FROM STURM, SYLVESTER, WITT AND WALL TO THE PRESENT DAY

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Introduction

- In 1829 Sturm proved a theorem calculating the number of real roots of a non-zero real polynomial $P(X) \in \mathbb{R}[X]$ in an interval $[a, b] \subset \mathbb{R}$, using the Euclidean algorithm in $\mathbb{R}[X]$ and counting sign changes.
- In 1853 Sylvester interpreted Sturm's theorem using continued fractions and the signature of a tridiagonal quadratic form. In fact, this was the first application of the signature!
- The survey paper of Étienne Ghys and A.R. http://arxiv.org/abs/1512.09258 *Signatures in algebra, topology and dynamics* includes a modern interpretation of the results of Sturm and Sylvester in terms of the "Witt group" of quadratic forms over the function field $\mathbb{R}(X)$.
- History, algebra, topology – and even some number theory!

Jacques Charles François Sturm (1803-1855)

Sturm's problem

- **Major problem in early 19th century** How many real roots of a degree n real polynomial $P(X) \in \mathbb{R}[X]$ are there in an interval $[a, b] \subset \mathbb{R}$?
- Sturm's 1829 formula for the numbers of roots involved the **Sturm sequences**: the remainders and quotients in the Euclidean algorithm (with sign change) in $\mathbb{R}[X]$ for finding the greatest common divisor of $P_0(X) = P(X)$, $P_1(X) = P'(X)$

  $$P_*(X) = (P_0(X), \ldots, P_n(X)) \ , \ Q_*(X) = (Q_1(X), \ldots, Q_n(X))$$

  with $\deg(P_{k+1}(X)) < \deg(P_k(X)) \leqslant n - k$ and

  $$P_{k-1}(X) + P_{k+1}(X) \ = \ P_k(X)Q_k(X) \ (1 \leqslant k \leqslant n) \ .$$

- **Simplifying assumption** $P(X)$ is generic: the real roots of $P_0(X), P_1(X), \ldots, P_n(X)$ are distinct and non-zero, so that $\deg(P_k(X)) = n - k$ and $P_n(X)$ is a non-zero constant.

Variation

- The **variation** of $p = (p_0, p_1, \ldots, p_n) \in (\mathbb{R}\backslash\{0\})^{n+1}$ is the number of sign changes $p_0 \to p_1 \to \cdots \to p_n$, which is expressed in terms of the sign changes $p_{k-1} \to p_k$ by

$$\mathrm{var}(p) \;=\; (n - \sum_{k=1}^{n} \mathrm{sign}(p_k/p_{k-1}))/2 \in \{0, 1, \ldots, n\} \ .$$

- Sturm's root-counting formula involved the variations of the Sturm functions $P_k(X)$ evaluated at 'regular' $x \in \mathbb{R}$.
- Call $x \in \mathbb{R}$ **regular** if $P_k(x) \neq 0$ $(0 \leqslant k \leqslant n-1)$, so that the variation

$$\mathrm{var}(P_*(x)) \;=\; \mathrm{var}(P_0(x), P_1(x), \ldots, P_n(x)) \in \{0, 1, \ldots, n\}$$

is defined.

Sturm's Theorem I.

- **Theorem** (1829) The number of real roots of a generic $P(X) \in \mathbb{R}[X]$ in $[a, b] \subset \mathbb{R}$ for regular $a < b$ is

$$|\{x \in [a,b] \,|\, P(x) = 0 \in \mathbb{R}\}| \;=\; \mathrm{var}(P_*(a)) - \mathrm{var}(P_*(b)) \;.$$

- **Idea of proof** The function

$$f \;:\; [a,b] \to \{0,1,\ldots,n\} \;;\; x \mapsto \mathrm{var}(P_*(a)) - \mathrm{var}(P_*(x))$$

jumps by $\begin{cases} 1 \\ 0 \end{cases}$ at root x of $P_k(X)$ if $k = \begin{cases} 0 \\ 1,2,\ldots,n. \end{cases}$

- For $k = 0$ the jump in f at a root x of $P_0(x)$ is 1, since for y close to x

$$P_0(y)P_1(y) = d/dy(P(y)^2)/2 = \begin{cases} < 0 & \text{if } y < x \\ > 0 & \text{if } y > x \;, \end{cases}$$

$$\mathrm{var}(P_0(y), P_1(y)) = \begin{cases} \mathrm{var}(+,-) = \mathrm{var}(-,+) = 1 & \text{if } y < x \\ \mathrm{var}(+,+) = \mathrm{var}(-,-) = 0 & \text{if } y > x \;. \end{cases}$$

Sturm's Theorem II.

