ROOTS OF POLYNOMIALS AND QUADRATIC FORMS

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Introduction

- In 1829 Sturm proved a theorem calculating the number of real roots of a non-zero real polynomial $P(X) \in \mathbb{R}[X]$ in an interval $[a, b] \subset \mathbb{R}$, using the Euclidean algorithm in $\mathbb{R}[X]$ and counting sign changes.
- In 1853 Sylvester interpreted Sturm's theorem using continued fractions and the signature of a tridiagonal quadratic form.
- The survey paper of Étienne Ghys and A.R. http://arxiv.org/abs/1512.09258 *Signatures in algebra, topology and dynamics* includes a modern interpretation of the results of Sturm and Sylvester in terms of the "Witt group" of stable isomorphism classes of invertible symmetric matrices.

Jacques Charles François Sturm (1803-1855)

Sturm's problem

- **Problem** How many real roots of $P(X) \in \mathbb{R}[X]$ are there in an interval $[a, b] \subset \mathbb{R}$? At the time, this was a major problem in analysis, algebra and numerical mathematics.
- **Sturm's formula** The Euclidean algorithm in $\mathbb{R}[X]$ for finding the greatest common divisor of $P_0(X) = P(X)$ and $P_1(X) = P'(X)$ gives the **Sturm sequences** of polynomials

  $$(P_*(X), Q_*(X)) = ((P_0(X), \ldots, P_n(X)), (Q_1(X), \ldots, Q_n(X)))$$

  with remainders $P_j(X)$ and quotients $Q_j(X)$, such that

  $$\deg(P_{j+1}(X)) < \deg(P_j(X)) \leqslant n - j \ (0 \leqslant j \leqslant n) \ ,$$
  $$P_{j-1}(X) + P_{j+1}(X) \ = \ P_j(X)Q_j(X) \ (1 \leqslant j \leqslant n) \ .$$
- Sturm's formula expressed the number of real roots of $P(X)$ in $[a, b]$ in terms of the variation ($=$ number of sign changes) in $P_*(a)$ and $P_*(b)$, assuming regularity.

The Euclidean algorithm

- The **Euclidean algorithm** for the greatest common divisor of integers $\pi_0 \geqslant \pi_1 \geqslant 1$ is the sequence pair

$$\pi_0 \geqslant \pi_1 > \cdots > \pi_n > \pi_{n+1} = 0 \ , \ \rho_0, \rho_1, \dots, \rho_n > 0$$

  with

$$\begin{aligned} \pi_{j-1} &= \pi_j \rho_j + \pi_{j+1} \ (1 \leqslant j \leqslant n) \ , \\ \rho_j &= \lfloor \pi_{j-1}/\pi_j \rfloor = \text{quotient when dividing } \pi_{j-1} \text{ by } \pi_j \ , \\ \pi_{j+1} &= \text{remainder} \ , \ \pi_n = \text{g.c.d.}(\pi_0, \pi_1) \ . \end{aligned}$$

- The sequences $(\pi_0/\pi_1, \pi_1/\pi_2, \dots, \pi_{n-1}/\pi_n)$, $(\rho_1, \rho_2, \dots, \rho_n)$ determine each other by

$$\frac{\pi_{j-1}}{\pi_j} = \rho_j + \cfrac{1}{\rho_{j+1} + \cfrac{1}{\rho_{j+2} + \ddots + \cfrac{1}{\rho_n}}} \ , \ \rho_j = \frac{\pi_{j-1}}{\pi_j} - \frac{\pi_{j+1}}{\pi_j} \ .$$

Euclidean pairs

- **Definition** A sequence $p_* = (p_0, p_1, \dots, p_n)$ of $p_j \in \mathbb{R}$ is **regular** if $p_j \neq 0 \in \mathbb{R}$ for $0 \leqslant j \leqslant n$.
- **Definition** A **Euclidean pair** (p_*, q_*) consists of two regular sequences $p_* = (p_0, p_1, \dots, p_n)$, $q_* = (q_1, q_2, \dots, q_n)$ in $\mathbb{R}$ satisfying the identities

$$p_{j-1} + p_{j+1} \;=\; p_j q_j \in \mathbb{R} \;(1 \leqslant j \leqslant n,\; p_{n+1} = 0)\ .$$

