A TOPOLOGIST'S VIEW OF SYMMETRIC AND QUADRATIC FORMS

Andrew Ranicki (Edinburgh)

http://www.maths.ed.ac.uk/~aar

Patterson 60++, Göttingen, 27 July 2009

The mathematical ancestors of S.J.Patterson

Augustus Edward Hough Love
Eidgenössische Technische Hochschule Zürich

$\downarrow$

G. H. (Godfrey Harold) Hardy
University of Cambridge

$\downarrow$

Mary Lucy Cartwright
University of Oxford (1930)

$\downarrow$

Walter Kurt Hayman

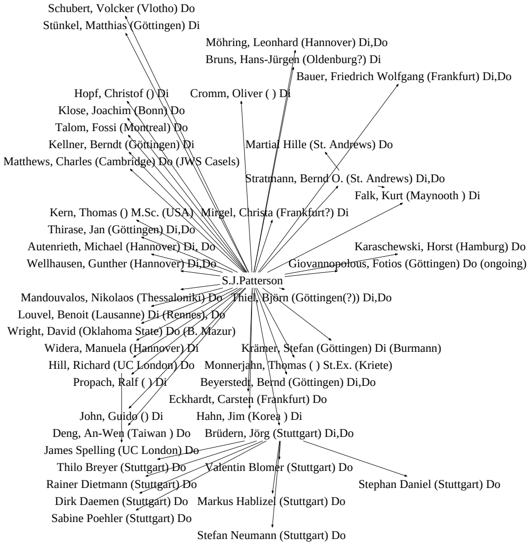
$\downarrow$

Alan Frank Beardon

$\downarrow$

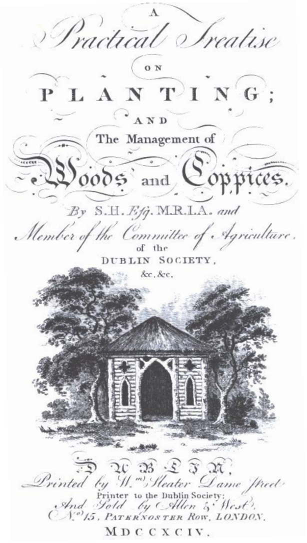
Samuel James Patterson
University of Cambridge (1975)

The 35 students and 11 grandstudents of S.J.Patterson



Paddy with Carla Ranicki at the Göttingen Wildgehege, 1985

Irish roots: a practical treatise on planting Woods ...


A
Practical Treatise
ON
PLANTING;
AND
The Management of
Woods and Coppices.
By S.H. Esq. M.R.I.A. and
Member of the Committee of Agriculture,
of the
DUBLIN SOCIETY,
&c. &c.

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Symmetric forms

- **Slogan 1** It is a fact of sociology that topologists are interested in quadratic forms – Serge Lang.
- Let A be a commutative ring, or more generally a noncommutative ring with an involution.
- **Slogan 2** Topologists like quadratic forms over group rings!
- **Definition** For $\epsilon = 1$ or -1 an ϵ-**symmetric form** (F, λ) over A is a f.g. free A-module F with a bilinear pairing $\lambda : F \times F \to A$ such that

$$\lambda(x, y) \;=\; \epsilon\lambda(y, x) \in A \;\; (x, y \in F) \ .$$

- The form (F, λ) is **nonsingular** if the A-module morphism

$$\lambda \;:\; F \to F^* \;=\; \mathrm{Hom}_A(F, A) \ ; \ x \mapsto (y \mapsto \lambda(x, y))$$

is an isomorphism.

The $(-)^n$-symmetric form of a $2n$-manifold

- **Slogan 3** Manifolds have ϵ-symmetric forms over $\mathbb{Z}$ and $\mathbb{Z}_2$, given algebraically by Poincaré duality and cup/cap products, and geometrically by intersections.
- $\mathbb{Z}$ in oriented case, $\mathbb{Z}_2$ in general. An m-dimensional manifold M^m is **oriented** if the tangent m-plane bundle τ_M is oriented, in which case the homology and cohomology are related by the Poincaré duality isomorphisms $H^*(M) \cong H_{m-*}(M)$.
- An oriented $2n$-dimensional manifold M^{2n} has a $(-)^n$-symmetric **intersection form** over $\mathbb{Z}$

$$\lambda \ : \ F^n(M) \times F^n(M) \to \mathbb{Z} \ ; \ (x, y) \mapsto \langle x \cup y, [M] \rangle$$

  with $F^n(M) = H^n(M)/\{\text{torsion}\}$ a f.g. free $\mathbb{Z}$-module.
- **Geometric interpretation** If $K^n, L^n \subset M^{2n}$ are oriented n-dimensional submanifolds which intersect transversely in an oriented 0-dimensional manifold $K \cap L$ then $[K], [L] \in H_n(M) \cong H^n(M)$ are such that