- For $k = 1, 2, \ldots, n$ the jump in f at a root x of $P_k(x)$ is 0.
- $k = n$ trivial, since $P_n(X)$ is non-zero constant.
- For $k = 1, 2, \ldots, n-1$ the numbers $P_{k-1}(x)$, $P_{k+1}(x) \neq 0 \in \mathbb{R}$ have opposite signs since

$$P_{k-1}(x) + P_{k+1}(x) \ = \ P_k(x)Q_k(x) \ = \ 0 \ .$$

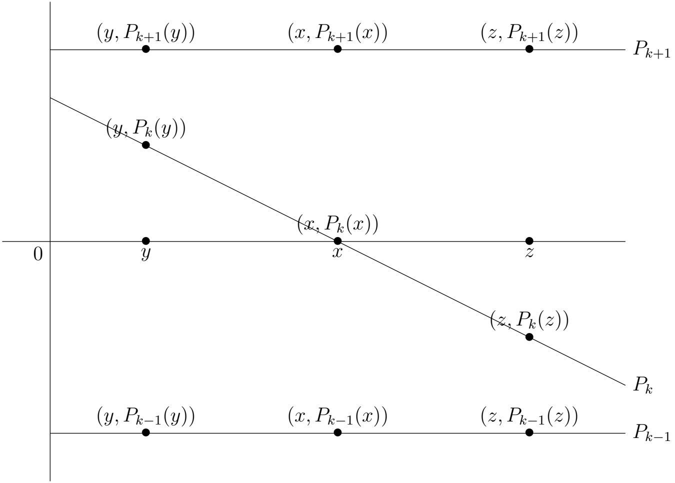
- For y, z close to x with $y < x < z$

$$\begin{aligned}
&\mathrm{sign}(P_{k-1}(y)) = -\mathrm{sign}(P_{k+1}(y)) \\
&\qquad\qquad = \mathrm{sign}(P_{k-1}(z)) = -\mathrm{sign}(P_{k+1}(z)) \ , \\
&\mathrm{var}(P_{k-1}(y), P_k(y), P_{k+1}(y)) \\
&\qquad\qquad = \mathrm{var}(P_{k-1}(z), P_k(z), P_{k+1}(z)) = 1 \ ,
\end{aligned}$$

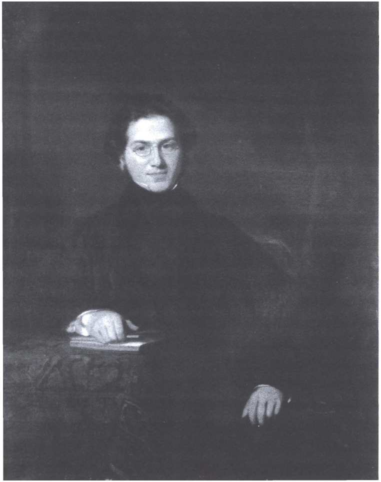
  that is

$$\mathrm{var}(+,+,-) = \mathrm{var}(+,-,-) = \mathrm{var}(-,+,+) = \mathrm{var}(-,-,+) = 1.$$

Sturm's theorem III.



James Joseph Sylvester (1814-1897)



Sylvester's papers related to Sturm's theorem

- *On the relation of Sturm's auxiliary functions to the roots of an algebraic equation.* (1841)
- *A demonstration of the theorem that every homogeneous quadratic polynomial is reducible by real orthogonal substitutions to the form of a sum of positive and negative squares.* (1852)
- *On a remarkable modification of Sturm's Theorem* (1853)
- *On a theory of the syzygetic relations of two rational integral functions, comprising an application to the theory of Sturm's functions, and that of the greatest algebraical common measure.* (1853)

The signature

- **Definition** The **signature** of a symmetric $n \times n$ matrix $S = (s_{ij})_{1 \leqslant i,j \leqslant n}$ is
$$\tau(S) \ = \ \tau_+(S) - \tau_-(S) \in \{-n, -n+1, \dots, n-1, n\}$$
with $\tau_+(S)$ (resp. $\tau_-(S)$) the number of positive (resp. negative) eigenvalues.
- **Law of Inertia** (Sylvester (1852)) For any invertible $n \times n$ matrix $A = (a_{ij})$ with transpose $A^* = (a_{ji})$
$$\tau(A^*SA) \ = \ \tau(S) \ .$$
- **Theorem** (Sylvester (1853), Jacobi (1857), Gundelfinger (1881), Frobenius (1895))
The signature of a symmetric $n \times n$ matrix S in $\mathbb{R}$ with the principal minors $\mu_k = \mu_k(S) = \det(s_{ij})_{1 \leqslant i,j \leqslant k}$ non-zero is
$$\tau(S) \ = \ \sum_{k=1}^{n} \mathrm{sign}(\mu_k/\mu_{k-1}) \ = \ n - 2\,\mathrm{var}(\mu) \ .$$

The tridiagonal symmetric matrix (Jacobi, Sylvester)

