# THE QUADRATIC FORM $\mathbf{E}_8$ AND EXOTIC HOMOLOGY MANIFOLDS

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- The Bryant-Mio-Ferry-Weinberger construction of $2n$-dimensional exotic homology manifolds for $2n \geqslant 6$ with Quinn index 9 starts with

$$E_8 \times T^{2n} \neq 0 \in L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$$

and proceeds by controlled Wall realization.

- Edwards challenge: find an <u>explicit</u> $(-)^n$-quadratic form over $\mathbb{Z}[\mathbb{Z}^{2n}]$ realizing the non-zero element $E_8 \times T^{2n}$.

## The surgery obstruction groups

- Wall (1970) defined the surgery obstruction groups $L_m(\Lambda)$ of a ring with involution $\Lambda$, with

$$L_m(\Lambda) = L_{m+4}(\Lambda) .$$

- The surgery obstruction of an $m$-dimensional normal map $(f,b) : M \to X$ is an element

$$\sigma_*(f,b) \in L_m(\mathbb{Z}[\pi_1(X)])$$

with $\sigma_*(f,b) = 0$ if (and for $m \geqslant 5$ only if) $(f,b)$ is normal bordant to a homotopy equivalence.

## $\mathbf{L}_{2n}(\Lambda)$

- $L_{2n}(\Lambda)$ = the Witt group of nonsingular $(-)^n$-quadratic forms $(K, \lambda, \mu)$ over $\Lambda$, with $K$ a f.g. free $\Lambda$-module,

  $$\lambda = (-)^n \lambda^* : K \to K^* = \mathrm{Hom}_\Lambda(K, \Lambda)$$

  and $\mu$ a $(-)^n$-quadratic refinement of $\lambda$.

- For $n$-connected $(f, b) : M^{2n} \to X$

  $$\sigma_*(f, b) = (K_n(M), \lambda, \mu) \in L_{2n}(\mathbb{Z}[\pi_1(X)])$$

  with $K_n(M) = \ker(\widetilde{f}_* : H_n(\widetilde{M}) \to H_n(\widetilde{X}))$ the kernel (stably) f.g. free $\mathbb{Z}[\pi_1(X)]$-module.

- For a finitely presented group $\pi$ every form $(K, \lambda, \mu)$ over $\mathbb{Z}[\pi]$ is realized as $\sigma_*(f, b)$ for some $(f, b) : M^{2n} \to X$ with $\pi_1(X) = \pi$.

- If $\pi$ has no 2-torsion and $n$ is even, then $\lambda$ determines $\mu$.

# $\mathbf{E}_8$

- Nonsingular quadratic form $(\mathbb{Z}^8, E_8)$ over $\mathbb{Z}$

$$E_8 = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Positive definite, rank = signature = 8.

- For $m \geqslant 2$ $E_8$ is realized in the $PL$ category as the surgery obstruction

$$\sigma_*(f_0, b_0) = (\mathbb{Z}^8, E_8) = 1 \in L_{4m}(\mathbb{Z}) = \mathbb{Z}$$

of $2m$-connected $4m$-dimensional normal map

$$(f_0, b_0) : M_0^{4m} \to S^{4m}$$

with $M_0$ the Milnor $E_8$-plumbing of 8 $\tau_{S^{2m}}$'s.

## $\mathbf{E}_8 \times \mathbf{T}^{2n}$

- $E_8 \times T^{2n}$ is the surgery obstruction of the $2m$-connected $(4m+2n)$-dimensional normal map

  $(g,c) = (f_0,b_0)\times 1 : M_0^{4m}\times T^{2n} \to S^{4m}\times T^{2n}$ .

  $$E_8 \times T^{2n} = \sigma_*(g,c) = (0,\ldots,0,1) \neq 0$$
  $$\in L_{4m+2n}(\mathbb{Z}[\mathbb{Z}^{2n}]) = L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}]) =$$
  $$L_{2n}(\mathbb{Z}) \oplus \cdots \oplus \binom{2n}{k} L_{2n-k}(\mathbb{Z}) \oplus \cdots \oplus L_0(\mathbb{Z}).$$

- $E_8 \times T^{2n}$ is represented by the kernel $(-)^n$-quadratic form $(K_{2m+n}(M_1),\lambda,\mu)$ of any bordant $(2m+n)$-connected normal map $(g_1,c_1) : M_1^{4m+2n} \to S^{4m}\times T^{2n}$.

