MANIFOLDS AND DUALITY

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- Classification of manifolds
- Uniqueness Problem
- Existence Problem
- Quadratic algebra
- Applications

Manifolds

- An n-dimensional manifold M^n is a topological space which is locally homeomorphic to $\mathbb{R}^n$.
  - compact, oriented, connected.

- Classification of manifolds up to homeomorphism.
  - For $n = 1$: circle
  - For $n = 2$:
    sphere, torus, ..., handlebody.
  - For $n \geq 3$: in general impossible.

The Uniqueness Problem

- *Is every homotopy equivalence of n-dimensional manifolds*

$$f \ : \ M^n \ \to \ N^n$$

homotopic to a homeomorphism?

  - For $n = 1, 2$: Yes.

  - For $n \geq 3$: in general No.

The Poincaré conjecture

- *Every homotopy equivalence $f : M^3 \to S^3$ is homotopic to a homeomorphism.*
  - Stated in 1904 and still unsolved!

- **Theorem**
  ($n \geq 5$: Smale, 1960, $n = 4$: Freedman, 1983)
  Every homotopy equivalence $f : M^n \to S^n$ is homotopic to a homeomorphism.

Old solution of the Uniqueness Problem

- Surgery theory works best for $n \geq 5$.
  - From now on let $n \geq 5$.

- **Theorem**
  (Browder, Novikov, Sullivan, Wall, 1970)
  A homotopy equivalence $f : M^n \to N^n$ is homotopic to a homeomorphism if and only if two obstructions vanish.

- The 2 obstructions of surgery theory:

  1. In the <u>topological K-theory</u> of vector bundles over N.

  2. In the <u>algebraic L-theory</u> of quadratic forms over the fundamental group ring $\mathbb{Z}[\pi_1(N)]$.

Traditional surgery theory

- Advantage:
  - Suitable for <u>computations</u>.

- Disadvantages:
  - <u>Inaccessible</u>.
  - A <u>complicated</u> mix of topology and algebra.
  - Passage from a homotopy equivalence to the obstructions is <u>indirect</u>.
  - Obstructions are <u>not independent</u>.

Wall's programme

- "The theory of quadratic structures on chain complexes should provide a simple and satisfactory algebraic version of the whole setup."
  - C.T.C.Wall, *Surgery on compact manifolds*, 1970

- Such a theory is now available.
  - Ranicki, *Algebraic L-theory and topological manifolds*, 1992

Siebenmann's theorem

- The kernel groups of a map $f : M \to N$ are the relative homology groups
$$K_r(x) \ = \ H_{r+1}(f^{-1}(x) \to \{x\}) \ (x \in N) \ .$$

- Exact sequence
$$\dots \to K_r(x) \to H_r(f^{-1}(x)) \to H_r(\{x\}) \to K_{r-1}(x) \to \dots \ .$$

- $K_*(x) = 0$ for a homeomorphism f.

- **Theorem** (Siebenmann, 1972) *A homotopy equivalence $f : M^n \to N^n$ with*
$$K_*(x) \ = \ 0 \ (x \in N)$$
is homotopic to a homeomorphism.

New solution of the Uniqueness Problem

- The total surgery obstruction $s(f)$ of a homotopy equivalence $f : M^n \to N^n$ is the cobordism class of
  - the sheaf of $\mathbb{Z}$-module chain complexes
  - with n-dimensional Poincaré duality
  - over N
  - with stalk homology $K_*(x)$ $(x \in N)$.
- Cobordism and Poincaré duality are algebraic.
- **Theorem** *A homotopy equivalence f is homotopic to a homeomorphism if and only if $s(f) = 0$.*

Poincaré duality

- **Theorem** (Poincaré, 1895)
  The homology and cohomology of a compact oriented n-dimensional manifold M are isomorphic:

$$H^{n-r}(M) \;\cong\; H_r(M)\ (r = 0, 1, 2, \ldots)\ .$$

- **Definition** (Browder, 1962)
  An $\underline{n\text{-dimensional duality space}}$ X is a space with isomorphisms:

$$H^{n-r}(X) \;\cong\; H_r(X)\ (r = 0, 1, 2, \ldots)\ .$$

The Existence Problem

- *Is an n-dimensional duality space X homotopy equivalent to an n-dimensional manifold?*
  - For $n = 1, 2$: Yes.
  - For $n \geq 3$: in general No.

Old solution of the Existence Problem

- **Theorem**
  (Browder, Novikov, Sullivan, Wall, 1970)
  An n-dimensional duality space X is homotopy equivalent to an n-dimensional manifold if and only if 2 obstructions vanish.

- The 2 obstructions (as for Uniqueness):

  1. In the topological K-theory of vector bundles over X.

  2. In the algebraic L-theory of quadratic forms over the fundamental group ring $\mathbb{Z}[\pi_1(X)]$.

- Same (dis)advantages as for the old solution of the Uniqueness Problem.