$$\lambda([K], [L]) \ = \ |K \cap L| \in \mathbb{Z} \ .$$

The ϵ-symmetric Witt group

- A **lagrangian** for a nonsingular ϵ-symmetric form (F, λ) is a direct summand $L \subset F$ such that
  - $\lambda(L, L) = 0$, so that $L \subset L^{\perp} = \ker(\lambda| : F \to L^*)$
  - $L = L^{\perp}$
- A form is **metabolic** if it admits a lagrangian.
- **Example** For any ϵ-symmetric form (L^*, ν) the nonsingular ϵ-symmetric form $(F, \lambda) = (L \oplus L^*, \begin{pmatrix} 0 & 1 \\ \epsilon & \nu \end{pmatrix})$ with

$$\lambda \ : \ F \times F \to A \ ; \ ((x_1, y_1), (x_2, y_2)) \mapsto y_2(x_1) + \epsilon y_1(x_2) + \nu(y_1)(y_2)$$

  is metabolic, with lagrangian L.
- The ϵ-**symmetric Witt group of** A is the Grothendieck-type group

$$L^0(A, \epsilon) = \frac{\{\text{isomorphism classes of nonsingular } \epsilon\text{-symmetric forms over } A\}}{\{\text{metabolic forms}\}}$$

Why do topologists like Witt groups?

- **Slogan 4** Topologists like Witt groups because we need them in the Browder-Novikov-Sullivan-Wall surgery theory classification of manifolds.
- Trivially, the stable classification of symmetric and quadratic forms over a ring A is easier than the isomorphism classification.
- Nontrivially, the stable classification is just about possible for the group rings $A = \mathbb{Z}[\pi]$ of interesting groups π.
  - The Witt groups of quadratic forms over group rings $A = \mathbb{Z}[\pi_1(M)]$ play a central role in the Wall obstruction theory for non-simply-connected manifolds M.
  - Algebra and number theory are used to compute Witt groups of $\mathbb{Z}[\pi]$ for finite groups π.
  - Geometry and topology are used to compute Witt groups of $\mathbb{Z}[\pi]$ for infinite groups π. Novikov, Borel and Farrell-Jones conjectures.

The signature of symmetric forms over $\mathbb{R}$ and $\mathbb{Z}$

- **Theorem** (Sylvester, 1852) Every nonsingular 1-symmetric form (F,λ) over $\mathbb{R}$ is isomorphic to

$$\bigoplus_p(\mathbb{R},1)\oplus\bigoplus_q(\mathbb{R},-1)$$

with $p+q=\dim_{\mathbb{R}}(F)$.

- **Definition** The **signature** of (F,λ) is

$$\text{signature}\,(F,\lambda)~=~p-q\in\mathbb{Z}~.$$

- **Corollary 1** Two nonsingular 1-symmetric forms (F,λ), (F',λ') over $\mathbb{R}$ are isomorphic if and only if $(p,q)=(p',q')$, if and only if

$$\dim_{\mathbb{R}}(F)~=~\dim_{\mathbb{R}}(F')~,~\text{signature}\,(F,\lambda)~=~\text{signature}\,(F',\lambda')~.$$

- **Corollary 2** The signature defines isomorphisms

$$L^0(\mathbb{R},1)\xrightarrow{\cong}\mathbb{Z}~;~(F,\lambda)\mapsto\text{signature}\,(F,\lambda)~,$$

$$L^0(\mathbb{Z},1)\xrightarrow{\cong}\mathbb{Z}~;~(F,\lambda)\mapsto\text{signature}\,\mathbb{R}\otimes_{\mathbb{Z}}(F,\lambda)~.$$

Cobordism

- **Definition** Oriented m-dimensional manifolds M, M' are **cobordant** if

$$M \cup -M' \;=\; \partial N$$

  is the boundary of an oriented $(m+1)$-dimensional manifold N, where $-M'$ is M' with the opposite orientation.
- The m-**dimensional oriented cobordism group** Ω_m is the abelian group of cobordism classes of oriented m-dimensional manifolds, with addition by disjoint union.
- **Examples**

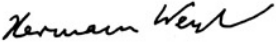
$$\Omega_0 \;=\; \mathbb{Z}\;,\; \Omega_1 \;=\; \Omega_2 \;=\; \Omega_3 \;=\; 0\;.$$

- **Slogan 5** The Witt groups of symmetric and quadratic forms are the algebraic analogues of the cobordism groups of manifolds.

