# THE GEOMETRIC HOPF INVARIANT

Michael Crabb (Aberdeen)

Andrew Ranicki (Edinburgh)

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## The algebraic theory of surgery

- *"The chain complex theory offers many advantages ...*
  *a simple and satisfactory algebraic version of the whole setup.*
  *I hope it can be made to work."*
  C.T.C. Wall, *Surgery on Compact Manifolds* (1970)
- The chain complex theory developed in *The algebraic theory of surgery* (R., 1980) expressed surgery obstruction of a normal map $(f, b) : M \to X$ from an $m$-dimensional manifold $M$ to an $m$-dimensional geometric Poincaré complex $X$ as the cobordism class of a quadratic Poincaré complex $(C, \psi)$

  $$\sigma_*(f, b) \ = \ (C, \psi) \in L_m(\mathbb{Z}[\pi_1(X)])$$

  with $C$ a f.g. free $\mathbb{Z}[\pi_1(X)]$-module chain complex such that

  $$H_*(C) \ = \ K_*(M) \ = \ \ker(\widetilde{f}_* : H_*(\widetilde{M}) \to H_*(\widetilde{X}))$$

  and $\psi : H^*(C) \cong H_{m-*}(C)$ an algebraic Poincaré duality.
- Originally, it was necessary to make $(f, b)$ highly-connected by preliminary surgeries below the middle dimension.

## Advantages and a disadvantage

- The algebraic theory of surgery did indeed offer the advantages predicted by Wall in 1970.
- However, the identification $\sigma_*(f, b) = (C, \psi)$ was not as nice as could have been wished for!
- The chain homotopy theoretic treatment of the Wall self-intersection function counting double points

$$\mu(g : S^n \looparrowright M^{2n}) \in \frac{\mathbb{Z}[\pi_1(M)]}{\{x - (-)^n x^{-1} \,|\, x \in \pi_1(M)\}}$$

  was too indirect, making use of Wall's result that for $n \geqslant 3$ $\mu(g) = 0$ if and only if $g$ is regular homotopic to an embedding – proved by the Whitney trick for removing double points.
- Need to count double points of immersions using $\pi_1(M) \times \mathbb{Z}_2$-equivariant homotopy theory, specifically an equivariant version of the geometric Hopf invariant.

## Unstable vs. stable homotopy theory

- The stabilization map
$$[X,Y] \to \{X;Y\} \;=\; \varinjlim_k [\Sigma^k X, \Sigma^k Y] \;=\; [X, \Omega^\infty \Sigma^\infty Y]$$
is in general not an isomorphism!
- Terminology: for any space $X$ let $X^+ = X \sqcup \{+\}$ (disjoint union) and $X^\infty = X \cup \{\infty\}$ (one point compactification).
- $\Omega^\infty \Sigma^\infty Y/Y$ is filtered, with $k$th filtration quotient
$$F_k(Y) \;=\; E\Sigma_k^+ \wedge_{\Sigma_k} (\bigwedge_k Y) \;.$$
- The Thom space of a $j$-plane bundle $\mathbb{R}^j \to E(\nu) \to M$ is $T(\nu) = E(\nu)^\infty$, and $F_k(T(\nu)) = T(e_k(\nu))$ with
$$\mathbb{R}^{jk} \to E(e_k(\nu)) = E\Sigma_k \times_{\Sigma_k} \prod_k E(\nu) \to E\Sigma_k \times_{\Sigma_k} \prod_k M \;.$$
- For an immersion $f : M^m \looparrowright N^n$ with $\nu_f : M \to BO(n-m)$ and Umkehr map $F : \Sigma^\infty N^+ \to \Sigma^\infty T(\nu_f)$ the adjoint $N^+ \to \Omega^\infty \Sigma^\infty T(\nu_f)$ sends $k$-tuple points of $f$ to $F_k(T(\nu_f))$.