- **Definition** The **tridiagonal symmetric matrix** of $q = (q_1, q_2, \ldots, q_n)$ is

$$\mathrm{Tri}(q) \;=\; \begin{pmatrix} q_1 & 1 & 0 & \ldots & 0 \\ 1 & q_2 & 1 & \ldots & 0 \\ 0 & 1 & q_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & q_n \end{pmatrix}$$

- **Tridiagonal Signature Theorem** For $q \in \mathbb{R}^n$ the signature of $\mathrm{Tri}(q)$ is

$$\tau(\mathrm{Tri}(q)) \;=\; \sum_{k=1}^{n} \mathrm{sign}(\mu_k/\mu_{k-1}) \;=\; n - 2\,\mathrm{var}(\mu)$$

assuming $\mu_k = \mu_k(\mathrm{Tri}(q)) = \det(\mathrm{Tri}(q_1, q_2, \ldots, q_k)) \neq 0 \in \mathbb{R}$.

Continued fractions and the Sturm functions

- The **improper continued fraction** of $(q_1, q_2, \ldots, q_n)$ is

$$[q_1, q_2, \ldots, q_n] \;=\; q_1 - \cfrac{1}{q_2 - \ddots - \cfrac{1}{q_n}}$$

  assuming there are no divisions by 0.
- The continued fraction expansion of $P(X)/P'(X)$ is

$$\frac{P(X)}{P'(X)} \;=\; [Q_1(X), Q_2(X), \ldots, Q_n(X)] \in \mathbb{R}(X)$$

  with $Q_1(X), Q_2(X), \ldots, Q_n(X)$ the Sturm quotients.
- The Sturm remainders $(P_0(X), P_1(X), \ldots, P_n(X))$ are the numerators in the **reverse convergents**

$$[Q_{k+1}(X), Q_{k+2}(X), \ldots, Q_n(X)] = \frac{P_k(X)}{P_{k+1}(X)} \in \mathbb{R}(X)\ (0 \leqslant k \leqslant n-1$$

$$P_k(X)/P_n(X) = \det(\mathrm{Tri}(Q_{k+1}(X), Q_{k+2}(X), \ldots, Q_n(X))).$$

Sylvester's Duality Theorem (1853)

- The **convergents** of $[Q_1(X), Q_2(X), \ldots, Q_n(X)] \in \mathbb{R}(X)$ are

$$\begin{aligned} & [Q_1(X), Q_2(X), \ldots, Q_k(X)] \\ & \qquad = \frac{P_k^*(X)}{\det(\mathrm{Tri}(Q_2(X), Q_3(X), \ldots, Q_k(X)))} \end{aligned}$$

with numerators the minors of $\mathrm{Tri}(Q_1(X), Q_2(X), \ldots, Q_n(X))$

$$\begin{aligned} P_k^*(X) &= \mu_k(\mathrm{Tri}(Q_1(X), Q_2(X), \ldots, Q_n(X))) \\ &= \det(\mathrm{Tri}(Q_1(X), Q_2(X), \ldots, Q_k(X))) \in \mathbb{R}[X] . \end{aligned}$$

- **Sylvester's Duality Theorem** Let $x \in \mathbb{R}$ be regular for $P(X)$. The variations of the sequence of the numerators of the convergents and reverse convergents are equal

$$\mathrm{var}(P_0(x), P_1(x), \ldots, P_n(x)) = \mathrm{var}(P_0^*(x), P_1^*(x), \ldots, P_n^*(x)) .$$

Sylvester's reformulation of Sturm's Theorem

- **Theorem** (S.-S.) The number of real roots of $P(X) \in \mathbb{R}[X]$ in an interval $[a, b]$ with regular $a < b$ can be calculated from the signatures of the tridiagonal symmetric matrices $\mathrm{Tri}(Q_*(x))$ for $x = a$ and b

$$\begin{aligned} &\mathrm{var}(P_0(a), P_1(a), \dots, P_n(a)) - \mathrm{var}(P_0(b), P_1(b), \dots, P_n(b)) \\ = \ &\big(\tau(\mathrm{Tri}(Q_*(b))) - \tau(\mathrm{Tri}(Q_*(a)))\big)/2 \in \{0, 1, 2, \dots, n\} \ . \end{aligned}$$

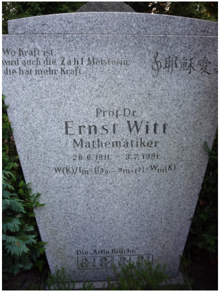
- **Proof** For any regular $x \in [a, b]$

$$\begin{aligned} &\mathrm{var}(P_0(x), P_1(x), \dots, P_n(x)) \\ = \ &\mathrm{var}(P_0^*(x), P_1^*(x), \dots, P_n^*(x)) \text{ (by the Duality Theorem)} \\ = \ &\big(n - \tau(\mathrm{Tri}(Q_*(x)))\big)/2 \in \{0, 1, 2, \dots, n\} \ . \end{aligned}$$

Sylvester's musical inspiration for the Duality Theorem

As an artist delights in recalling the particular time and atmospheric effects under which he has composed a favourite sketch, so I hope to be excused putting upon record that it was in listening to one of the magnificent choruses in the 'Israel in Egypt' that, unsought and unsolicited, like a ray of light, silently stole into my mind the idea (simple, but previously unperceived) of the equivalence of the Sturmian residues to the denominator series formed by the reverse convergents. The idea was just what was wanting,—the key-note to the due and perfect evolution of the theory.