The signature of manifolds

- **Slogan 6** Don't be ashamed to apply quadratic forms to topology!
- The **signature** of an oriented $4k$-dimensional manifold M^{4k} is

$$\text{signature}(M^{4k}) \;=\; \text{signature}(F^{2k}(M),\lambda) \in L^0(\mathbb{Z},1) \;=\; \mathbb{Z}\ .$$

- The signature of a manifold was first defined by Weyl in a 1923 paper http://www.maths.ed.ac.uk/~aar/surgery/weyl.pdf published in Spanish in South America to spare the author the shame of being regarded as a topologist. Here is Weyl's own signature: 

- **Theorem** (Thom, 1952, Hirzebruch, 1953) The signature is a cobordism invariant, determined by the tangent bundle τ_M

$$\sigma \;:\; \Omega_{4k} \to \mathbb{Z}\ ;\ M \mapsto \text{signature}(M^{4k}) \;=\; \langle \mathcal{L}(\tau_M),[M]\rangle\ .$$

If $M = \partial N$ is the boundary of an oriented $(4k+1)$-manifold N then $L = \text{im}(F^{2k}(N) \to F^{2k}(M))$ is a lagrangian of $(F^{2k}(M),\lambda)$, which is thus metabolic and has signature 0. σ is an isomorphism for $k=1$, onto for $k \geqslant 2$, with $\text{signature}(\mathbb{CP}^2 \times \mathbb{CP}^2 \times \cdots \times \mathbb{CP}^2) = 1$.

Quadratic forms

- **Definition** An ϵ-**quadratic form** (F,λ,μ) over A is an ϵ-symmetric form (F,λ) with a function

$$\mu \ : \ F \to Q_\epsilon(A) \ = \ \mathrm{coker}(1-\epsilon : A \to A)$$

such that for all $x,y \in F$, $a \in A$
  - $\lambda(x,x) \ = \ (1+\epsilon)\mu(x) \in A$
  - $\mu(ax) \ = \ a^2\mu(x) \ , \ \mu(x+y)-\mu(x)-\mu(y) \ = \ \lambda(x,y) \in Q_\epsilon(A)$.
- **Proposition** (Tits 1966, Wall 1970) The pairs (λ,μ) are in one-one correspondence with equivalence classes of $\psi \in \mathrm{Hom}_A(F,F^*)$ such that

$$\lambda(x,y) \ = \ \psi(x)(y)+\epsilon\psi(y)(x) \in A \ , \ \mu(x) \ = \ \psi(x)(x) \in Q_\epsilon(A) \ .$$

Equivalence: $\psi \sim \psi'$ if $\psi'-\psi = \chi-\epsilon\chi^*$ for some $\chi \in \mathrm{Hom}_A(F,F^*)$.
- An ϵ-symmetric form (F,λ) is a fixed point of the ϵ-duality

$$\lambda \in \ker(1-\epsilon* : \mathrm{Hom}_A(F,F^*) \to \mathrm{Hom}_A(F,F^*)) \ = \ H^0(\mathbb{Z}_2;\mathrm{Hom}_A(F,F^*))$$

while an ϵ-quadratic form (F,λ,μ) is an orbit

$$(\lambda,\mu) \ = \ [\psi] \in \mathrm{coker}(1-\epsilon*) \ = \ H_0(\mathbb{Z}_2;\mathrm{Hom}_A(F,F^*)) \ .$$

The ϵ-quadratic forms $H_\epsilon(L, \alpha, \beta)$

- **Definition** Given $(-\epsilon)$-symmetric forms (L, α), (L^*, β) over A define the nonsingular ϵ-quadratic form over A

$$\begin{aligned} H_\epsilon(L, \alpha, \beta) &= (L \oplus L^*, \lambda, \mu) \ , \\ \lambda((x_1, y_1), (x_2, y_2)) &= y_2(x_1) + \epsilon y_1(x_2) \ , \\ \mu(x, y) &= \alpha(x)(x) + \beta(y)(y) + y(x) \end{aligned}$$

  with L, L^* complementary lagrangians in the ϵ-symmetric form $(L \oplus L^*, \lambda)$.
- **Proposition** A nonsingular ϵ-quadratic form (F, λ, μ) is isomorphic to $H_\epsilon(L, \alpha, \beta)$ if and only if the ϵ-symmetric form (F, λ) is metabolic.
- **Proof** If $L \subset F$ is a lagrangian of (F, λ) and $\lambda = \psi + \epsilon\psi^*$ then there exists a complementary lagrangian $L^* \subset F$ for (F, λ), and

$$\psi = \begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix} , \ \psi + \epsilon\psi^* = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix} \ : \ F = L \oplus L^* \to F^* = L^* \oplus L$$

  with $\alpha + \epsilon\alpha^* = 0 : L \to L^*$, $\beta + \epsilon\beta^* = 0 : L^* \to L$.