## The Hopf invariant (I.)

- (Hopf, 1931) Isomorphism $H : \pi_3(S^2) \cong \mathbb{Z}$ via linking numbers of $S^1 \sqcup S^1 \hookrightarrow S^3$.
- (Freudenthal, 1937) Suspension map for pointed space $X$

  $$E \ : \ \pi_n(X) \to \pi_{n+1}(\Sigma X) \, ; \, (f : S^n \to X) \mapsto (\Sigma f : S^{n+1} \to \Sigma X) \, .$$

  ($E$ for **Einhängung**). If $X$ is $(m-1)$-connected then $E$ is an isomorphism for $n \leqslant 2m-2$ and surjective for $n = 2m-1$.
- (G.W.Whitehead, 1950) *EHP* exact sequence

  $$\cdots \longrightarrow \pi_n(X) \xrightarrow{E} \pi_{n+1}(\Sigma X) \xrightarrow{H} \pi_n(X \wedge X) \xrightarrow{P} \pi_{n-1}(X) \longrightarrow \cdots$$

  for any $(m-1)$-connected space $X$, with $n \leqslant 3m-2$.
- For $X = S^m$, $n = 2m$

  $$H \ = \ \text{Hopf invariant} \ : \ \pi_{2m+1}(S^{m+1}) \to \pi_{2m}(S^m \wedge S^m) = \mathbb{Z} \, .$$

## The quadratic construction

- Given an inner product space $V$ let $LV = V$ with $\mathbb{Z}_2$-action
$$T \ : \ LV \to LV \ ; \ v \mapsto -v$$
with restriction $T : S(LV) \to S(LV)$.
- The **quadratic construction** on pointed space $X$ is
$$Q_V(X) \ = \ S(LV)^+ \wedge_{\mathbb{Z}_2} (X \wedge X)$$
with $T : X \wedge X \to X \wedge X; (x,y) \mapsto (y,x)$. The projection
$$\overline{Q}_V(X) \ = \ S(LV)^+ \wedge (X \wedge X) \to Q_V(X)$$
is a double cover away from the base point.
- $Q_{\mathbb{R}^0}(X) = \{\text{pt.}\}$, $Q_{\mathbb{R}^1}(X) = X \wedge X$.
- $Q_{\mathbb{R}^k}(S^0) = S(L\mathbb{R}^k)^+/\mathbb{Z}_2 = (\mathbb{RP}^{k-1})^+$.
- For $V = \mathbb{R}^\infty$ write
$$Q(X) \ = \ Q_{\mathbb{R}^\infty}(X) \ = \ \varinjlim_k Q_{\mathbb{R}^k}(X) \ .$$

## The Hopf invariant (II.)

- (James, 1955) Stable homotopy equivalence for connected $X$

$$\Omega\Sigma X \simeq_s \bigvee_{k=1}^{\infty}(X \wedge \cdots \wedge X) .$$

- (Snaith, 1974) Stable homotopy equivalence

$$\Omega^\infty\Sigma^\infty X \simeq_s \bigvee_{k=1}^{\infty} E\Sigma_k^+ \wedge_{\Sigma_k} (X \wedge \cdots \wedge X) .$$

  for connected $X$. Group completion for disconnected $X$.
- For $k = 2$ a stable homotopy projection

$$\Omega^\infty\Sigma^\infty X \to Q(X) = E\Sigma_2^+ \wedge_{\Sigma_2} (X \wedge X) .$$

  However, until now it was only defined for connected $X$, and was not natural in $X$.

