to [T^n, TOP/PL] = H^3(T^n; \mathbb{Z}_2)$$

  is onto, so for any $x \neq 0 \in H^3(T^n; \mathbb{Z}_2)$ there exists an element $y \in L_{n+1}(\mathbb{Z}[\mathbb{Z}^n])$ with $[y] = x$. For $n \geqslant 5$ realize $y = \sigma_*(f)$ as the rel $\partial$ surgery obstruction of a $PL$ normal map

$$(f; 1, g) : (W^{n+1}; T^n, \tau^n) \to T^n \times (I; \{0\}, \{1\})$$

  with $g : \tau^n \to T^n$ homotopic to a homeomorphism $h$, and

$$s(g) \ = \ \kappa(h) \ = \ x \neq 0 \in \mathbb{S}^{PL}(T^n) = H^3(T^n; \mathbb{Z}_2) \ .$$

- The homotopy equivalence $g$ is not $PL$ split at $T^3 \subset T^n$ with $\langle x, [T^3] \rangle = 1 \in \mathbb{Z}_2$, since $g^{-1}(T^3) = T^3 \# \Sigma^3$ with $\Sigma^3 =$ Poincaré homology sphere with Rochlin invariant $\alpha(\Sigma^3) = 1$. $g$ is not homotopic to a $PL$ homeomorphism.

## A counterexample to the Combinatorial Triangulation Conjecture from the surgery theory point of view

- For $x \neq 0 \in H^3(T^n;\mathbb{Z}_2)$, $y \in L_{n+1}(\mathbb{Z}[\mathbb{Z}^n])$, $n \geqslant 5$ use the *PL* normal map $(f;1,g):(W^{n+1};T^n,\tau^n) \to T^n \times (I;\{0\},\{1\})$ with $g:\tau^n \to T^n$ homotopic to a homeomorphism $h$ to define a compact $(n+1)$-dimensional *TOP* manifold

$$M^{n+1} \ = \ W/\{x \sim h(x)\,|\, x \in \tau^n\}$$

  with a *TOP* normal map $F:M \to T^{n+1}$ such that

  $\sigma_*(F) = (y,0) \in L_{n+1}(\mathbb{Z}[\mathbb{Z}^{n+1}]) = L_{n+1}(\mathbb{Z}[\mathbb{Z}^n]) \oplus L_n(\mathbb{Z}[\mathbb{Z}^n])$ .

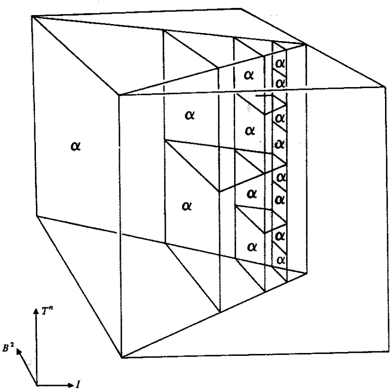
- The combinatorial triangulation obstruction of $M$ is

$$\kappa(M) = \delta(x) \neq 0 \in \mathrm{im}(\delta : H^3(T^n;\mathbb{Z}_2) \to H^4(M;\mathbb{Z}_2)) \ .$$

  $\nu_M^{TOP}:M \to BTOP$ does not have a *PL* lift, so $M$ does not have a *PL* structure, and is not homotopy equivalent to a compact $(n+1)$-dimensional *PL* manifold.

## The original counterexample to the Manifold Hauptvermutung and the Combinatorial Triangulation Conjecture

Elementary construction in Siebenmann's 1970 ICM paper:



## Some applications of TOP surgery theory for finite fundamental groups

- The surgery obstruction groups $L_*(\mathbb{Z}[\pi])$ have been computed for many finite groups $\pi$ using algebraic number theory and representation theory, starting with Wall (1970–).
- Solution of the topological space form problem:
  The classification of free actions of finite groups on spheres. (Madsen, Thomas, Wall 1977)
- Solution of the deRham problem:
  The topological classification of linear representations of cyclic groups. (Cappell-Shaneson, 1981, Hambleton-Pedersen, 2005)