The ϵ-quadratic Witt group

- **Definition** A nonsingular ϵ-quadratic form (F, λ, μ) is **hyperbolic** if there exists a lagrangian L for (F, λ) such that $\mu(L) = \{0\} \subseteq Q_\epsilon(A)$.
- **Proposition** Every hyperbolic form is isomorphic to $H_\epsilon(L, 0, 0) = (L \oplus L^*, \lambda, \mu)$ for some f.g. free A-module L, with

$$\lambda = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix} : F \times F \to A \; ; \; ((x_1, y_1), (x_2, y_2)) \mapsto y_2(x_1) + \epsilon y_1(x_2) \; ,$$
$$\mu : F \to Q_\epsilon(A) \; ; \; (x, y) \mapsto y(x) \; .$$

- **Definition** The **ϵ-quadratic Witt group of A** is

$$L_0(A, \epsilon) = \frac{\{\text{isomorphism classes of nonsingular } \epsilon\text{-quadratic forms over } A\}}{\{\text{hyperbolic forms}\}}$$

- The 4-periodic surgery obstruction groups $L_n(A)$ of Wall (1970) are

$$L_{2k}(A) = L_0(A, (-)^k) \; ,$$
$$L_{2k+1}(A) = L_1(A, (-)^k) = \varinjlim_j \mathrm{Aut}(H_{(-)^k}(A^j, 0, 0))^{ab} / \{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\} \; .$$

The forgetful map

- Forgetting the ϵ-quadratic structure defines a map

$$L_0(A,\epsilon) \to L^0(A,\epsilon)\ ;\ (F,\lambda,\mu) \mapsto (F,\lambda)\ .$$

- The kernel of the forgetful map is generated by

$$H_\epsilon(L,\alpha,\beta) \in \ker(L_0(A,\epsilon) \to L^0(A,\epsilon))\ .$$

- **Proposition** If $1/2 \in A$

$$\epsilon\text{-quadratic forms over } A \ = \ \epsilon\text{-symmetric forms over } A$$

and the forgetful map is an isomorphism

$$L_0(A,\epsilon) \xrightarrow{\cong} L^0(A,\epsilon)\ .$$

- **Proof** An ϵ-symmetric form (F,λ) over A has a unique ϵ-quadratic function

$$\mu\ :\ F \to Q_\epsilon(A)\ ;\ x \mapsto \lambda(x,x)/2\ .$$

Quadratic forms over $\mathbb{Z}_2$

- **Theorem** (Dickson, 1901) A nonsingular 1-quadratic form (F, λ, μ) over $\mathbb{Z}_2$ with $\dim_{\mathbb{Z}_2} F = 2g$ is isomorphic to

$$\begin{array}{ll} \text{either} & H_1(\bigoplus\limits_{g} \mathbb{Z}_2, 0, 0) \\ \text{or} & H_1(\mathbb{Z}_2, 1, 1) \oplus H_1(\bigoplus\limits_{g-1} \mathbb{Z}_2, 0, 0)\ . \end{array}$$

- The two cases are distinguished by the subsequent Arf invariant, and the Theorem gives

$$L_0(\mathbb{Z}_2, 1) \ = \ \mathbb{Z}_2\ .$$

- In fact, Dickson obtained such a classification for nonsingular 1-quadratic forms over any finite field of characteristic 2.

The signature of quadratic forms over $\mathbb{Z}$

- **Theorem** (van der Blij, 1958) The signature of a nonsingular 1-symmetric form (F, λ) over $\mathbb{Z}$ is such that

$$\text{signature}(F, \lambda) \;\equiv\; \lambda(v, v) \pmod 8$$

for any $v \in F$ such that $\lambda(x, x) \equiv \lambda(x, v) \pmod 2$ $(x \in F)$.

- For nonsingular 1-quadratic form (F, λ, μ) can take $v = 0 \in F$, so

$$\text{signature}(F, \lambda) \;\equiv\; 0 \pmod 8 \;.$$

- **Example** signature$(\mathbb{Z}^8, E_8) = 8$, with exact sequence

$$0 \longrightarrow L_0(\mathbb{Z}, 1) = \mathbb{Z} \xrightarrow{\;8\;} L^0(\mathbb{Z}, 1) = \mathbb{Z} \longrightarrow \mathbb{Z}_8 \longrightarrow 0 \;.$$

- **Theorem** (R., 1980) For any A, ϵ both the composites of

$$L_0(A, \epsilon) \to L^0(A, \epsilon) \;;\; (F, \lambda, \mu) \mapsto (F, \lambda) \;,$$
$$L^0(A, \epsilon) \to L_0(A, \epsilon) \;;\; (F, \lambda) \mapsto (\mathbb{Z}^8, E_8) \otimes (F, \lambda)$$

are multiplication by 8, so $L^0(A, \epsilon)$, $L_0(A, \epsilon)$ only differ in 8-torsion.

Čahit Arf (1910-1997)

- Turkish number theorist, student of Hasse in Göttingen, 1937-38
- A banker's view of the Arf invariant over $\mathbb{Z}_2$

- 10 Turkish Lira = €4.75

The Arf invariant I.