- **Example** For integers $\pi_0 \geqslant \pi_1 \geqslant 1$ the Euclidean algorithm sequences $(\pi_0, \pi_1, \dots, \pi_n)$, $(\rho_1, \rho_2, \dots, \rho_n)$ determine a Euclidean pair (p_*, q_*) by

$$p_j \;=\; (-1)^{j(j-1)/2}\pi_j\ ,\;\; q_j \;=\; \rho_j\ .$$

Variation and regularity

- **Definition** The **variation** of a regular sequence $p_* = (p_0, p_1, \ldots, p_n)$ in $\mathbb{R}$ is

$$\begin{aligned} \mathrm{var}(p_*) &= \text{number of changes of sign in } p_* \\ &= \Big(n - \sum_{j=1}^{n} \mathrm{sign}(p_{j-1}/p_j)\Big)/2 \in \{0, 1, \ldots, n\} \ . \end{aligned}$$

- **Definition** A polynomial $P(X) \in \mathbb{R}[X]$ is **regular** if it has no repeated roots.
- **Definition** A point $t \in \mathbb{R}$ is **regular** for $P(X) \in \mathbb{R}[X]$ if

$$P_*(t) \ = \ (P_0(t), P_1(t), \ldots, P_n(t)) \ ,$$

is a regular sequence in $\mathbb{R}$.

Sturm's Theorem (1829)

- **Theorem** The number of real roots of a regular $P(X) \in \mathbb{R}[X]$ in $[a, b] \subset \mathbb{R}$ for regular $a < b$ is

$$|\{x \in [a, b] \mid P(x) = 0 \in \mathbb{R}\}| \;=\; \mathrm{var}(P_*(a)) - \mathrm{var}(P_*(b)) \;.$$

- **Idea of proof** Let $a = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = b$ be the partition of $[a, b]$ at the points $t_1 < t_2 < \cdots < t_{N-1}$ which are not regular. For each $i \in \{1, 2, \ldots, N-1\}$ there is a unique $j_i \in \{0, 1, \ldots, n-1\}$ such that

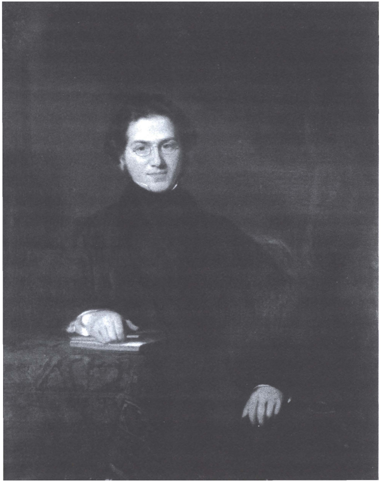
$$P_{j_i}(t_i) \;=\; 0 \;,\; P_k(t_i) \neq 0 \text{ for } k \neq j_i \;.$$

The function

$$[a, b] \to \{0, 1, \ldots, n\} \;;\; t \mapsto \mathrm{var}(P_*(a)) - \mathrm{var}(P_*(t))$$

is constant for $t \in (t_i, t_{i+1})$. The jump is 1 at t_i with $j_i = 0$, i.e. at the real roots of $P(X)$. The jump is 0 at t_i with $j_i \geqslant 1$, since $p_{j_i-1}(t_i) + p_{j_i+1}(t_i) = p_{j_i}(t_i)q_{j_i}(t_i) = 0$ with the first two terms $\neq 0$.

James Joseph Sylvester (1814-1897)



Sylvester's papers related to Sturm's theorem

- *On the relation of Sturm's auxiliary functions to the roots of an algebraic equation.* (1841)
- *A demonstration of the theorem that every homogeneous quadratic polynomial is reducible by real orthogonal substitutions to the form of a sum of positive and negative squares.* (1852)
- *On a remarkable modification of Sturm's Theorem* (1853)
- *On a theory of the syzygetic relations of two rational integral functions, comprising an application to the theory of Sturm's functions, and that of the greatest algebraical common measure.* (1853)
- Sylvester used continued fractions to express Sturm's formula in terms of the signatures of tridiagonal symmetric forms. In fact, the signature was developed for just this purpose!

Cauchy's Spectral Theorem (1829)

- **Definition** The **transpose** of an $n \times n$ matrix $A = (a_{ij})$ is

$$A^* = (a_{ji}) .$$

- **Definition** The symmetric $n \times n$ matrices S, T in $\mathbb{R}$ are **orthogonally congruent** if

$$T = A^* S A$$

for an $n \times n$ matrix A which is orthogonal, $A^* A = I_n$.

- **Spectral Theorem**
(i) The eigenvalues of symmetric S are real.
(ii) Symmetric S, T are orthogonally congruent if and only if they have the same eigenvalues.