Ernst Witt (1911–1991)



- "Artin fractions":

$$\frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}, \quad \frac{2\cancel{6}}{\cancel{6}5} = \frac{2}{5}, \quad \frac{1\cancel{9}}{\cancel{9}5} = \frac{1}{5}, \quad \frac{4\cancel{9}}{\cancel{9}8} = \frac{4}{8}.$$

- $(x, y, z) = (1, 6, 4)$, $(2, 6, 5)$, $(1, 9, 5)$ and $(4, 9, 8)$ are the only single-digit solutions of

$$\frac{10x + y}{10y + z} = \frac{x}{z}.$$

Fractions

- Let R be a commutative ring, and $S \subset R$ a multiplicative subset of non-zero divisors, with $1 \in S$.
- The **localization of R inverting S** is the ring of fractions

$$S^{-1}R \;=\; \{r/s \,|\, r \in R,\ s \in S\}$$

  with natural injection

$$R \to S^{-1}R \;;\; r \mapsto r/1 \;.$$

- The comparison of the classifications of symmetric matrices over R and $S^{-1}R$ is a fundamental technique of algebra, topology - and number theory!

Minkowski, Hasse and Witt

- Minkowski (1880's) related the classification of symmetric matrices over the integers $\mathbb{Z}$ and rationals $\mathbb{Q} = (\mathbb{Z}\backslash\{0\})^{-1}\mathbb{Z}$.
- Hasse (1920's) related the classification of symmetric matrices over R and K for the ring of algebraic integers $R = \mathcal{O}_K$ in an algebraic number field $K = (R\backslash\{0\})^{-1}R$.
- Witt (1937) introduced the "Witt group" $W(K)$ of a field K to be the Grothendieck group (avant la lettre) of equivalence classes of invertible symmetric matrices over K.
- Witt's computation of $W(K)$ for $\mathrm{char}(K) \neq 2$ gave a uniform treatment of the invariants of Minkowski and Hasse for an algebraic number field K.
- Relation between $W(R)$ and $W(S^{-1}R)$ given by the "localization exact sequence". $R = \mathbb{R}[X]$ and $S^{-1}R = \mathbb{R}(X)$ relevant to Sturm's theorem.

Symmetric forms

- Let R be a commutative ring.
- A **symmetric form over** R (V, ϕ) is a f.g. free R-module V together with a symmetric bilinear pairing $\phi : V \times V \to R$.
- (V, ϕ) essentially the same as a symmetric $n \times n$ matrix $S = (s_{ij})$ with

$$s_{ij} \;=\; s_{ji} \in R \;,\; n \;=\; \dim_R V \;.$$

- (V, ϕ) is **nonsingular** if the adjoint R-module morphism

$$\phi \;:\; V \to V^* = \mathrm{Hom}_R(V, R) \,;\; v \mapsto (w \mapsto \phi(v, w) = \phi(w, v))$$

  is an isomorphism. The form is nonsingular if and only if the matrix is invertible.

The symmetric Witt group $W(R)$

- The **symmetric Witt group** $W(R)$ is the abelian group of equivalence classes of nonsingular symmetric forms (V,ϕ) over R with

  (i) $(V,\phi) \sim (V',\phi')$ if there exists an isomorphism $f : V \to V'$ such that $\phi'(f(v),f(w)) = \phi(v,w)$ for all $v,w \in V$,

  (ii) $(V,\phi) \oplus (V,-\phi) \sim 0$ for any (V,ϕ) .

  Addition by

  $$(V_1,\phi_1) + (V_2,\phi_2) = (V_1 \oplus V_2, \phi_1 \oplus \phi_2) \in W(R) \ .$$

- (Sylvester, 1852) By the Law of Inertia the signature map is an isomorphism

  $$\tau \ : \ W(\mathbb{R}) \to \mathbb{Z} \ ; \ (V,\phi) \mapsto \tau(V,\phi) \ .$$

- (Serre, 1962) $\tau : W(\mathbb{Z}) \to \mathbb{Z}$ is an isomorphism.