- Let K be a field of characteristic 2. A nonsingular 1-symmetric form (F, λ) over K is metabolic if and only if $\dim_K(F) \equiv 0 (\bmod 2)$. The function $L^0(K, 1) \to \mathbb{Z}_2; (F, \lambda) \mapsto \dim_K(F)$ is an isomorphism, and the forgetful map $L_0(K, 1) \to L^0(K, 1)$ is 0.
- The **Arf invariant** of a nonsingular 1-quadratic form (F, λ, μ) over K is
$$\mathrm{Arf}(F, \lambda, \mu) \; = \; \sum_{i=1}^{g} \mu(a_i)\mu(b_i) \in \mathrm{coker}(1 - \psi^2 : K \to K)$$
for any symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ of F, with
$$\lambda(a_i, a_j) \; = \; \lambda(b_i, b_j) \; = \; 0 \; , \; \lambda(a_i, b_j) \; = \; 1 \text{ if } i = j, \; = \; 0 \text{ if } i \neq j$$
and $\psi^2 : K \to K; x \mapsto x^2$ the Frobenius endomorphism.
- **Proposition** For 1-symmetric forms $\alpha = \alpha^* : L \to L^*$, $\beta = \beta^* : L^* \to L$ over K there exist $u \in L^*, v \in L$ with $\alpha(x)(x) = u(x) \in K$ $(x \in L)$, $\beta(y)(y) = y(v) \in K$ $(y \in L^*)$, and
$$\mathrm{Arf}(H_1(L, \alpha, \beta)) \; = \; u(v) \in \mathrm{coker}(1 - \psi^2 : K \to K) \; .$$

The Arf invariant II.

- **Definition** A field K of characteristic 2 is **perfect** if $\psi^2 : K \to K$ is an automorphism, i.e. every $k \in K$ has a square root $\sqrt{k} \in K$.
- **Theorem** (Arf, 1941) If K is perfect then
  - (i) Every nonsingular 1-quadratic form over K is isomorphic to one of the type $H_1(L, \alpha, \beta)$.
  - (ii) There is an isomorphism $H_1(L, \alpha, \beta) \cong H_1(L', \alpha', \beta')$ if and only if
  $$\dim_{\mathbb{Z}_2}(L) \;=\; \dim_{\mathbb{Z}_2}(L') \;,\; \mathrm{Arf}(H_1(L, \alpha, \beta)) \;=\; \mathrm{Arf}(H_1(L', \alpha', \beta')) \;.$$
  - (iii) The Arf invariant defines an isomorphism
  $$\mathrm{Arf} \;:\; L_0(K, 1) \xrightarrow{\;\cong\;} \mathrm{coker}(1 - \psi^2) \;;\; (F, \lambda, \mu) \mapsto \mathrm{Arf}(F, \lambda, \mu) \;.$$
- **Example** For $K = \mathbb{Z}_2$ have isomorphism
$$\mathrm{Arf} \;:\; L_0(\mathbb{Z}_2, 1) \xrightarrow{\;\cong\;} \mathrm{coker}(1 - \psi^2 : \mathbb{Z}_2 \to \mathbb{Z}_2) \;=\; \mathbb{Z}_2 \;.$$

5 formulae for the Arf invariant over $\mathbb{Z}_2$

- **Formula 1** (Klingenberg+Witt, 1954) The Arf invariant of the nonsingular 1-quadratic form $H_1(L,\alpha,\beta)$ over $\mathbb{Z}_2$ is

$$\mathrm{Arf}(F,\lambda,\mu) \;=\; \mathrm{trace}(\beta\alpha : L \to L) \in \mathbb{Z}_2\ .$$

- **Formula 2** (M.Kneser, 1954) Centre of Clifford algebra.
- **Formula 3** (W.Browder, 1972) The majority vote

$$\mathrm{Arf}(F,\lambda,\mu) \;=\; \mathrm{majority}\{\mu(x)\,|\,x \in F\} \in \mathbb{Z}_2 \;=\; \{0,1\}\ .$$

- **Formula 4** (E.H.Brown, 1972) Gauss sum

$$\mathrm{Arf}(F,\lambda,\mu) \;=\; \Big(\sum_{x\in F} e^{\pi i\mu(x)}\Big)/\sqrt{|F|} \in \mathbb{Z}_2 \;=\; \{1,-1\}\ .$$