## The stable $\mathbb{Z}_2$-equivariant homotopy groups

- Given pointed $\mathbb{Z}_2$-spaces $X, Y$ let $[X, Y]_{\mathbb{Z}_2}$ be the set of $\mathbb{Z}_2$-equivariant homotopy classes of $\mathbb{Z}_2$-equivariant maps $X \to Y$.
- The **stable $\mathbb{Z}_2$-equivariant homotopy group** is

$$\{X; Y\}_{\mathbb{Z}_2} = \varinjlim_k [\Sigma^{k,k} X, \Sigma^{k,k} Y]_{\mathbb{Z}_2}$$

with

$$T : \Sigma^{k,k} X = S^k \wedge S^k \wedge X \to \Sigma^{k,k} X \; ; \; (s, t, x) \mapsto (t, s, x) \; ,$$
$$T : \Sigma^{k,k}(Y \wedge Y) \to \Sigma^{k,k}(Y \wedge Y) \; ; \; (s, t, y_1, y_2) \mapsto (t, s, y_2, y_1) \; .$$

- **Example** By the $\mathbb{Z}_2$-equivariant Pontrjagin-Thom isomorphism $\{S^0; S^0\}_{\mathbb{Z}_2}$ = the cobordism group of 0-dimensional framed $\mathbb{Z}_2$-manifolds (= finite $\mathbb{Z}_2$-sets). The decomposition of finite $\mathbb{Z}_2$-sets as fixed $\cup$ free determines

$$\{S^0; S^0\}_{\mathbb{Z}_2} \cong \mathbb{Z} \oplus \mathbb{Z} \; ; \; D = D^{\mathbb{Z}_2} \cup (D - D^{\mathbb{Z}_2}) \mapsto \left( |D^{\mathbb{Z}_2}|, \frac{|D| - |D^{\mathbb{Z}_2}|}{2} \right)$$

## The relative difference

- For any inner product space $V$ there is a cofibration
$$S^0=\{0\}^+\to V^\infty\to V^\infty/\{0\}^+=\Sigma S(V)^+$$
with $S(V)=\{v\in V\mid \|v\|=1\}$ and
$$\Sigma S(V)^+\xrightarrow{\cong} V^\infty/\{0\}^+\ ;\ (t,u)\mapsto [t,u]=\frac{tu}{1-t}.$$
- For maps $p,q:V^\infty\wedge X\to Y$ such that $p(0,x)=q(0,x)\in Y$ $(x\in X)$ define the **relative difference** map
$$\delta(p,q)\ :\ \Sigma S(V)^+\wedge X\to Y\ ;$$
$$(t,u,x)\mapsto\begin{cases}p([1-2t,u],x) & \text{if } 0\leqslant t\leqslant 1/2\\ q([2t-1,u],x) & \text{if } 1/2\leqslant t\leqslant 1\ .\end{cases}$$
- The homotopy class of $\delta(p,q)$ is the obstruction to the existence of a rel $0^\infty\wedge X$ homotopy $p\simeq q:V^\infty\wedge X\to Y$. Barratt-Puppe sequence
$$\cdots\to[\Sigma S(V)^+\wedge X,Y]\to[V^\infty\wedge X,Y]\to[X,Y]$$

## $\mathbb{Z}_2$-equivariant stable homotopy theory = fixed-point + fixed-point-free

- **Theorem** For any pointed spaces $X, Y$ there is a split short exact sequence of abelian groups

$$0 \to \{X; Q(Y)\} \xrightarrow{1+T} \{X; Y \wedge Y\}_{\mathbb{Z}_2} \xrightarrow{\rho} \{X; Y\} \to 0$$

  with an $S$-duality isomorphism

$$\{X; Q(Y)\} \cong \varinjlim_{V, \dim(V) < \infty} [\Sigma S(LV)^+ \wedge V^\infty \wedge X, V^\infty \wedge LV^\infty \wedge (Y \wedge Y)]_{\mathbb{Z}_2} .$$

- $\rho$ is given by the $\mathbb{Z}_2$-fixed points, split by

$$\sigma \ : \ \{X; Y\} \to \{X; Y \wedge Y\}_{\mathbb{Z}_2} \ ; \ F \mapsto \Delta_Y F \ .$$

- The injection $1+T$ is induced by projection $S(L\mathbb{R}^\infty)^+ \to 0^\infty$

$$1+T \ : \ \{X; Q(Y)\} = \{X; \overline{Q}(Y)\}_{\mathbb{Z}_2} \to \{X; Y \wedge Y\}_{\mathbb{Z}_2}$$

  split by

$$\delta \ : \ \{X; Y \wedge Y\}_{\mathbb{Z}_2} \to \{X; Q(Y)\} \ ; \ G \mapsto \delta(G, \sigma\rho(G)) \ .$$