Linking forms

- An (R, S)-**module** T is a f.g. homological dimension 1 R-module such that $S^{-1}T = 0$, so that

$$T \; = \; \text{coker}(\sigma : R^n \to R^n) \; (\det(\sigma) \in S) \; .$$

- A **symmetric linking form over** (R, S) (T, λ) is an (R, S)-module T with a symmetric bilinear pairing

$$\lambda : T \times T \to S^{-1}R/R \; .$$

- (T, λ) is **nonsingular** if the adjoint R-module morphism

$$T \to \text{Hom}_R(T, S^{-1}R/R) \; ; \; x \mapsto (y \mapsto \lambda(x, y))$$

  is an isomorphism.
- The **Witt group** $W(R, S)$ of nonsingular symmetric linking forms over (R, S) is defined by analogy with $W(R)$.

The localization exact sequence of Witt groups

- **Theorem** (R. 1980) For any commutative ring R and multiplicative system $S \subset R$ the Witt groups of R and $S^{-1}R$ are related by exact sequence

  $$W(R) \longrightarrow W(S^{-1}R) \xrightarrow{\partial} W(R,S) \ .$$

  The boundary map ∂ given by the "dual lattice" construction

  $$\begin{aligned}\partial S^{-1}(V,\phi) &= (\mathrm{coker}(\phi : V \to V^*), (f,g) \mapsto f(\phi^{-1}(g))) \\ &= (V^{\#}/V, (v/s, w/t) \mapsto \phi(v,w)/st)\end{aligned}$$

  with $V^{\#} = \{v/s \in S^{-1}V \mid \phi(v)/s \in V^* \subset S^{-1}V^*\}$.
- **Example** For any $r, s \in S$

  $$\begin{aligned}&\partial(K, r/s) \\ &= (R/(rs), 1/rs : R/(rs) \times R/(rs) \to S^{-1}R/R; (x,y) \mapsto xy/rs)\end{aligned}$$

  $(= (R/(r), 1/r) \oplus (R/(s), 1/s)$ for coprime $r, s \in S$.)

W(Dedekind ring)

- (Milnor, 1970) The localization exact sequence for a Dedekind ring R with quotient field $K = S^{-1}R$ ($S = R\backslash\{0\}$) is
$$0 \longrightarrow W(R) \longrightarrow W(K) \xrightarrow{\partial} W(R,S) = \bigoplus_{\pi \triangleleft R \text{ prime}} W(R/\pi) \ .$$
∂ is split onto for a principal ideal domain R.
- **Example** For $R = \mathbb{Z} \subset S^{-1}R = \mathbb{Q}$ (R,S)-modules = finite abelian groups, $W(\mathbb{Z}) = \mathbb{Z}$ and $W(\mathbb{Q}) = \mathbb{Z} \oplus W(\mathbb{Z},S)$ with
$$\begin{aligned} W(\mathbb{Z},S) &= \bigoplus_{p \text{ prime}} W(\mathbb{F}_p) \\ &= W(\mathbb{F}_2) \oplus \bigoplus_{4q+1 \text{ prime}} W(\mathbb{F}_{4q+1}) \oplus \bigoplus_{4r+3 \text{ prime}} W(\mathbb{F}_{4r+3}) \\ &= \mathbb{Z}_2 \oplus \bigoplus_{4q+1 \text{ prime}} (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus \bigoplus_{4r+3 \text{ prime}} \mathbb{Z}_4 \ . \end{aligned}$$
- For an odd prime p $\left(\dfrac{-1}{p}\right) = \begin{cases} 1 & \text{if } p = 4q+1 \\ -1 & \text{if } p = 4r+3. \end{cases}$

The Euclidean algorithm and the localization exact sequence

- For any R, S let $p_0, p_1 \in S$ be coprime, verified by the Euclidean algorithm with 'abstract Sturm sequences' $p = (p_0, p_1, \dots, p_n) \in S^{n+1}$, $q = (q_1, q_2, \dots, q_n) \in R^n$

$$p_k q_k \;=\; p_{k-1} + p_{k+1} \;(1 \leqslant k \leqslant n)$$

  with $p_n = \text{g.c.d}(p_0, p_1) = 1$, $p_{n+1} = 0$.
- **Proposition** (Ghys-R.,2016)
  The Sturm sequences lift $(R/(p_0), p_1/p_0) \in W(R, S)$ to $(S^{-1}R^n, \text{Tri}(q)) \in W(S^{-1}R)$, with

$$(S^{-1}R^n, \text{Tri}(q)) \;=\; \bigoplus_{k=1}^{n} (S^{-1}R, p_{k-1}/p_k) \in W(S^{-1}R) \;,$$

$$\begin{aligned} \partial(S^{-1}R^n, \text{Tri}(q)) \;&=\; \partial(S^{-1}R, p_0/p_1) \\ &=\; (R/(p_0), p_1/p_0) \in W(R, S) \;. \end{aligned}$$