- **Formula 5** (Lannes, 1981) If $v \in F$ is such that

$$\mu(x) \;=\; \lambda(x,v) \in \mathbb{Z}_2\ (x \in L)$$

then

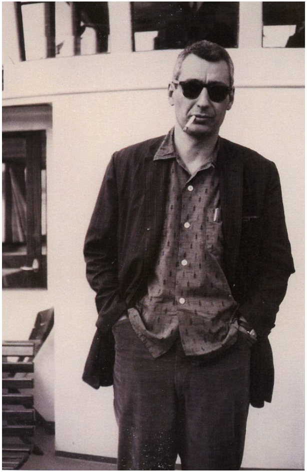
$$\mathrm{Arf}(F,\lambda,\mu) \;=\; \mu(v) \in \mathbb{Z}_2\ .$$

Framed manifolds

- A **framing** of an m-dimensional differentiable manifold M^m is an embedding $M \times \mathbb{R}^j \subset \mathbb{R}^{m+j}$ (j large). Equivalent to a stable trivialization of the tangent bundle τ_M as given by a vector bundle isomorphism
$$\delta\tau_M \ : \ \tau_M \oplus \epsilon^j \ \cong \ \epsilon^{m+j} \ .$$
- **Slogan 7** Framed manifolds have $\pm$-quadratic forms.
- **Theorem** (Pontrjagin, 1955) (i) Isomorphism between the m-**dimensional framed cobordism group** Ω_m^{fr} and the stable homotopy group
$$\pi_m^S = \varinjlim_j \pi_{m+j}(S^j) \xrightarrow{\cong} \Omega_m^{fr} \ ; \ (f : S^{m+j} \to S^j) \mapsto M^m \ = \ f^{-1}(\mathrm{pt.}) \ .$$
- (ii) $\Omega_0^{fr} = \mathbb{Z}$, $\Omega_1^{fr} = \mathbb{Z}_2$ (Hopf invariant), $\Omega_2^{fr} \ = \ \mathbb{Z}_2$ (Arf invariant).
- (iii) The Arf invariant of $M^2 \times \mathbb{R}^j \subset \mathbb{R}^{j+2}$ was defined using the quadratic form $(H_1(M;\mathbb{Z}_2), \lambda, \mu)$ over $\mathbb{Z}_2$ with
$$\mu(S^1 \subset M) \ = \ \mathrm{Hopf}(S^1 \times \mathbb{R} \times \mathbb{R}^j \subset M \times \mathbb{R}^j \subset \mathbb{R}^{j+2}) \in \Omega_1^{fr} \ = \ \mathbb{Z}_2 \ .$$

Michel Kervaire (1927-2007)

- French topologist, student of Hopf in Zürich.
- Worked in New York and Geneva.



The quadratic form of a framed $(4k+2)$-manifold

- **Theorem** (Kervaire, K-Milnor, Browder, Brown, ..., 1960's) A framed $(4k+2)$-dimensional manifold $(M^{4k+2}, \delta\tau_M)$ has a nonsingular 1-quadratic form $(H_{2k+1}(M;\mathbb{Z}_2), \lambda, \mu)$ over $\mathbb{Z}_2$, with μ determined by $\delta\tau_M$
- General construction uses the embedding $M^{4k+2} \times \mathbb{R}^j \subset \mathbb{R}^{j+4k+2}$, the Umkehr map
$$(\mathbb{R}^{j+4k+2})^\infty \;=\; S^{j+4k+2} \to (M^{4k+2} \times \mathbb{R}^j)^\infty \;=\; \Sigma^j M^+$$
and functional Steenrod squares.
- The normal bundle of an embedding $x : S^{2k+1} \subset M^{4k+2}$ is a $(2k+1)$-plane vector bundle ν_x over S^{2k+1} with a stable trivialization $\delta\nu_x : \nu_x \oplus \epsilon^j \cong \epsilon^{j+2k+1}$. Such pairs are classified by a $\mathbb{Z}_2$-invariant, and
$$\mu(x) \;=\; (\delta\nu_x, \nu_x) \in \pi_{2k+2}(BO(j+2k+1), BO(2k+1)) \;=\; \mathbb{Z}_2 \;.$$
- Can also define $\mu(x) \in \mathbb{Z}_2$ geometrically using the self-intersections of immersions $x : S^{2k+1} \looparrowright M^{4k+2}$ determined by the framing.