Sylvester's Law of Inertia

- **Definition** Let S be a symmetric $n \times n$ matrix in $\mathbb{R}$.
  (i) The **positive index** $\tau_+(S) \geqslant 0$ of S is the dimension of a maximal subspace $V_+ \subseteq \mathbb{R}^n$ such that $S(x,x) > 0$ for all $x \in V_+ \backslash \{0\}$.
  (ii) The **negative index** $\tau_-(S) \geqslant 0$ of S is the dimension of a maximal subspace $V_- \subseteq \mathbb{R}^n$ such that $S(x,x) < 0$ for all $x \in V_- \backslash \{0\}$.
- **Definition** Symmetric $n \times n$ matrices S, T are **linearly congruent** if
  $$T \ = \ A^* S A$$
  for an invertible $n \times n$ matrix A.
- **Law of Inertia** (1852) S, T are linearly congruent if and only if
  $$(\tau_+(S), \tau_-(S)) \ = \ (\tau_+(T), \tau_-(T)) \ .$$

The signature

- **Definition** The **signature** of a symmetric $n \times n$ matrix S in $\mathbb{R}$ is

$$\tau(S) \;=\; \tau_+(S) - \tau_-(S) \in \{-n, -n+1, \dots, n-1, n\}\ .$$

- The following conditions on S are equivalent:
  - S is invertible,
  - $\tau_+(S) + \tau_-(S) = n$
  - the eigenvalues constitute a regular sequence $\lambda_* = (\lambda_1, \lambda_2, \dots, \lambda_n)$, i.e. each $\lambda_j \neq 0$.
- **Proposition** For invertible S

$$\tau(S) \;=\; \sum_{j=1}^{n} \mathrm{sign}(\lambda_j) \;=\; n - 2\,\mathrm{var}(\mu_*)$$

with $\mu_j = \lambda_1 \lambda_2 \dots \lambda_j$ $(1 \leqslant j \leqslant n)$ and $\mu_0 = 1$.

The principal minors and the Sylvester-Jacobi-Gundelfinger-Frobenius Theorem

- **Definition** The **principal minors** of an $n \times n$ matrix $S = (s_{ij})_{1 \leqslant i,j \leqslant n}$ in $\mathbb{R}$ are the determinants of the principal submatrices $S(k) = (s_{ij})_{1 \leqslant i,j \leqslant k}$

$$\mu_k(S) \ = \ \det(S(k)) \in \mathbb{R} \ (1 \leqslant k \leqslant n) \ .$$

  For $k = 0$ set $\mu_0(S) = 1$.
- **Theorem** (Sylvester (1853), Jacobi (1857), Gundelfinger (1881), Frobenius (1895))
  The signature of a symmetric $n \times n$ matrix S in $\mathbb{R}$ with the principal minors $\mu_k = \mu_k(S)$ constituting a regular sequence $\mu_* = (\mu_0, \mu_1, \dots, \mu_n)$ is

$$\tau(S) \ = \ \sum_{k=1}^{n} \mathrm{sign}(\mu_k/\mu_{k-1}) \ = \ n - 2\,\mathrm{var}(\mu_*) \ .$$

- There is a proof in the survey, using "plumbing" of matrices.

The tridiagonal symmetric matrix

- **Definition** The **tridiagonal symmetric matrix** of a sequence $q_* = (q_1, q_2, \ldots, q_n)$ in $\mathbb{R}$ is

$$\mathrm{Tri}(q_*) \;=\; \begin{pmatrix} q_1 & 1 & 0 & \ldots & 0 \\ 1 & q_2 & 1 & \ldots & 0 \\ 0 & 1 & q_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & q_n \end{pmatrix}$$

- Sylvester observed that every continued fraction is the ratio of successive principal minors $\mu_k = \mu_k(\mathrm{Tri}(q_*))$

$$\mu_k/\mu_{k-1} \;=\; q_k - \cfrac{1}{q_{k-1} - \cfrac{1}{q_{k-2} - \ddots - \cfrac{1}{q_1}}}$$

and $\tau(\mathrm{Tri}(q_*)) = \sum\limits_{k=1}^{n} \mathrm{sign}(\mu_k/\mu_{k-1}) = n - 2\,\mathrm{var}(\mu_*)$.