The Theorem of Galewski and Stern

- The kernel groups $K_r(x)$ of an n-dimensional duality space X fit into the exact sequence

$$\cdots \to K_r(x) \to H^{n-r}(\{x\}) \to H_r(X, X\backslash\{x\}) \to K_{r-1}(x) \to \dots \ .$$

- $K_*(x) = 0$ for a manifold.

- **Theorem** (Galewski and Stern, 1977) *A polyhedral duality space X with*

  $K_*(x) = 0$ $(x \in X)$ *(a homology manifold)*

  is homotopy equivalent to a manifold.

New solution of the Existence Problem

- The total surgery obstruction $s(X)$ of and n-dimensional duality space X is the cobordism class of
  - the sheaf of $\mathbb{Z}$-module chain complexes
  - with $(n-1)$-dimensional Poincaré duality
  - over X
  - with stalk homology $K_*(x)$ $(x \in X)$.
- Cobordism and Poincaré duality are algebraic.
- **Theorem** *A duality space X is homotopy equivalent to a manifold if and only if $s(X) = 0$.*

Quadratic algebra

- Chain complexes with the homological properties of manifolds and duality spaces.

- An <u>n-dimensional duality complex</u> is a chain complex

  $$C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} C_{n-2} \to \ldots \to C_0 \ (d^2 = 0)$$

  with isomorphisms

  $$H^{n-r}(C) \cong H_r(C) \ (r = 0, 1, 2, \ldots) \ .$$

  – generalized quadratic forms

- <u>cobordism</u> of duality complexes

Local and global duality complexes

X = connected space

- The global surgery group $L_n(\mathbb{Z}[\pi_1(X)])$ of Wall is the cobordism group of n-dimensional duality complexes of $\mathbb{Z}[\pi_1(X)]$-modules.
  - Generalized Witt groups.

- The local surgery group $H_n(X;\mathbb{L}(\mathbb{Z}))$ is the cobordism group of n-dimensional duality complexes of $\mathbb{Z}$-module sheaves over X.
  - Generalized homology with coefficients $L_*(\mathbb{Z})$.

The surgery exact sequence

- **Theorem** *The local and global surgery groups are related by the exact sequence*

$$\ldots \to H_n(X;\mathbb{L}(\mathbb{Z})) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_n(X) \to H_{n-1}(X;\mathbb{L}(\mathbb{Z})) \to \ldots .$$

- The <u>assembly map</u> A is the passage from local to global duality.

- The <u>structure group</u> $\mathbb{S}_n(X)$ is the cobordism group of $(n-1)$-dimensional <u>local</u> duality complexes over X which are <u>globally</u> underlinetrivial.

The total surgery obstructions

- **Uniqueness**: the total surgery obstruction of a homotopy equivalence $f : M^n \to N^n$

$$s(f) \in \mathbb{S}_{n+1}(N) .$$

  - $s(f)$ is the cobordism class of the n-dimensional globally trivial local duality complex with stalk homology the kernels $K_*(x)$ $(x \in N)$.

- **Existence**: the total surgery obstruction of an n-dimensional duality space X

$$s(X) \in \mathbb{S}_n(X) .$$

  - $s(X)$ is the cobordism class of the $(n-1)$-dimensional globally trivial local duality complex with stalk homology the kernels $K_*(x)$ $(x \in X)$.

Topology and homotopy theory

- The difference between the <u>topology</u> of manifolds and the <u>homotopy theory</u> of duality spaces = the difference between the cobordism theories of the <u>local</u> and <u>global</u> duality complexes.

$$\begin{array}{ccc} \text{manifolds} & \longrightarrow & \text{local duality} \\ \big\downarrow & & \big\downarrow{\scriptstyle A} \\ \text{duality spaces} & \longrightarrow & \text{global duality} \end{array}$$

- <u>Converse of Poincaré duality</u>:
  A duality space with sufficient local duality is homotopy equivalent to a manifold.

The Novikov and Borel conjectures

- The Novikov conjecture on the homotopy invariance of the higher signatures is <u>algebraic</u>:
  - $A : H_*(B\pi;\mathbb{L}(\mathbb{Z})) \to L_*(\mathbb{Z}[\pi])$ is rationally injective, for every group π.

- The Borel conjecture on the existence and uniqueness of aspherical manifolds is <u>algebraic</u>:
  - $A : H_*(B\pi;\mathbb{L}(\mathbb{Z})) \to L_*(\mathbb{Z}[\pi])$ is an isomorphism if $B\pi$ is a duality space.

- The various solution methods can now be turned into <u>algebra</u>:
  - topology, geometry, analysis (C^*-algebra), index theorems, ...

Applications

- algebraic computations of the L-groups
  - number theory
- singular spaces
  - algebraic varieties
- differential geometry
  - hyperbolic geometry
- non-compact manifolds
  - controlled topology
- 3- and 4-dimensional manifolds