SURGERY ON TREES

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- Homotopy vs. homeomorphism

- Transversality

- Codimension 1 submanifolds

- Seifert-Van Kampen Theorem

- Mayer-Vietoris exact sequence

- Algebraic K- and L-theory

Homotopy vs. homeomorphism

- A class of manifolds is rigid if every homotopy equivalence of manifolds in the class is homotopic to a homeomorphism.

- Main example of a rigid class:
  hyperbolic manifolds, including all oriented 2-dimensional manifolds.

- Borel conjecture: aspherical manifolds are rigid.

- In general, manifolds are not rigid and there are many distinct homeomorphism classes of manifolds within a homotopy type.

Surgery obstruction theory

- Surgery theory provides systematic obstruction theory in dimensions $n \geq 5$ for deciding if a homotopy equivalence of n-dimensional manifolds $f : M^n \to X$ is homotopic to a homeomorphism.

- Obstructions involve the algebraic K- and L-theory of modules and quadratic forms over the group ring $\mathbb{Z}[\pi]$ of the fundamental group $\pi = \pi_1(X)$.

Codimension 1 methods in topology

- Study of 3-manifolds via surfaces $N^2 \subset M^3$.

- The Eilenberg–Steenrod excision axiom for homology is a codimension 1 transversality requirement.

- Codimension 1 submanifolds $N^{n-1} \subset M^n$ play a central role in:

  - Novikov's proof of the topological invariance of the rational Pontrjagin classes (1966)

  - Kirby-Siebenmann structure theory of high dimensional topological manifolds (1969)

  - Chapman's proof of the topological invariance of Whitehead torsion (1974)

  - <u>controlled topology</u>.

Geometric transversality

- Let X be a space with a subspace

$$Y \times \mathbb{R} \subset X \ .$$

  Identify $Y = Y \times \{0\} \subset Y \times \mathbb{R}$.

- Transversality theorem: every map from an n-dimensional manifold

$$f : M^n \to X$$

  is homotopic to a map which is transverse at $Y \subset X$, with

$$N^{n-1} = f^{-1}(Y) \subset M^n$$

  a codimension 1 submanifold.

Splitting homotopy equivalences

- A homotopy equivalence $f : M^n \to X$ splits along $Y \subset X$ if it is homotopic to one for which the restrictions

$$f| \ : N^{n-1} = f^{-1}(Y) \to Y \ ,$$
$$f| \ : M \backslash N \ = \ f^{-1}(X \backslash Y) \to X \backslash Y$$

  are also homotopy equivalences.

- Codimension 1 splitting necessary along all $Y \subset X$ (and sometimes sufficient) for f to be homotopic to a homeomorphism.

- Waldhausen (1969) proved that Haken 3-dimensional manifolds are rigid, using codimension 1 splitting methods.

Non-splitting homotopy equivalences

- In general, homotopy equivalences do not split along codimension 1 submanifolds, with both K- and L-theory obstructions.

- Farrell and Hsiang (1970) proved that for $n \geq 6$ a homotopy equivalence $f : M^n \to X \times S^1$ splits along $X \times \{\text{pt.}\} \subset X \times S^1$ if and only if $\tau(f) \in \text{im}(\text{Wh}(\pi) \to \text{Wh}(\pi \times \mathbb{Z}))$, $\pi = \pi_1(X)$ (Whitehead torsion).

- Cappell (1972) used a partial computation of the L-theory of the infinite dihedral group $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$ to construct homotopy equivalences $h : M^{4k+1} \to \mathbb{RP}^{4k+1} \# \mathbb{RP}^{4k+1}$ for $k \geq 1$ which do not split along the separating codimension 1 submanifold in the connected sum $S^{4k} \subset \mathbb{RP}^{4k+1} \# \mathbb{RP}^{4k+1}$.

Object of the exercise

- Invent algebra which is sufficiently flexible to have the geometric transversality properties of manifolds.

- Identify the difference between homotopy equivalences and homeomorphisms of manifolds with the extent to which the K- and L-groups of the fundamental group ring $\mathbb{Z}[\pi]$ have this flexibility.

- Computations in algebraic K- and L-theory are used in two directions, to prove that:

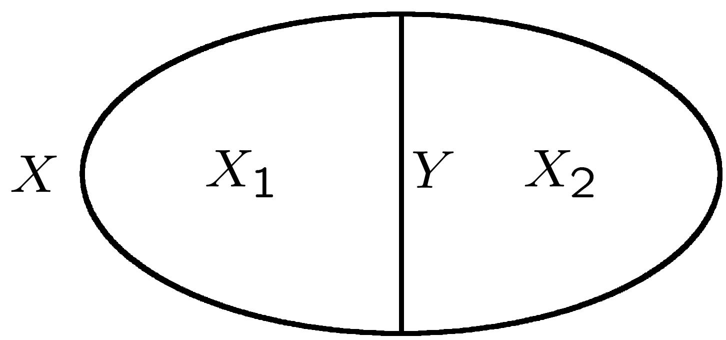
  - some homotopy equivalences of manifolds are definitely homotopic to homeomorphisms

  - others are definitely not homotopic to homeomorphisms.