## The Novikov Conjecture

- The higher signatures of a compact oriented $n$-dimensional $TOP$ manifold $M$ with fundamental group $\pi_1(M) = \pi$ are

$$\sigma_x(M) = \langle \mathcal{L}(M) \cup f^*(x), [M] \rangle \in \mathbb{Q}$$

  with $x \in H^{n-4*}(K(\pi,1);\mathbb{Q})$, $f : M \to K(\pi,1)$ a classifying map for the universal cover.
- Conjecture (N., 1969) The higher signatures are homotopy invariant, that is

$$\sigma_x(M) \ = \ \sigma_x(N) \in \mathbb{Q}$$

  for any homotopy equivalence $h : M \to N$ of $TOP$ manifolds and any $x \in H^{n-4*}(K(\pi,1);\mathbb{Q})$.
- Equivalent to the rational injectivity of the assembly map $A : H_*(K(\pi,1);\mathbb{L}(\mathbb{Z})) \to L_*(\mathbb{Z}[\pi])$. Trivial for finite $\pi$.
- Solved for a large class of infinite groups $\pi$, using algebra, topology, differential geometry and analysis ($C^*$-algebra methods).

## The Borel Conjecture

- A topological space $X$ is aspherical if $\pi_i(X) = 0$ for $i \geqslant 2$, or equivalently $X \simeq K(\pi, 1)$ with $\pi = \pi_1(X)$. If $X$ is a Poincaré duality space then $\pi$ is infinite torsionfree.
- Borel Conjecture Every aspherical $n$-dimensional Poincaré duality space $X$ is homotopy equivalent to a compact $n$-dimensional $TOP$ manifold, with homotopy rigidity
$$\mathbb{S}^{TOP}(X \times D^k, \partial) = 0 \text{ for } k \geqslant 0 .$$
- For $n \geqslant 5$ the Conjecture is equivalent to the assembly map $A : H_*(X; \mathbb{L}(\mathbb{Z})) \to L_*(\mathbb{Z}[\pi])$ being an isomorphism for $* \geqslant n+1$, and $s(X) = 0 \in \mathbb{S}_n(X) = \mathbb{Z}$.
- Many positive results on the Borel Conjecture starting with $X = T^n$, $\pi = \mathbb{Z}^n$, and the closely related Novikov Conjecture (especially Farrell-Jones, 1986–). Solutions use K.-S. $TOP$ manifold structure theorems, controlled algebra and differential geometry.

## Controlled algebra/topology

- The development of high-dimensional *TOP* manifolds since 1970 has centred on the applications of a mixture of algebra and topology, called controlled algebra, in which the size of permitted algebraic operations is measured in a control (metric) space.
- For example, homeomorphisms of *TOP* manifolds can be approximated by bounded/controlled homotopy equivalences. Also, there are bounded/controlled analogues for homeomorphisms of the Whitehead and Hurewicz theorems for recognizing homotopy equivalences as maps inducing isomorphisms in the homotopy and homology groups.
- Key ingredient: codimension 1 splitting theorems.

## Approximating homeomorphisms
## I. Homotopy conditions

- A CE map of manifolds $f : M \to N$ is a map such that the point-inverses

$$f^{-1}(x) \subset M \quad (x \in N)$$

  are contractible, or equivalently
  - $f$ is a homotopy equivalence such that the restrictions

$$f| : f^{-1}(U) \to U \quad (U \subseteq N \text{ open})$$

    are also homotopy equivalences.
- Theorem (Siebenmann, 1972) For $n \geqslant 5$ a map $f : M \to N$ of $n$-dimensional $TOP$ manifolds is $CE$ if and only if $f$ is a limit of homeomorphisms.

## Approximating homeomorphisms
## II. Topological conditions

- The tracks of a homotopy $h : f_0 \simeq f_1 : X \to Y$ are the paths

$$[0,1] \to Y \ ; \ t \mapsto h(x,t) \ (x \in X)$$

  from $h(x,0) = f_0(x)$ to $h(x,1) = f_1(x)$.
- If $\alpha$ is an open cover of a space $N$ then a map $f : M \to N$ is an $\alpha$-equivalence if there exist a homotopy inverse $g : N \to M$ and homotopies $gf \simeq 1 : M \to M$, $fg \simeq 1 : N \to N$ with each track contained in some $U \in \alpha$.
- Theorem (Chapman, Ferry, 1979) If $n \geqslant 5$ and $N^n$ is a $TOP$ manifold, then for any open cover $\alpha$ of $N$ there exists an open cover $\beta$ of $N$ such that any $\beta$-equivalence is $\alpha$-close to a homeomorphism.