The Kervaire invariant of a framed $(4k+2)$-manifold

- **Definition** The **Kervaire invariant** of $(M^{4k+2}, \delta\tau_M)$ is
$$\text{Kervaire}(M, \delta\tau_M) \ = \ \text{Arf}(H_{2k+1}(M;\mathbb{Z}_2), \lambda, \mu) \in \mathbb{Z}_2$$
defining a function
$$\text{K} = \text{Kervaire} \ : \ \Omega^{fr}_{4k+2} \ = \ \pi^S_{4k+2} \to L_{4k+2}(\mathbb{Z}, 1) \ = \ L_0(\mathbb{Z}_2, 1) \ = \ \mathbb{Z}_2 \ .$$
- **Example** For $k = 0, 1, 3$ K is onto: there exists a framing $\delta\tau_M$ of $M = S^{2k+1} \times S^{2k+1}$ with $K(M) = 1$.
- **Theorem** (K, 1960) For $k = 2$ $K = 0$ and there exists a 10-dimensional PL (= piecewise linear) manifold without differentiable structure.
- **Theorem** (K-Milnor, 1963) (i) For $k \geqslant 2$ every framed $4k$-manifold M has signature$(M) = 0$ and is framed cobordant to an exotic sphere.
(ii) A framed $(4k+2)$-manifold M is framed cobordant to an exotic sphere if and only if $K(M) = 0 \in \mathbb{Z}_2$. Thus $K(M)$ is a surgery obstruction.
- Google: 4,500 hits for *Arf invariant*, and 4,000 hits for *Kervaire invariant*.

The Kervaire invariant problem

- **Problem** (1963) For which dimensions $4k+2$ is the function $K : \Omega^{fr}_{4k+2} \to \mathbb{Z}_2$ onto?
- **Slogan 8** The Kervaire invariant problem is a key to understanding the homotopy groups of spheres.
- K is onto for $4k+2 = 2, 6, 14, 30, 62$.
- Browder (1969) If K is onto then

$$4k+2 \ = \ 2^i - 2 \text{ for some } i \geqslant 2 \ .$$

- Two independent solutions have been announced:
  - Akhmetev (2008): heavy duty geometry, K is onto for a finite number of dimensions.
  - Hopkins-Hill-Ravenel (2009): heavy duty algebraic topology, if K is onto then $4k+2 \in \{2, 6, 14, 30, 62, 126\}$. The case $4k+2 = 126$ is still unresolved.
- http://www.maths.ed.ac.uk/˜aar/atiyah80.htm
- http://www.math.rochester.edu/u/faculty/doug/kervaire.html

ϵ-quadratic and ϵ-symmetric structures on chain complexes

- Define the ϵ-symmetric and ϵ-quadratic forms on an A-module F

$$\begin{aligned} \mathrm{Sym}(F,\epsilon) &= \ker(1-T_\epsilon : \mathrm{Hom}_A(F,F^*) \to \mathrm{Hom}_A(F,F^*))\ , \\ \mathrm{Quad}(F,\epsilon) &= \mathrm{coker}(1-T_\epsilon : \mathrm{Hom}_A(F,F^*) \to \mathrm{Hom}_A(F,F^*)) \end{aligned}$$

  with T_ϵ the ϵ-duality involution $T_\epsilon\lambda(x)(y) = \epsilon\lambda(y)(x)$.
- **Slogan 9** Use chain complexes to model manifolds in algebra!
- Given an A-module chain complex C define the $\mathbb{Z}_2$-hypercohomology and $\mathbb{Z}_2$-hyperhomology

$$\begin{aligned} Q^n(C,\epsilon) &= H^n(\mathbb{Z}_2; C\otimes_A C) = H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C\otimes_A C))\ , \\ Q_n(C,\epsilon) &= H_n(\mathbb{Z}_2; C\otimes_A C) = H_n(W\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C\otimes_A C)) \end{aligned}$$

  with $T(x\otimes y) = \epsilon y\otimes x$ and W the free $\mathbb{Z}[\mathbb{Z}_2]$-resolution of $\mathbb{Z}$

$$W\ :\ \ldots \longrightarrow \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2]$$

- **Example** If $C_r = 0$ for $r \neq 0$

$$Q^0(C,\epsilon) = \mathrm{Sym}(C_0^*,\epsilon)\ ,\ Q_0(C,\epsilon) = \mathrm{Quad}(C_0^*,\epsilon)\ .$$

The generalized ϵ-symmetric Witt groups $L^n(A,\epsilon)$

- The ϵ-**symmetric** L-**groups** $L^n(A,\epsilon)$ are the algebraic cobordism groups of n-dimensional f.g. free A-module chain complexes C with a class $\phi \in Q^n(C,\epsilon)$ inducing a Poincaré duality

$$H^{n-*}(C) \cong H_*(C) .$$

- **Example** $L^0(A,\epsilon)$ is the Witt group of nonsingular ϵ-symmetric forms.
- $L^*(A,1)$ = the Mishchenko symmetric L-groups
- **Example** An oriented n-dimensional manifold M with universal cover $\widetilde{M}$ has a symmetric signature

$$\sigma^*(M) = (C(\widetilde{M}),\phi) \in L^n(\mathbb{Z}[\pi_1(M)],1) .$$

- Generalization of the signature: the special case $n = 4k$, $\pi_1(M) = \{1\}$

$$\sigma^*(M) = \text{signature}(M) \in L^{4k}(\mathbb{Z},1) = \mathbb{Z} .$$

The generalized ϵ-quadratic Witt groups $L_n(A,\epsilon)$

- The ϵ-**quadratic** L-**groups** $L_n(A)$ are the algebraic cobordism groups of n-dimensional f.g. free A-module chain complexes C with a class $\psi \in Q_n(C,\epsilon)$ inducing a Poincaré duality