## The geometric Hopf invariant $h(F)$ (I.)

- Let $X, Y$ be pointed spaces. The **geometric Hopf invariant** of a stable map $F : \Sigma^\infty X \to \Sigma^\infty Y$ is the stable map

$$h(F) \ = \ \delta((F \wedge F)\Delta_X, \Delta_Y F) \ : \ \Sigma^\infty X \to \Sigma^\infty Q(Y) \ .$$

- The injection $1 + T : \{X; Q(Y)\} \hookrightarrow \{X; Y \wedge Y\}_{\mathbb{Z}_2}$ sends the stable homotopy class of $h(F)$ to the stable $\mathbb{Z}_2$-equivariant homotopy class of

$$(1 + T)h(F) \ = \ \Delta_Y F - (F \wedge F)\Delta_X \ : \ X \to Y \wedge Y \ .$$

- *The stable homotopy class of $h(F)$ is the primary obstruction to the desuspension of $F$.*
- Good naturality properties: if $\pi$ is a group, $X, Y$ are $\pi$-spaces and $F$ is $\pi$-equivariant then $h(F)$ is $\pi$-equivariant.

## The geometric Hopf invariant $h(F)$ (II.)

- **Proposition** The geometric Hopf invariant of $F : \Sigma^\infty X \to \Sigma^\infty Y$

$$\begin{aligned} h(F) \in \ker(\rho : \{X;Y\wedge Y\}_{\mathbb{Z}_2} \to \{X;Y\}) \\ = \ \mathrm{im}(1+T : \{X;Q(Y)\} \hookrightarrow \{X;Y\wedge Y\}_{\mathbb{Z}_2}) \end{aligned}$$

  has the following properties:

  (i) If $F \in \mathrm{im}([X,Y] \to \{X;Y\})$ then $h(F) = 0$.

  (ii) For $F_1, F_2 : \Sigma^\infty X \to \Sigma^\infty Y$

  $$h(F_1 + F_2) \ = \ h(F_1) + h(F_2) + (F_1 \wedge F_2)\Delta_X \ .$$

  (iii) For $F : \Sigma^\infty X \to \Sigma^\infty Y$, $G : \Sigma^\infty Y \to \Sigma^\infty Z$

  $$h(GF) \ = \ (G \wedge G)h(F) + h(G)F \ .$$

  (iv) If $X = S^{2m}$, $Y = S^m$, $F : S^{2m+\infty} \to S^{m+\infty}$ then

  $$h(F) \ = \ \text{mod 2 Hopf invariant}\,(F) \in \{S^{2m};Q(S^m)\} = \mathbb{Z}_2 \ .$$

  (v) $h : \{X;Y\} \to \{X;Q(Y)\}; F \mapsto h(F)$ is the James-Hopf double point map.

## The Main Theorem

- **Theorem** The quadratic Poincaré complex $(C,\psi)$ of an $m$-dimensional normal map $(f,b) : M \to X$ has

$$\psi \ = \ (e\otimes e)(h(F)/\pi)[X] \\ \in Q_m(C) \ = \ H_m(C(S(L\mathbb{R}^\infty))\otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C\otimes_{\mathbb{Z}[\pi]} C))$$

  with $\pi=\pi_1(X)$, $[X]\in H_m(X)$ the fundamental class, and

$$h(F)/\pi \ : \ H_m(X)\to H_m(S(L\mathbb{R}^\infty)\times_{\mathbb{Z}_2}(\widetilde{M}\times_\pi \widetilde{M}))$$

  the $\pi$-equivariant geometric Hopf invariant. Here $F:\Sigma^\infty\widetilde{X}^+\to\Sigma^\infty\widetilde{M}^+$ is the stable $\pi$-equivariant map inducing the Umkehr $f^!: C(\widetilde{X})\to C(\widetilde{M})$ determined by $b:\nu_M\to\nu_X$, and $e=\text{inclusion}: C(\widetilde{M})\to C=\mathcal{C}(f^!)$.
- The $m$-dimensional quadratic Poincaré complex $(C,\psi)$ has a direct connection with double points of immersions $S^n \looparrowright M^m$, particularly for $m=2n$.