The Witt group $W(\mathbb{R}(X))$

- (R., 1998) The Witt group localization exact sequence for $\mathbb{R}[X] \subset \mathbb{R}(X)$ splits

$$0 \longrightarrow W(\mathbb{R}[X]) = \mathbb{Z} \longrightarrow W(\mathbb{R}(X)) \xrightarrow{\partial} W(\mathbb{R}[X], S) \longrightarrow 0$$

  with $W(\mathbb{R}[X], S)$ the Witt group of nonsingular symmetric linking forms $(T, \lambda : T \times T \to \mathbb{R}(X)/\mathbb{R}[X])$ on f.g. S-torsion $\mathbb{R}[X]$-modules T (= finite-dimensional $\mathbb{R}$-vector space T with an endomorphism $X : T \to T$) .

- $$\begin{aligned} W(\mathbb{R}[X], S) &= \bigoplus_{\pi \lhd \mathbb{R}[X] \text{ prime}} W(\mathbb{R}[X]/\pi) \\ &= \bigoplus_{x \in \mathbb{R}} W(\mathbb{R}[X]/(X-x)) \oplus \bigoplus_{z \in \mathcal{H}} W(\mathbb{R}[X]/(X-z)(X-\overline{z})) \\ &= \mathbb{Z}[\mathbb{R}] \oplus \mathbb{Z}_2[\mathcal{H}] \ (\mathcal{H} = \text{upper half plane} \subset \mathbb{C}) \ . \end{aligned}$$

The Witt group interpretation of Sturm-Sylvester theorem

- Suppose that $P(X) \in \mathbb{R}[X]$ is a degree n polynomial with g real roots $\{x_1, x_2, \dots, x_g\} \subset \mathbb{R}$ and $2h$ complex roots $\{z_1, z_2, \dots, z_h\} \cup \{\overline{z}_1, \overline{z}_2, \dots, \overline{z}_h\} \subset \mathcal{H} \cup \overline{\mathcal{H}}$, with $n = g + 2h$ and $\mathcal{H}=$ complex upper half plane.
- Let $P_*(X) = (P_0(X), \dots, P_n(X))$, $Q_*(X) = (Q_1(X), \dots, Q_n(X))$ be the Sturm functions of $P(X)$.
- **Theorem** (Ghys-R., 2016) The location of the roots of $P(X)$ can be read off from the Witt class of the nonsingular symmetric form $(\mathbb{R}(X), P(X)/P'(X))$ over $\mathbb{R}(X)$

$$
\begin{aligned}
&(\mathbb{R}(X), P(X)/P'(X)) = (\mathbb{R}(X)^n, \mathrm{Tri}(Q_*(X))) \\
&= \bigoplus_{i=1}^{g} (\mathbb{R}(X), X - x_i) \oplus \bigoplus_{j=1}^{h} (\mathbb{R}(X), (X - z_j)(X - \overline{z}_j)) \\
&\qquad\qquad\qquad\qquad \oplus - (\mathbb{R}(X)^h, 1) \\
&= \sum_{i=1}^{g} 1.x_j + \sum_{j=1}^{h} 1.z_j - h.1 \in W(\mathbb{R}(X)) = \mathbb{Z}[\mathbb{R}] \oplus \mathbb{Z}_2[\mathcal{H}] \oplus \mathbb{Z}\ .
\end{aligned}
$$

Terry Wall (1936–)

Surgery

- (Thom, Milnor, 1950's) A **surgery** on a closed n-dimensional manifold L uses an embedding

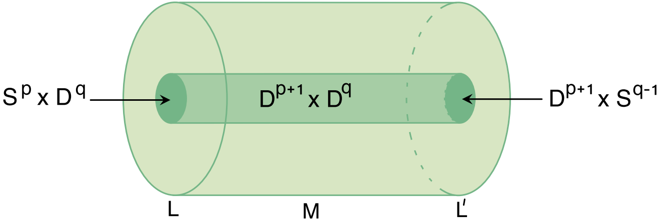
$$S^p \times D^q \subset L \ (p + q = n)$$

to construct a new closed n-dimensional manifold

$$L' = (L \backslash S^p \times D^q) \cup D^{p+1} \times S^{q-1} .$$

- The **trace** of the surgery is the cobordism $(M; L, L')$ with

$$M = L \times I \cup D^{p+1} \times D^q .$$



Manifolds, intersections and linking

- An oriented $2i$-dimensional manifold with boundary $(M, \partial M)$ has an **intersection** $(-1)^i$-symmetric form over $\mathbb{Z}$ $(H_i(M)/\text{torsion}, \phi_M)$ over $(\mathbb{Z}, \mathbb{Z}\backslash\{0\})$

$$\phi_M(N_1^i \subset M, N_2^i \subset M) \ = \ N_1 \cap N_2 \in \mathbb{Z} \ .$$