Sylvester's mathematical inspiration

- For a Euclidean pair (p_*, q_*) the regular sequences $(p_0/p_1, p_1/p_2, \ldots, p_{n-1}/p_n)$ and q_* determine each other by

$$\frac{p_{j-1}}{p_j} = q_j - \cfrac{1}{q_{j+1} - \cfrac{1}{q_{j+2} - \ddots - \cfrac{1}{q_n}}} \ , \ q_j = \frac{p_{j-1}}{p_j} + \frac{p_{j+1}}{p_j} \ .$$

- For his modification of Sturm's theorem Sylvester needed an expression for $\tau(\mathrm{Tri}(q_*))$ in terms of p_*. He could not obtain it directly, so he reversed $q_* = (q_1, q_2, \ldots, q_n)$ to define

$$q'_* = (q_n, q_{n-1}, \ldots, q_1)$$

with $\dfrac{p_{j-1}}{p_j} = \dfrac{\mu_{n-j+1}(\mathrm{Tri}(q'_*))}{\mu_{n-j}(\mathrm{Tri}(q'_*))}$ and $\tau(\mathrm{Tri}(q'_*)) = n - 2\,\mathrm{var}(p_*)$. He then observed that $\mathrm{Tri}(q_*)$, $\mathrm{Tri}(q'_*)$ are linearly congruent, so that $\tau(\mathrm{Tri}(q_*)) = \tau(\mathrm{Tri}(q'_*)) = n - 2\,\mathrm{var}(p_*)$.

Sylvester's modification of Sturm

- **Theorem** (1853) For a Euclidean pair $(p_*, q_*) = ((p_0, p_1, \ldots, p_n), (q_1, q_2, \ldots, q_n))$

$$\tau(\mathrm{Tri}(q_*)) \;=\; \sum_{k=1}^{n} \mathrm{sign}(p_k/p_{k-1}) \;=\; n - 2\,\mathrm{var}(p_*)\ .$$

For regular $P(X) \in \mathbb{R}[X]$ and regular $a < b$ with Sturm sequences $(P_*(X), Q_*(X))$ the number of real roots in $[a, b]$ is

$$\begin{aligned} &|\{x \in [a, b] \mid P(x) = 0 \in \mathbb{R}\}| \\ &\qquad = \mathrm{var}(P_*(a)) - \mathrm{var}(P_*(b)) \\ &\qquad = \big(\tau(\mathrm{Tri}(Q_*(b))) - \tau(\mathrm{Tri}(Q_*(a)))\big)/2\ . \end{aligned}$$

Proof of Sylvester's Theorem I.

- (i) The principal minors $\mu'_k = \mu_k(\mathrm{Tri}(q'_*))$ of the tridiagonal symmetric $n \times n$ matrix

$$\mathrm{Tri}(q'_*) \;=\; \begin{pmatrix} q_n & 1 & 0 & \dots & 0 \\ 1 & q_{n-1} & 1 & \dots & 0 \\ 0 & 1 & q_{n-2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & q_1 \end{pmatrix}$$

constitute a regular sequence $(\mu'_1, \mu'_2, \dots, \mu'_n)$ such that

$$\frac{\mu'_{j-1}}{\mu'_j} \;=\; \frac{p_{n-j+1}}{p_{n-j}} \;(1 \leqslant j \leqslant n)\;.$$

(ii) The signature of $\mathrm{Tri}(q'_*)$ is

$$\tau(\mathrm{Tri}(q'_*)) \;=\; n - 2\,\mathrm{var}(\mu'_*) \;=\; n - 2\,\mathrm{var}(p_*)\;.$$

Proof of Sylvester's Theorem II.

- (iii) The invertible $n \times n$ matrix

$$A = \begin{pmatrix} 0 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 1 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

is such that

$$\mathrm{Tri}(q_*) = A^*\mathrm{Tri}(q'_*)A ,$$

so that by the Law of Inertia

$$\tau(\mathrm{Tri}(q_*)) = \tau(\mathrm{Tri}(q'_*)) = n - 2\,\mathrm{var}(p_*) .$$

Sylvester's musical inspiration

As an artist delights in recalling the particular time and atmospheric effects under which he has composed a favourite sketch, so I hope to be excused putting upon record that it was in listening to one of the magnificent choruses in the 'Israel in Egypt' that, unsought and unsolicited, like a ray of light, silently stole into my mind the idea (simple, but previously unperceived) of the equivalence of the Sturmian residues to the denominator series formed by the reverse convergents. The idea was just what was wanting,—the key-note to the due and perfect evolution of the theory.

A magnificent chorus from *Israel in Egypt*