- In order to find an explicit form $(K,\lambda,\mu)$ for $E_8\times T^{2n}$ need to work out $(K_{2m+n}(M_1),\lambda,\mu)$ by algebraic surgery below the middle dimension.

## The algebraic theory of surgery

- Theorem (R., 1980) $L_m(\Lambda)$ is isomorphic to the cobordism group of $m$-dimensional quadratic Poincaré complexes over $\Lambda$. The surgery obstruction of $(f,b) : M^m \to X$ is the cobordism class

$$\sigma_*(f,b) = (C,\psi) \in L_m(\mathbb{Z}[\pi_1(X)])$$

of an $m$-dimensional quadratic Poincaré complex $(C,\psi)$.

- $C = \mathcal{C}(f^! : C(\widetilde{X}) \to C(\widetilde{M}))$ with

$$f^! : C(\widetilde{X}) \simeq C(\widetilde{X})^{m-*} \xrightarrow{f^*} C(\widetilde{M})^{m-*} \simeq C(\widetilde{M})$$

- $\widetilde{X}$ = universal cover of $X$, $\widetilde{M} = f^*\widetilde{X}$

- $H_*(C) = K_*(M) = \ker(f_* : H_*(\widetilde{M}) \to H_*(\widetilde{X}))$

- $(1+T)\psi_0 : C^{m-*} = \mathrm{Hom}_\Lambda(C,\Lambda) \xrightarrow{\simeq} C$.

## The instant surgery obstruction

- Corollary The surgery obstruction of a $2n$-dimensional normal map $(f,b) : M \to X$ is given by

$$\sigma_*(f,b) = (K,\lambda,\mu) \in L_{2n}(\mathbb{Z}[\pi_1(X)])$$

  with

$$K = \text{coker}(\begin{pmatrix} d^* & 0 \\ (-)^{n+1}(1+T)\psi_0 & d \end{pmatrix} : C^{n-1} \oplus C_{n+2} \to C^n \oplus C_{n+1})$$

$$\lambda = \begin{pmatrix} (1+T)\psi_0 & d \\ (-)^n d^* & 0 \end{pmatrix}, \mu = \begin{pmatrix} \psi_0 & d \\ 0 & 0 \end{pmatrix}$$

  the $(-)^n$-quadratic form determined by $(C,\psi)$.

- If $(f,b)$ is $n$-connected the usual Wall kernel form $(\lambda,\mu)$ on $K = K_n(M)$.

- In general, $(f,b)$ is not $n$-connected.

## Almost $(-)^n$ symmetric forms

- Clauwens.

- An almost $(-)^n$-symmetric form $(A, \alpha)$ over a ring with involution $\Lambda$ is a f.g. free $\Lambda$-module $A$ together with a nonsingular sesquilinear pairing $\alpha : A \to A^*$ such that
$1 + (-)^{n+1}\alpha^{-1}\alpha^* : A \to A$ is nilpotent.

- A $2n$-dimensional manifold $N$ with a Poincaré duality cellular chain isomorphism on the universal cover $\widetilde{N}$

$$\phi_0 = [N] \cap - : C(\widetilde{N})^{2n-*} \cong C(\widetilde{N})$$

has an almost $(-)^n$-symmetric form over $\mathbb{Z}[\pi_1(N)]$

$$(C^n(\widetilde{N}), \alpha = \phi_0 + d\phi_1)$$

with $\phi_1 : \phi_0 \simeq \phi_0^*$ a chain homotopy and $d : C_{n+1}(\widetilde{N}) \to C_n(\widetilde{N})$ the differential.

## Quadratic $\otimes$ almost symmetric

- In a $(-)^m$-quadratic form $(K, \lambda, \mu)$ over $\Lambda$ the $(-)^m$-quadratic function $\mu$ corresponds to an equivalence class of $\Lambda$-module morphisms $\psi : K \to K^*$ such that
$$\lambda = \psi + (-)^m \psi^* : K \to K^*$$
with $\psi \sim \psi'$ if $\psi' = \psi + \chi + (-)^{m+1}\chi^*$.

