ALGEBRAIC TRANSVERSALITY FOR TOPOLOGICAL MANIFOLDS

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- Geometric transversality = cutting and pasting manifolds.

- Algebraic transversality = cutting and pasting algebraic Poincaré complexes.

- Success/failure of geometric transversality for homotopy equivalences of manifolds = success/failure of algebraic transversality.

- Codimension 1 transversality is the key to everything!

Geometric transversality

- Let $(X, Y \subset X)$ be a codimension k pair of spaces, meaning that Y has a k-dimensional bundle neighbourhood E in X

$$X = E \cup Z \ .$$

- $E = k$-plane vector bundle $Y \to BO(k)$ or TOP analogue, with zero section $Y \subset E$.

- Transversality Theorem (Sard, Thom, 1952) Every map from an n-dimensional manifold $f : M^n \to X$ is homotopic to a map which is transverse at $Y \subset X$, with

$$N^{n-k} = f^{-1}(Y) \subset M^n$$

a codimension k submanifold.

- Extreme example: $X =$ simplicial complex, $Y =$ dual cell $D(\sigma, X)$ of k-simplex $\sigma \in X$.

Codimension k splitting

- A homotopy equivalence $f : M^n \to X$ splits along $Y \subset X$ if it is homotopic to one with

$$f| \ = \ g \ : \ N^{n-k} = f^{-1}(Y) \to Y \ ,$$
$$f| \ : \ M\backslash N \ = \ f^{-1}(X\backslash Y) \to X\backslash Y$$

  homotopy equivalences.

- For $n-k \geq 5$ splitting obstruction $s_Y(f)$ entirely in algebraic K- and L-theory, which keep track of changes in $N \subset M$ by ambient surgeries (= handle exchanges).

- (Browder-Casson-Sullivan 1969) For $k \geq 3$ splitting obstruction = surgery obstruction

$$s_Y(f) \ = \ \sigma_*(g) \in L_{n-k}(\mathbb{Z}[\pi_1(Y)]) \ .$$

- (Cappell-Shaneson 1974) For $k = 1, 2$ homology splitting obstructions = knot and link cobordism invariants.

Codimension 1 splitting in manifold topology

Splitting along codimension 1 submanifolds $N^{n-1} \subset M^n$ and Bass-Heller-Swan computation $Wh(\mathbb{Z}^n) = 0$ essential for all topological applications using torus T^n, including :

- Novikov's proof of topological invariance of the rational Pontrjagin classes (1966)

- Kirby-Siebenmann structure theory of high dimensional topological manifolds (1969)

- Chapman's proof of topological invariance of Whitehead torsion (1974)

- bounded and controlled topology (1978–)

Quadratic complexes

- Quadratic structure group of chain complex C in additive chain duality category

$$\psi \in Q_n(C) \ = \ H_n(\mathbb{Z}_2; \mathrm{Hom}(C^*, C)) \ .$$

- An element $\psi \in Q_n(C)$ represented by $\psi_s : C^r \to C_{n-r-s}$ $(s \geq 0)$ such that up to signs

$$d\psi_s + \psi_s d^* + \psi_{s+1} + \psi^*_{s+1} = 0 : C^r \to C_{n-r-s-1} \ .$$

- An n-dimensional quadratic complex (C, ψ) is a pair $(C, \psi \in Q_n(C))$. The complex is Poincaré if the chain map $(1+T)\psi_0 : C^{n-*} \to C$ is a chain equivalence – algebraic mimicry of manifold duality.

- Quadratic Poincaré pair $(C \to D, (\delta\psi, \psi))$ with P.-Lefschetz $(D/C)^{n+1-*} \simeq D$ – algebraic mimicry of manifold with boundary.

The philosophy of algebraic transversality

- Regard quadratic Poincaré complexes as geometric objects in their own right, as algebraic analogues of manifolds.

- Investigate cut and paste properties of quadratic Poincaré complexes.

- Apply this algebra to the topology of manifolds.

- Algebraic transversality applies in various contexts: over a single ring with involution A, a commutative square, amalgamated free products and HNN extensions, a space X, bounded/controlled algebra, ...