## The difference of diagonals

- For any space $X$ the diagonal map
$$\Delta_X \;:\; X \to X \wedge X \;;\; x \mapsto (x,x)$$
is $\mathbb{Z}_2$-equivariant.
- For any inner product space $V$ define the $\mathbb{Z}_2$-equivariant homeomorphism
$$\kappa_V \;:\; LV^\infty \wedge V^\infty \to V^\infty \wedge V^\infty \;;\; (x,y) \mapsto (x+y, -x+y)\ .$$
- Given a map $F : V^\infty \wedge X \to V^\infty \wedge Y$ define the **noncommutative** square of $\mathbb{Z}_2$-equivariant maps
$$\begin{array}{ccc} LV^\infty \wedge V^\infty \wedge X & \xrightarrow{1 \wedge \Delta_X} & LV^\infty \wedge V^\infty \wedge X \wedge X \\ {\scriptstyle 1 \wedge F}\Big\downarrow & (\kappa_V^{-1} \wedge 1)(F \wedge F)(\kappa_V \wedge 1) & \Big\downarrow \\ LV^\infty \wedge V^\infty \wedge Y & \xrightarrow{1 \wedge \Delta_Y} & LV^\infty \wedge V^\infty \wedge Y \wedge Y \end{array}$$

## The unstable geometric Hopf invariant $h_V(F)$ (I.)

- **Definition** The **unstable geometric Hopf invariant** of a map $F : V^\infty \wedge X \to V^\infty \wedge Y$ is the $\mathbb{Z}_2$-equivariant relative difference map

$$h_V(F) \;=\; \delta(p,q) \;:\; \Sigma S(LV)^+ \wedge V^\infty \wedge X \to LV^\infty \wedge V^\infty \wedge Y \wedge Y$$

of the $\mathbb{Z}_2$-equivariant maps

$$p \;=\; (1 \wedge \Delta_Y)(1 \wedge F)\;,\; q \;=\; (\kappa_V^{-1} \wedge 1)(F \wedge F)(\kappa_V \wedge \Delta_X)\;: \\ LV^\infty \wedge V^\infty \wedge X \to LV^\infty \wedge V^\infty \wedge Y \wedge Y$$

with

$$p(0,v,x) \;=\; q(0,v,x) \;=\; (0,w,y,y) \;\; (F(v,x) = (w,y))\;, \\ \Sigma S(LV)^+ \;=\; LV^\infty/0^\infty \;=\; (LV\backslash\{0\})^\infty\;.$$

## The unstable geometric Hopf invariant $h_V(F)$ (II.)

- **Proposition** The unstable geometric Hopf invariant

$$h_V \ : \ [V^\infty \wedge X, V^\infty \wedge Y] \to$$
$$\{\Sigma S(LV)^+ \wedge V^\infty \wedge X; LV^\infty \wedge V^\infty \wedge Y \wedge Y\}_{\mathbb{Z}_2} = \{X; Q_V(Y)\}$$

has the following properties:

(i) If $F \in \mathrm{im}([X,Y] \to [V^\infty \wedge X, V^\infty \wedge Y])$ then $h_V(F) = 0$.

(ii) For $F_1, F_2 : V^\infty \wedge X \to V^\infty \wedge Y$

$$h_V(F_1 + F_2) \ = \ h_V(F_1) + h_V(F_2) + (F_1 \wedge F_2)\Delta_X \ .$$

(iii) For $F : V^\infty \wedge X \to V^\infty \wedge Y$, $G : V^\infty \wedge Y \to V^\infty \wedge Z$

$$h_V(GF) \ = \ (G \wedge G)h_V(F) + h_V(G)F \ .$$

(iv) $h(F) = \varinjlim_k h_{V \oplus \mathbb{R}^k}(\Sigma^k F)$ for $F : V^\infty \wedge X \to V^\infty \wedge Y$.