- An oriented $(2i-1)$-dimensional manifold with boundary $(L, \partial L)$ has a $(-1)^i$-symmetric **linking** form over $(\mathbb{Z}, \mathbb{Z}\backslash\{0\})$ $(\text{torsion } H_{i-1}(L), \lambda_L)$ with

$$\lambda_L(N_1^{i-1} \subset L, N_2^{i-1} \subset L) = \frac{\delta N_1 \cap N_2}{s} \in \mathbb{Q}/\mathbb{Z}$$

if $\delta N_1^i \subset L$ extends $\partial \delta N_1 = \bigcup\limits_s N_1 \subset L$ for some $s \geqslant 1$.

- For $(M^{2i}, \partial M)$ with even i can define **signature** $\tau(M) = (H_i(M)/\text{torsion}, \phi_M) \in W(\mathbb{Z}) = \mathbb{Z}$ and

$$\partial : W(\mathbb{Q}) \to W(\mathbb{Z}, \mathbb{Z}\backslash\{0\});$$

$$(H_i(M;\mathbb{Q}), \phi_M) \mapsto (\text{torsion}(H_{i-1}(\partial M)), \lambda_{\partial M}) \ .$$

The lens spaces

- For any coprime $a, c \in \mathbb{Z}$ define the **lens space**

$$L(c,a) \ = \ S^1 \times D^2 \cup_A S^1 \times D^2$$

using any $b, d \in \mathbb{Z}$ such that $ad - bc = 1$, with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ realized by

$$A \ : \ S^1 \times S^1 \to S^1 \times S^1 \ ; \ (z,w) \mapsto (z^a w^b, z^c w^d) \ .$$

- $L(c,a)$ is a closed oriented 3-dimensional manifold with symmetric linking form $(H_1(L(c,a)),\lambda) = (\mathbb{Z}_c, a/c)$.
- Surgery on $S^1 \times D^2 \subset L(c,a)$ results in a cobordism $(M(c,a); L(c,a), L(a,c))$ with

$$M(c,a) \ = \ L(c,a) \times I \cup D^2 \times D^2 \ ,$$

$$-L(a,c) \ = \ (L(c,a) \backslash S^1 \times D^2) \cup D^2 \times S^1 \ .$$

Symmetric intersection form $(H_2(M(c,a)),\phi) = (\mathbb{Z}, ac)$.

Topological proof of the Sylvester Duality Theorem I.

- (Hirzebruch, 1962) For coprime $c > a > 0$ the Euclidean algorithm for g.c.d.$(a, c) = 1$

$$p_0 = c \ , \ p_1 = a \ , \ \dots \ , \ p_n = 1 \ , \ p_{n+1} = 0 \ ,$$
$$p_k q_k = p_{k-1} + p_{k+1} \ (1 \leqslant k \leqslant n) \ .$$

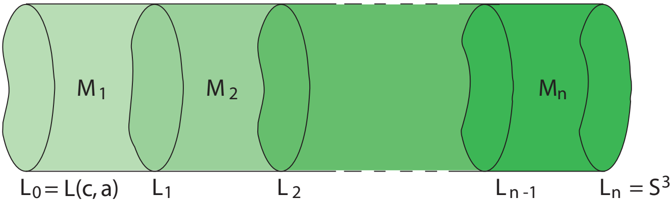
  determines an expression of the lens space $L(c, a) = \partial M$ as the boundary of an oriented 4-dimensional manifold M with intersection form $(H_2(M), \phi) = (\mathbb{Z}^n, \mathrm{Tri}(q))$.
- The continued fraction $a/c = [q_1, q_2, \dots, q_n]$ is realized topologically by a sequence of cobordisms of lens spaces

$$(M, \partial M) = (M_1; L_0, L_1) \cup (M_2; L_1, L_2) \cup \dots \cup (M_n \cup D^4; L_{n-1}, \emptyset)$$

  with

$$L_0 = L(p_0, p_1) = L(c, a) \ , \ L_n = L(p_n, p_{n+1}) = L(1, 0) = S^3 \ ,$$
$$L_k = L(p_k, p_{k+1}) = -L(p_k, p_{k-1}) \ (1 \leqslant k \leqslant n) \ .$$

Topological proof of the Sylvester Duality Theorem II.

- 

- M is obtained by glueing together the cobordisms $(M_k; L_{k-1}, L_k)$ for $k = 1, 2, \ldots, n$ ("plumbing") with

$$L_{k-1} = L(p_{k-1}, p_k)\ ,\ \ M_k = M(p_{k-1}, p_k)$$

$$(M, \partial M)$$

$$= \ (M_1; L_0, L_1) \cup (M_2; L_1, L_2) \cup \cdots \cup (M_n \cup D^4; L_{n-1}, \emptyset)\ .$$

Topological proof of the Sylvester Duality Theorem III.