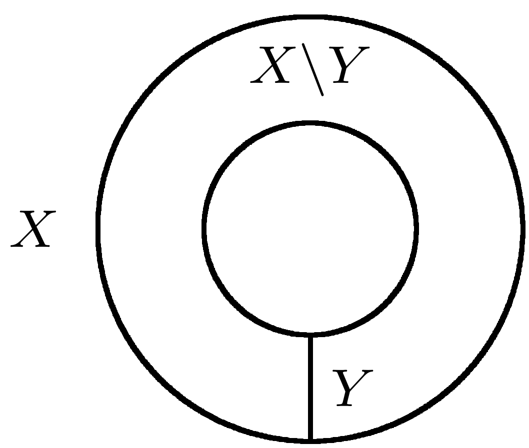
The two cases

- X, Y connected, $Y \times \mathbb{R} \subset X$.

- Case A: The complement $X \backslash Y = X_1 \cup X_2$ is disconnected.



- Case B: The complement $X \backslash Y$ is connected.



The Seifert–van Kampen Theorem

- The fundamental group of a connected space X with a connected subspace $Y \subset X$ is determined by the fundamental groups of Y, $X \backslash Y$.

- Case A: amalgamated free product

$$\pi_1(X) \;=\; \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2)$$

  – Example: $X = S^1 \vee S^1$, $Y = \{\text{pt.}\}$.

- Case B: HNN extension

$$\pi_1(X) \;=\; \pi_1(X \backslash Y) *_{\pi_1(Y)} \{t\}$$

  – Example: $X = S^1$, $Y = \{\text{pt.}\}$.

- Amalgamated free products and HNN extensions are the groups which act on trees with quotient I and S^1 (Bass-Serre).

The Mayer–Vietoris exact sequence

- The homology of X is determined by the homologies of Y, $X \backslash Y$ by the Mayer–Vietoris exact sequence.

  – Case A: $X = X_1 \cup_Y X_2$

  $$\begin{aligned} \ldots \ \to \ & H_n(Y) \to H_n(X_1) \oplus H_n(X_2) \\ \to \ & H_n(X) \to H_{n-1}(Y) \to \ldots \ . \end{aligned}$$

  – Case B:

  $$\begin{aligned} \ldots \ \to \ & H_n(Y) \to H_n(X \backslash Y) \\ \to \ & H_n(X) \to H_{n-1}(Y) \to \ldots \end{aligned}$$

  with two inclusions $Y \to X \backslash Y$.

- Proved by codimension 1 transversality on cycle level.

The Whitehead group

- The Whitehead group Wh(π) of a group π is the abelian group of equivalence classes of invertible $k \times k$ matrices with entries in $\mathbb{Z}[\pi]$ for all integers $k \geq 1$, modulo the equivalence relation generated by Gaussian elimination and stabilization by direct sum with identity matrices. (Whitehead, 1939)

- A nonsingular matrix over $\mathbb{Z}[\pi]$ can be reduced to the identity matrix by elementary row and column operations if and only if it represents 0 in the Whitehead group Wh(π).

Higman

- Initiated the algebraic computation of the Whitehead group
  - Units in group rings (1940)

- $\mathrm{Wh}(\{1\}) = 0$ by Gaussian elimination for matrices with entries in $\mathbb{Z}$.

- $\mathrm{Wh}(\mathbb{Z}) = 0$, $\mathrm{Wh}(\mathbb{Z}_5) \neq 0$.

Whitehead torsion

- A homotopy equivalence $f : M \to N$ of compact polyhedra has a Whitehead torsion $\tau(f) \in \mathrm{Wh}(\pi_1(N))$ such that:
  - $\tau(f) = \tau(g)$ if $f, g\colon M \to N$ homotopic,
  - $\tau(f) = 0$ if f is a homeomorphism.
- There exist 3-dimensional lens space manifolds M, N which are homotopy equivalent but not homeomorphic, with homotopy equivalences $f : M \to N$ such that $\tau(f) \neq 0$. (Reidemeister, Whitehead, 1930's).
- Generalized Whitehead groups $\mathrm{Wh}_n(\pi)$ defined for $n \in \mathbb{Z}$ (Bass for $n \leq -1$, Quillen for $n \geq 3$) with $\mathrm{Wh}_1(\pi) = \mathrm{Wh}(\pi)$, and $\mathrm{Wh}_0(\pi) = \widetilde{K}_0(\mathbb{Z}[\pi])$ the reduced projective class group.