(v) If $V = \mathbb{R}$, $X = S^{2m}$, $Y = S^m$, $F : S^{2m+1} \to S^{m+1}$ then

$$h_{\mathbb{R}}(F) \ = \ \text{Hopf invariant}\,(F) \in \{S^{2m}; Q_{\mathbb{R}}(S^m)\} = \mathbb{Z} \ (m \geqslant 0) \ .$$

For $m = 0$ $h_{\mathbb{R}}(F) = d(d-1)/2 \in \mathbb{Z}$, $d = \deg(F : S^1 \to S^1)$.

## The unstable geometric Hopf invariant $h_V(F)$ (III.)

- The $\mathbb{Z}_2$-equivariant cofibration sequence

$$S(LV)^+ \to \{0\}^+ \to LV^\infty$$

induces the Barratt-Puppe exact sequence

$$\begin{aligned} \dots \to \{X; Q_V(Y)\} &= \{X; S(LV)^+ \wedge_{\mathbb{Z}_2} (Y \wedge Y)\}_{\mathbb{Z}_2} \\ &\to \{X; Y \wedge Y\}_{\mathbb{Z}_2} \to \{X; LV^\infty \wedge Y \wedge Y\}_{\mathbb{Z}_2} \to \dots \end{aligned}$$

- For any $F : V^\infty \wedge X \to V^\infty \wedge Y$

$$\begin{aligned} \Delta_Y F - (F \wedge F)\Delta_X \ &= \ [h_V(F)] \\ &\in \mathrm{im}(\{X; Q_V(Y)\} \to \{X; Y \wedge Y\}_{\mathbb{Z}_2}) \\ &= \ \ker(\{X; Y \wedge Y\}_{\mathbb{Z}_2} \to \{X; LV^\infty \wedge Y \wedge Y\}_{\mathbb{Z}_2}) \ . \end{aligned}$$

## The universal example of a $k$-stable map

- For any pointed space $X$ evaluation defines a $k$-stable map
$$e \ : \ \Sigma^k(\Omega^k\Sigma^k X) \to \Sigma^k X \ ; \ (s,\omega) \mapsto \omega(s)$$
with $\mathrm{adj}(e) = 1 : \Omega^k\Sigma^k X \to \Omega^k\Sigma^k X$. The unstable geometric Hopf invariant of $e$ defines a stable map
$$h_{\mathbb{R}^k}(e) \ : \ \Omega^k\Sigma^k X \to Q_{\mathbb{R}^k}(X) = S(L\mathbb{R}^k)^+ \wedge_{\mathbb{Z}_2} (X \wedge X)$$
which is a stable splitting of the Dyer-Lashof map
$$Q_{\mathbb{R}^k}(X) \to \Omega^k\Sigma^k X \ .$$

- For any $k$-stable map $F : \Sigma^k Y \to \Sigma^k X$ the stable homotopy class of the composite
$$h_{\mathbb{R}^k}(F) : Y \xrightarrow{\mathrm{adj}(F)} \Omega^k\Sigma^k X \xrightarrow{h_{\mathbb{R}^k}(e)} Q_{\mathbb{R}^k}(X)$$
is the primary obstruction to a $k$-fold desuspension of $F$, i.e. to the compression of $\mathrm{adj}(F)$ into $X \subset \Omega^k\Sigma^k X$.