## Approximating homeomorphisms
## III. Metric conditions

- For $\delta > 0$ a $\delta$-map $f : M \to N$ of metric spaces is a map such that for every $x \in N$

$$\text{diameter}(f^{-1}(x)) < \delta \ .$$

- Theorem (Ferry, 1979) If $n \geqslant 5$ and $N^n$ is a *TOP* manifold, then for any $\epsilon > 0$ there exists $\delta < \epsilon$ such that any surjective $\delta$-map $f : M^n \to N^n$ of $n$-dimensional *TOP* manifolds is homotopic through $\epsilon$-maps to a homeomorphism.
  - Squeezing.

## Metric algebra

- $X$ = metric space, with metric $d : X \times X \to \mathbb{R}^+$.
- An $X$-controlled group = a free abelian group $\mathbb{Z}[A]$ with basis $A$ and a labelling function

$$A \to X \ ; \ a \mapsto x_a \ .$$

- A morphism $f = (f(a,b)) : \mathbb{Z}[A] \to \mathbb{Z}[B]$ of $X$-controlled groups is a matrix with entries $f(a,b) \in \mathbb{Z}$ indexed by the basis elements $a \in A$, $b \in B$. The diameter of $f : \mathbb{Z}[A] \to \mathbb{Z}[B]$ is

$$\text{diameter}(f) \ = \ \sup \ d(x_a, x_b) \geqslant 0$$

  with $a \in A$, $b \in B$ such that $f(a,b) \neq 0$.
- For morphisms $f : \mathbb{Z}[A] \to \mathbb{Z}[B]$, $g : \mathbb{Z}[B] \to \mathbb{Z}[C]$

$$\text{diameter}(gf) \leqslant \text{diameter}(f) + \text{diameter}(g)$$

# Controlled algebra

- (Quinn, 1979–) Controlled algebraic $K$- and $L$-theory, with diameter $< \epsilon$ for small $\epsilon > 0$.
- Many applications to high-dimensional $TOP$ manifolds, e.g. controlled $h$-cobordism theorem, mapping cylinder neighbourhoods, stratified sets and group actions.
- (Controlled Hurewicz for homeomorphisms) If $n \geqslant 5$ and $N^n$ is a $TOP$ manifold, then for any $\epsilon > 0$ there exists $\delta < \epsilon$ such that if $f : M^n \to N^n$ induces a $\delta$-epsilon chain equivalence then $f$ is homotopic to a homeomorphism.
- Disadvantage: condition diameter $< \epsilon$ is not functorial, since diameter of composite $< 2\epsilon$. Hard to compute the controlled obstruction groups.

## Recognizing topological manifolds

- Theorem (Edwards, 1978) For $n \neq 4$ the polyhedron $|K|$ of a simplicial complex $K$ is an $n$-dimensional $TOP$ manifold if and only if the links of $\sigma \in K$ are simply-connected and have the homology of $S^{n-|\sigma|-1}$.
- Theorem (Quinn, 1987) For $n \geqslant 5$ a topological space $X$ is an $n$-dimensional $TOP$ manifold if and only if it is an $n$-dimensional $ENR$ homology manifold with the disjoint disc property and 'resolution obstruction'

$$i(X) = 0 \in L_0(\mathbb{Z}) = \mathbb{Z} \ .$$

## Bounded surgery theory

- A morphism $f : A \to B$ of $X$-groups is bounded if

$$\text{diameter}(f) < \infty$$

- (Ferry-Pedersen, 1995–) Algebraic $K$- and $L$-theory of $X$-controlled groups with bounded morphisms.
- Bounded surgery theory is functorial: the composite of bounded morphisms is bounded. Easier to compute bounded than the controlled obstruction groups. Realization of quadratic forms as in the compact theory.
- $\mathbb{R}^n$-bounded surgery simplifies proof of $TOP/PL \simeq K(\mathbb{Z}_2, 3)$, replacing non-simply-connected compact $PL$ manifolds in

$$\pi_k(TOP/PL) \;=\; \mathbb{S}_*^{PL}(T^n \times D^k, \partial) \;\; (k < n,\;\; n \geqslant 5)$$

by simply-connected non-compact $PL$ manifolds in

$$\pi_k(TOP/PL) \;=\; \mathbb{S}^{\mathbb{R}^n-\text{bounded}-PL}(\mathbb{R}^n \times D^k, \partial) \; .$$

## The future

- More accessible proofs of the Kirby-Siebenmann results in dimensions $n \geqslant 5$.
- A grand unified theory of topological manifolds, controlled topology and sheaf theory.
- A proof/disproof of the Triangulation Conjecture.
- The inclusion of the dimensions $n \leqslant 4$ in the big picture.

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