- The <u>product</u> of $(K, \lambda, \mu)$ and an almost $(-)^n$-symmetric form $(A, \alpha)$ over $\Lambda'$ is the $(-)^{m+n}$-quadratic form over $\Lambda \otimes_{\mathbb{Z}} \Lambda'$
$$(K, \lambda, \mu) \otimes (A, \alpha) = (K \otimes_{\mathbb{Z}} A, \lambda', \mu')$$
$$\lambda' = \psi' + (-)^{m+n}\psi'^* \ , \ \psi' = \psi \otimes \alpha.$$

- Product of Witt groups
$$L_{2m}(\Lambda) \otimes AL^{2n}(\Lambda') \to L_{2m+2n}(\Lambda \otimes \Lambda')$$
with $AL^{2n}(\Lambda')$ the Witt group of almost $(-)^n$-symmetric forms over $\Lambda'$.

## The surgery product formula

- Theorem (R., Clauwens, 1979)
  The surgery obstruction of a product normal map

$$(g,c) = (f,b) \times 1 : M^{2m} \times N^{2n} \to X \times N$$

  is the product

$$\begin{aligned} \sigma_*(g,c) &= \sigma_*(f,b) \otimes_{\mathbb{Z}} (C_n(\widetilde{N}),\alpha) \\ &\in L_{2m+2n}(\mathbb{Z}[\pi_1(X) \times \pi_1(N)]) \end{aligned}$$

  of surgery obstruction $\sigma_*(f,b) \in L_{2m}(\mathbb{Z}[\pi_1(X)])$ and the almost $(-)^n$-symmetric Witt class

$$(C_n(\widetilde{N}),\alpha) \in AL^{2n}(\mathbb{Z}[\pi_1(N)]) \ .$$

- Proof The instant surgery obstruction of $(g,c)$ is cobordant to the product $(K,\lambda,\mu)\otimes (C_n(\widetilde{N}),\alpha)$, with $(K,\lambda,\mu)$ the instant surgery obstruction of $(f,b)$.

## The almost $(-)^n$-symmetric form of $\mathbf{T}^{2n}$

- The standard $CW$ structure on $T^m$ has $C(\widetilde{T}^m)^{m-*} \cong C(\widetilde{T}^m)$ , $C_n(\widetilde{T}^m) = \binom{m}{n}\mathbb{Z}[\mathbb{Z}^m]$ with $\mathbb{Z}[\pi_1(T^m)] = \mathbb{Z}[\mathbb{Z}^m] = \mathbb{Z}[t_1, t_1^{-1}, \dots, t_m, t_m^{-1}]$.

- For $m = 2n$ the almost $(-)^n$-symmetric form $(C_n(\widetilde{T}^{2n}), \alpha)$ has rank $\binom{2n}{n}$, so that $E_8 \times T^{2n}$ is represented by a form of rank $8\binom{2n}{n}$. For $n = 1$

$$\alpha = \begin{pmatrix} 1 - t_1 & 1 \\ t_1 t_2 - t_1 - t_2 & 1 - t_2 \end{pmatrix}$$

- The rank $\binom{2n}{n}$ almost $(-1)^n$-symmetric form of $T^{2n}$ is a lot of work to write down for $n > 1$. Luckily $T^{2n} = T^2 \times \cdots \times T^2$ determines the Witt-equivalent form $\bigotimes\limits_{i=1}^{n} \alpha_i$ of rank $2^n$, with $\alpha_i$ the rank 2 form of $i$th $T^2$.

## An explicit form for $\mathbf{E}_8 \times \mathbf{T}^{2n}$

- Theorem The surgery obstruction

$$E_8 \times T^{2n} \in L_{2n}(\mathbb{Z}[\mathbb{Z}^{2n}])$$

is represented by the $(-)^n$-quadratic form of rank $2^{n+3}$

$$(\mathbb{Z}^8, E_8) \otimes (\mathbb{Z}[\mathbb{Z}^{2n}]^{2^n}, \bigotimes_{i=1}^{n} \alpha_i)$$