## Double points

- The **ordered double point set** of a map $f : M \to N$ is the free $\mathbb{Z}_2$-set

$$\overline{D}_2(f) \;=\; \{(x,y)\,|\,x \neq y \in M,\, f(x) = f(y) = N\}$$

with

$$T \;:\; \overline{D}_2(f) \to \overline{D}_2(f) \;;\; (x,y) \mapsto (y,x)\ .$$

- The **unordered double point set** is

$$D_2(f) \;=\; \overline{D}_2(f)/\mathbb{Z}_2\ .$$

- $f$ is an embedding if and only if $D_2(f) = \emptyset$.

## The Umkehr map of an immersion (I.)

- Let $f : M^m \looparrowright N^n$ be a generic immersion of closed manifolds with normal bundle $\nu_f : M \to BO(m-n)$. By the tubular neighbourhood theorem $f$ extends to a codimension 0 immersion $f' : E(\nu_f) \looparrowright N$. For $V = \mathbb{R}^k$ with $k \geqslant 2m-n+1$ there exists a map $e : V \times E(\nu_f) \to V$ such that

$$g \;=\; (e, f') \;:\; V \times E(\nu_f) \hookrightarrow V \times N \;;$$
$$(v,x) \mapsto (e(v,x), f'(x))$$

is an open codimension 0 embedding.
- The **Umkehr** map of $f$ is the stable map

$$F : (V \times N)^\infty = V^\infty \wedge N^+ \to (V \times E(\nu_f))^\infty = V^\infty \wedge T(\nu_f) \;;$$
$$(w,y) \mapsto \begin{cases} (v,x) & \text{if } (w,y) = g(v,x) \\ \infty & \text{if } (w,y) \notin \text{im}(g) \;. \end{cases}$$

## The Umkehr map of an immersion (II.)

- Let

$$G \;:\; V^{\infty} \wedge V^{\infty} \wedge N^{+} \to V^{\infty} \wedge V^{\infty} \wedge T(\nu_f \times \nu_f|_{\overline{D}_2(f)})$$

be the Umkehr map of the $\mathbb{Z}_2$-equivariant codimension 0 embedding

$$g \times g| \;:\; V \times V \times E(\nu_f \times \nu_f|_{\overline{D}_2(f)}) \hookrightarrow V \times V \times N \;.$$

$G$ represents an element

$$G \in \{N^{+}; T(\nu_f \times \nu_f|_{\overline{D}_2(f)})\}_{\mathbb{Z}_2} = \{N^{+}; T(\nu_f \times \nu_f|_{D_2(f)})\} \;.$$

- The map

$$H \;:\; D_2(f) \to S(LV) \times_{\mathbb{Z}_2} (M \times M) \;;$$
$$(x, y) \mapsto \Big(\frac{e(0,x) - e(0,y)}{\|e(0,x) - e(0,y)\|}, x, y\Big)$$

induces a map of Thom spaces

$$H : T(\nu_f \times \nu_f|_{D_2(f)}) \to T(S(LV) \times_{\mathbb{Z}_2} (E(\nu_f) \times E(\nu_f))) = Q_V(T(\nu_f)) \;.$$

## Capturing double points with homotopy theory

- **Theorem** The unstable geometric Hopf invariant of the Umkehr map $F : V^\infty \wedge N^+ \to V^\infty \wedge T(\nu_f)$ of an immersion $f : M^m \looparrowright N^n$ factors through the double point set $D_2(f)$

$$h_V(F) \ = \ HG \in \{N^+; Q_V(T(\nu_f))\}$$

with

$$h_V(F) \ : \ N^+ \xrightarrow{G} T(\nu_f \times \nu_f|_{D_2(f)}) \xrightarrow{H} Q_V(T(\nu_f)) \ .$$

- There is also a $\pi_1(M)$-equivariant version!
- **Corollary** If $f : M \looparrowright N$ is regular homotopic to an embedding $f_0 : M \hookrightarrow N$ with Umkehr map $F_0 : N^+ \to T(\nu_f)$ then $F$ is stably homotopic to $F_0$, and $h_V(F)$ is stably null-homotopic.
- *The geometric Hopf invariant is the primary homotopy theoretic method of capturing $D_2(f)$.*

# Manifolds are hot!