- The union $U_k = \bigcup_{j=1}^{k} M_j$ has
  $$(H_2(U_k;\mathbb{Q}),\phi_{U_k}) = \bigoplus_{j=1}^{k}(\mathbb{Q},p_{j-1}p_j)\ ,\ \tau(U_k) = \sum_{j=1}^{k}\mathrm{sign}(p_j/p_{j-1})$$
  with $p_j = \det(\mathrm{Tri}(q_{j+1},\dots,q_n))$.
- The union $V_k = \bigcup_{j=n-k+1}^{n} M_j$ has
  $$(H_2(V_k),\phi_{V_k}) = (\mathbb{Z}^k,\mathrm{Tri}(q_1,q_2,\dots,q_k))\ ,$$
  $$\tau(V_k) = \sum_{j=1}^{k}\mathrm{sign}(p_j^*/p_{j-1}^*) \text{ with } p_j^* = \det(\mathrm{Tri}(q_1,q_2,\dots,q_j))\ .$$
- It now follows from $M = U_n = V_n$ that
  $$\begin{aligned}\tau(M) &= \tau(\mathrm{Tri}(q_1,q_2,\dots,q_n)) \\ &= \sum_{j=1}^{n}\mathrm{sign}(p_j/p_{j-1}) = \sum_{j=1}^{n}\mathrm{sign}(p_j^*/p_{j-1}^*)\ .\end{aligned}$$

Surgery theory

- The 1960's Browder-Novikov-Sullivan-Wall surgery obstruction theory for classifying high dimensional manifolds within a homotopy type culminated in the development of Wall's algebraic **surgery obstruction groups** $L_n(R)$ for any ring with involution R.
- In the applications to topology $R = \mathbb{Z}[\pi]$ with π the fundamental group and the involution

$$\mathbb{Z}[\pi] \to \mathbb{Z}[\pi] \ ; \ \sum_{g \in \pi} n_g g \mapsto \sum_{g \in \pi} n_g g^{-1}$$

- For $n \geqslant 5$ a topological space X with n-dimensional Poincaré duality $H^{n-*}(X) \cong H_*(X)$ is homotopy equivalent to an n-dimensional topological manifold if and only if a certain algebraic L-theory obstruction related to $L_n(\mathbb{Z}[\pi_1(X)])$ vanishes.
- Every finitely presented group π can occur.

The algebraic L-groups I.

- The L-groups of a ring with involution R are abelian and are 4-periodic
$$L_n(R) \ = \ L_{n+4}(R) \ .$$
- Roughly speaking, modulo 2-primary torsion

$$L_n(R) =$$
$$\begin{cases} \text{Witt group of } (-1)^i\text{-symmetric forms over } R \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } n = 2i \\ (\text{automorphism group of } (-1)^i\text{-symmetric forms over } R)^{\text{ab}} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } n = 2i+1 \end{cases}$$

In particular, $L_{4*}(R) = W(R)$ modulo 2-primary torsion.

The algebraic L-groups II.

- (R., 1980) The algebraic L-groups $L_n(R)$ were expressed as the cobordism groups of n-dimensional f.g. free R-module chain complexes C with the Poincaré duality

$$H^{n-*}(C) \cong H_*(C)$$

  of an n-dimensional manifold.
- The Witt group localization exact sequence was extended to

$$\dots \longrightarrow L_{n+1}(R,S) \longrightarrow L_n(R) \longrightarrow L_n(S^{-1}R)$$
$$\xrightarrow{\partial} L_n(R,S) \longrightarrow L_{n-1}(R) \longrightarrow \dots$$

  for any ring with involution R and $S \subset R$ such that $R \to S^{-1}R$ is an injection of rings with involution.

The computation of $L_*(\mathbb{Z}[\pi])$

- In the 1970's Wall initiated the computations of $L_*(\mathbb{Z}[\pi])$ for many groups π.
- For finite π the computations use number theory, notably the "arithmetic square"

$$\begin{tikzcd}
\mathbb{Z}[\pi] \arrow[r] \arrow[d] & \widehat{\mathbb{Z}}[\pi] \arrow[d] \\
\mathbb{Q}[\pi] \arrow[r] & \widehat{\mathbb{Q}}[\pi]
\end{tikzcd}$$

  with $\widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}_n$ the profinite completion of $\mathbb{Z}$ and $\widehat{\mathbb{Q}} = (\widehat{\mathbb{Z}}\backslash\{0\})^{-1}\widehat{\mathbb{Z}}$ the quotient field, the finite adèles.
- The Novikov and Farrell-Jones conjectures predict $L_*(\mathbb{Z}[\pi])$ for infinite groups π. Verifications for many classes of groups, using group theory, differential geometry and topology.