$$H^{n-*}(C) \;\cong\; H_*(C)\ .$$

- $L_*(A,1) =$ the Wall surgery obstruction groups.
- **Example** $L_0(A,\epsilon)$ is the Witt group of nonsingular ϵ-symmetric forms.
- **Example** A degree 1 map of n-dimensional manifolds $f : M \to X$ with normal bundle map b has a **quadratic signature**

$$\sigma_*(f,b) \;=\; (C(\widetilde{f} : C(\widetilde{M} \to \widetilde{X}))_{*+1},\psi) \in L_n(\mathbb{Z}[\pi_1(X)],1)\ ,$$

the Wall surgery obstruction.
- Generalization of the Arf-Kervaire invariant: in the special case $n = 4k+2$, $X = S^{4k+2}$, $(M,\delta\tau_M)=$ framed manifold

$$\sigma_*(f,b) \;=\; \text{Kervaire}(M,\delta\tau_M) \in L_{4k+2}(\mathbb{Z},1) \;=\; \mathbb{Z}_2\ .$$

The L-groups $L_*(A,\epsilon)$, $L^*(A,\epsilon)$ and $\widehat{L}^*(A,\epsilon)$

- **Slogan 10** The ϵ-symmetric and ϵ-quadratic L-groups are related by the exact sequence

$$\dots \to L^{n+1}(A,\epsilon) \to \widehat{L}^{n+1}(A,\epsilon) \to L_n(A,\epsilon) \to L^n(A,\epsilon) \to \widehat{L}^n(A,\epsilon) \to \dots .$$

  The relative groups $\widehat{L}^*(A,\epsilon)$ (of exponent 8) are **homological invariants** of the ring A, not just Grothendieck-Witt groups.
- **Example** For a perfect field A of characteristic 2

$$\begin{aligned} \widehat{L}^1(A,1) &= \operatorname{coker}(1-\psi^2 : A \to A) = A/\{a - a^2 \mid a \in A\} , \\ \widehat{L}^0(A,1) &= \ker(1-\psi^2 : A \to A) = \{a \in A \mid a^2 = a\} = \mathbb{Z}_2 . \end{aligned}$$

- **Example** For $A = \mathbb{Z}$ recover van der Blij's theorem

$$\operatorname{coker}(L_0(\mathbb{Z},1) \to L^0(\mathbb{Z},1)) = \operatorname{coker}(8 : \mathbb{Z} \to \mathbb{Z}) \xrightarrow{\cong} \widehat{L}^0(\mathbb{Z},1) = \mathbb{Z}_8 ;$$

$$(F,\lambda) \mapsto \lambda(v,v) \equiv \text{signature}(F,\lambda)\ (\lambda(v,x) \equiv \lambda(x,x) \pmod 2 \forall x \in F).$$

The generalized Arf invariant

- **Definition** (Banagl and R., 2006) Given a $(-\epsilon)$-symmetric form (L,α) over a ring with involution A define the **generalized Arf group**

$$\mathrm{Arf}(L,\alpha) \ = \ \frac{\{\beta \in \mathrm{Hom}_A(L^*,L)\,|\,\beta^* = -\epsilon\beta\}}{\{\phi - \phi\alpha\phi^* + (\chi - \epsilon\chi^*)\,|\,\phi^* = -\epsilon\phi, \chi \in \mathrm{Hom}_A(L^*,L)\}}$$

- **Proposition** (i) The function $\beta \mapsto H_\epsilon(L,\alpha,\beta)$ defines a one-one correspondence between $\mathrm{Arf}(L,\alpha)$ and the isomorphism classes of nonsingular ϵ-quadratic forms (F,λ,μ) over A with a lagrangian L for the ϵ-symmetric form (F,λ) such that $\mu|_L = \alpha$, with

$$F \ = \ L \oplus L^* \ , \ \mu(x,y) \ = \ \alpha(x)(x) + \beta(y)(y) + y(x) \in Q_\epsilon(A) \ .$$

- (ii) The map $\mathrm{Arf}(L,\alpha) \to \ker(L_0(A,\epsilon) \to L^0(A,\epsilon)); \beta \mapsto H_\epsilon(L,\alpha,\beta)$ is an isomorphism if A is a perfect field of characteristic 2 and $(L,\alpha) = (A,1)$, with $\mathrm{Arf}(L,\alpha) = \mathrm{coker}(1-\psi^2 : A \to A)$.
- The generalized Arf invariant can be used to compute $L_*(\mathbb{Z}[D_\infty])$ with $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$ the infinite dihedral group.

References

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