Waldhausen's theorem

- Theorem (1976) The generalized Whitehead groups $Wh_*(X) = Wh_*(\pi_1(X))$ of a connected space X with connected $Y \subset X$ fit into exact sequences of the Mayer–Vietoris type:

  - Case A: $X \backslash Y$ disconnected $X = X_1 \cup_Y X_2$

$$\begin{aligned} \dots \;\to\;& \mathrm{Wh}_n(Y) \to \mathrm{Wh}_n(X_1) \oplus \mathrm{Wh}_n(X_2) \\ \to\;& \mathrm{Wh}_n(X) \to \mathrm{Wh}_{n-1}(Y) \oplus \mathrm{Nil}_n \to \dots\ . \end{aligned}$$

  - Case B: $X \backslash Y$ connected

$$\begin{aligned} \dots \;\to\;& \mathrm{Wh}_n(Y) \to \mathrm{Wh}_n(X \backslash Y) \\ \to\;& \mathrm{Wh}_n(X) \to \mathrm{Wh}_{n-1}(Y) \oplus \mathrm{Nil}_n \to \dots\ . \end{aligned}$$

- Corollary $\mathrm{Wh}_*(\pi) = 0$ for the fundamental groups π of Haken 3-manifolds.

Cappell's theorem

- The unitary nilpotent groups UNil_n are the obstructions to a Mayer-Vietoris exact sequence in the algebraic L-theory of groups acting on trees (= amalgamated free products and HNN extensions).

- Theorem (1976) For $n \geq 6$ a homotopy equivalence of n-dimensional manifolds $f : M^n \to X$ splits along a codimension 1 submanifold $Y \subset X$ if and only if Nil and UNil obstructions vanish.

- Proved geometrically, using the entire apparatus of the Browder-Novikov-Sullivan-Wall surgery theory.

Chain complexes

- The algebraic K-groups defined by chain complexes.

- "Whatever can be done for abstract K-groups can be done (usually with more difficulty) for the L-groups" (C.T.C.Wall)

- The algebraic L-groups defined by chain complexes with Poincaré duality.

Algebraic transversality

- Chain complexes over the group ring $\mathbb{Z}[\pi]$ of a group π which is an amalgamated free product $\pi_1 *_\rho \pi_2$ or an HNN extension $\pi_1 *_\rho \{t\}$ have the transversality properties of manifolds with these fundamental groups.

- Algebraic transversality: if C is a $\mathbb{Z}[\pi]$-module chain complex then C has corresponding codimension 1 subcomplex $D \subset C$ over $\mathbb{Z}[\rho]$.

- Can now prove Cappell's theorem using algebraic transversality for chain complexes with Poincaré duality.

State of the art

- The Nil and UNil groups were studied by Bass-Heller-Swan, Farrell and Hsiang in the 60's, Waldhausen, Cappell in the 70's, Connolly, Kozńiewski and R. in the 90's:

  - hard to compute,

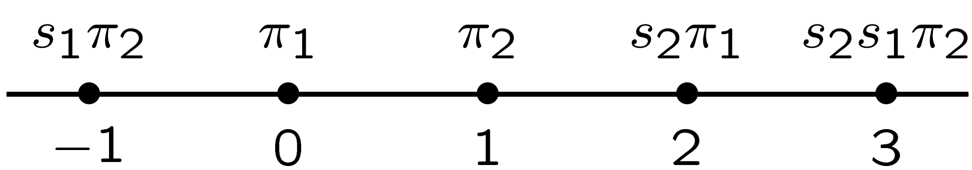
  - either 0 or infinitely generated,

  - obstructions to the integral Novikov conjecture.

- Connolly and R. have new results on the L-theory of D_∞.

Two trees

- <u>Case A</u>: $\pi = \pi_1 *_\rho \pi_2$ has tree T with
  $T^{(0)} \; = \; [\pi : \pi_1] \cup [\pi : \pi_2] \;\; , \;\; T^{(1)} \; = \; [\pi : \rho] \; .$
  - Example: Infinite dihedral group
    $D_\infty = \mathbb{Z}_2 *_{\{1\}} \mathbb{Z}_2 = \{s_1, s_2 \,|\, (s_1)^2, (s_2)^2\}$
    $T = \mathbb{R}$, $s_1(x) = -x$, $s_2(x) = 2-x$, $T/D_\infty = I$.



- <u>Case B</u>: $\pi = \pi_1 *_\rho \{t\}$ has tree T with
  $T^{(0)} \; = \; [\pi : \pi_1] \;\; , \;\; T^{(1)} \; = \; [\pi : \rho] \; .$
  - Example: Infinite cyclic group $\mathbb{Z} = \{t\}$
    $T = \mathbb{R} \;\; , \;\; t(x) = x + 1 \;\; , \;\; T/\mathbb{Z} = S^1 \; .$