over

$$\begin{aligned}\mathbb{Z}[\mathbb{Z}^{2n}] &= \mathbb{Z}[t_1, t_1^{-1}, \dots, t_{2n}, t_{2n}^{-1}] \\ &= \bigotimes_{i=1}^{n} \mathbb{Z}[t_{2i-1}, t_{2i-1}^{-1}, t_{2i}, t_{2i}^{-1}]\end{aligned}$$

with

$$\alpha_i = \begin{pmatrix} 1 - t_{2i-1} & 1 \\ t_{2i-1}t_{2i} - t_{2i-1} - t_{2i} & 1 - t_{2i} \end{pmatrix}$$

the almost $(-1)$-symmetric form of the $i$th $T^2$ in $T^{2n} = T^2 \times T^2 \times \cdots \times T^2$.

we shall be dealing with such a $U$ of dimension $2(k+1)$ topological manifold and a distinguished $2k$-dimensional homology class $h$ "dividing" $U$ in the following sense. There is a proper function $\varphi : U \to \mathbb{R}$ such that $h$ is contained in the image of the inclusion

homomorphism $H_{2k}(\varphi^{-1}[a,b]) \to H_{2k}(U)$ for some (and, hence for each) non-empty segment $[a,b]$, [redacted] $-\infty \le a \le b \le +\infty$. Given such an $h$ and a $U(p,q)$-flat bundle $X$ over $W$, we define the cup-pairing of $H^{2k}(U;X)$ on $h$ in the obvious way and denote it by $\sigma(h;X)$.

Localization **Lemma.** Let $U' \subset U$ be an

## A commission

- The nonsingular $(-1)$-quadratic form of the torus $T^2$ over $\mathbb{Z}[1/2][\mathbb{Z}^2] = \mathbb{Z}[1/2][t_1, t_1^{-1}, t_2, t_2^{-1}]$

$$\frac{\alpha - \alpha^*}{2} =$$

$$\begin{pmatrix} \dfrac{t_1^{-1} - t_1}{2} & \dfrac{1 - t_1^{-1}t_2^{-1} + t_1^{-1} + t_2^{-1}}{2} \\ \dfrac{-1 + t_1 t_2 - t_1 - t_2}{2} & \dfrac{t_2^{-1} - t_2}{2} \end{pmatrix}$$

was commissioned by Gromov in 1995.

The algebraic cobordism relation is stronger than the geometric one as it includes homotopy equivalences and so the group $HBrd_*B\Pi$ happily maps into $\mathrm{Witt}_*$. (See [Miš], [Kas], $[\mathrm{Ran}]_{\mathrm{ALT}}$, $[\mathrm{Ran}]_{\mathrm{LKLT}}$ and $[\mathrm{Ran}]_{\mathrm{NC}}$ for details and further references).

***Example.*** Let $\Pi = \mathbb{Z} \oplus \mathbb{Z}$ where $\mathbb{Q}(\Pi)$ equals the Laurent polynomial ring in the variables $t_i^{\pm 1}$, $i = 1, 2$. Then the (symplectic) form over $\mathbb{Q}(\Pi)$ corresponding to the 2-torus, (i.e. the Poincaré complex of this torus) can be given by the following invertible matrix $A$

$$A = \begin{pmatrix} ((t_2)^{-1} - t_2)/2 & (1 + (t_1)^{-1} - t_2 + (t_1)^{-1}t_2)/2 \\ (-1 - t_1 + (t_2)^{-1} - t_1(t_2)^{-1})/2 & ((t_1)^{-1} - t_1)/2 \end{pmatrix}$$

kindly communicated to me by Andrew Ranicki. It is not at all obvious that the class of $A$ does not vanish in $\mathrm{Witt}_2\,\mathbb{Q}(\Pi)$; but it is known to be non-zero even in $\mathbb{C}(\Pi) \supset \mathbb{Q}(\Pi)$ and in the $C^*$-algebra $C^*(\Pi) \supset \mathbb{C}(\Pi)$ as follows, for example, from Lusztig's theorem (see $8\frac{5}{8}$).

from p.120 of
M. Gromov, *Positive Curvature, Macroscopic Dimension, Spectral Gaps, and Higher Signatures*, in Functional Analysis on the Eve of the 21st Century, Vol. 2, Birkhaüser, 1995, 1–213.