Codimension 1 transversality

- Let $(X, Y \subset X)$ be a codimension 1 pair

$$X = (Y \times \mathbb{R}) \cup Z .$$

- A generalized homology theory h is a homotopy invariant functor with a Mayer-Vietoris exact sequence

$$\cdots \to h_n(Y) \to h_n(Z) \to h_n(X) \xrightarrow{\partial} h_{n-1}(Y) \to \ldots$$

  with ∂ sending an n-dimensional h-cycle x in X to $(n-1)$-dimensional cycle y in Y obtained by h-transversality, splitting x along $Y \subset X$.

- A homotopy invariant functor is a generalized homology theory if and only if it has codimension 1 transversality, i.e. if and only if it has Mayer-Vietoris exact sequences.

Codimension 1 algebraic transversality over A

- The <u>algebraic glueing</u> of $(n+1)$-dimensional quadratic Poincaré pairs $b = (C \to D, (\delta\psi, \psi))$, $b' = (C \to D', (\delta\psi', \psi))$ is $(n+1)$-dimensional quadratic Poincaré complex

$$b \cup -b' \; = \; (D \cup_C D', \delta\psi \cup_\psi -\delta\psi') \; .$$

- <u>Theorem</u> (R., 1980) For any $(n+1)$-dimensional quadratic Poincaré complex (E, θ) and chain map $f : D \to E$ there exist b, b' with $D' = C(f)^{n+1-*}$ and homotopy equivalence

$$b \cup b' \; \simeq \; (E, \theta) \; .$$

- This is the algebraic key to the controlled splitting theorem of Yamasaki (1987)

Codimension 1 algebraic transversality over Φ

- Given a commutative square of rings Φ

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ C & \longrightarrow & D \end{array}$$

  want Mayer-Vietoris exact sequences

  - in algebraic K-theory

$$\cdots \to K_n(A) \to K_n(B) \oplus K_n(C) \to K_n(D) \xrightarrow{\partial} K_{n-1}(A) \to \ldots$$

  - in algebraic L-theory

$$\cdots \to L_n(A) \to L_n(B) \oplus L_n(C) \to L_n(D) \xrightarrow{\partial} L_{n-1}(A) \to \ldots$$

- The Mayer-Vietoris exact sequences of cartesian and arithmetic squares obtained in 1970's early instances of algebraic transversality.

Algebraic K-theory transversality over $A[z, z^{-1}]$

- Higman (1940) trick for stabilizing matrices over $\mathbb{Z}[z, z^{-1}]$ to get linear entries only.

- Theorem (Bass-Heller-Swan $i = 0$, 1964, Bass $i \leq -1$, 1969, Quillen $i \geq 2$, 1972)

$$K_i(A[z, z^{-1}]) = K_i(A) \oplus K_{i-1}(A) \oplus \widetilde{\text{Nil}}_i(A) \oplus \widetilde{\text{Nil}}_i(A)$$

- Jump from K_1 to K_0 corresponds to the use of noncompact manifolds in understanding the topology of compact manifolds.

- Every finite f.g. free $A[z, z^{-1}]$-module chain complex C has algebraic fundamental domain $D \subset C$, a finite f.g. free A-module chain complex, with linear presentation

$$0 \to (D \cap zD)[z, z^{-1}] \xrightarrow{f - zg} D[z, z^{-1}] \to C \to 0$$

Algebraic L-theory transversality over $A[z,z^{-1}]$

- Theorem (Shaneson $i=2$, Novikov $i=1$, 1969, R. $i \leq 0$, 1972, 1992) For $-\infty \leq i \leq 2$

$$L_n^{\langle i \rangle}(A[z,z^{-1}]) \;=\; L_n^{\langle i \rangle}(A) \oplus L_{n-1}^{\langle i-1 \rangle}(A)$$

  with $L^{\langle 2 \rangle} = L^s$, $L^{\langle 1 \rangle} = L^h$, $L^{\langle 0 \rangle} = L^p$, and $L_n^{\langle -i \rangle}(A) = L_{n+i+1}(\mathbb{C}_{\mathbb{R}^{i+1}}(A))$ lower L-groups.

- Geometric proof used Farrell-Hsiang (1968) splitting theorem: a homotopy equivalence $h : M \to X \times S^1$ splits along $X \subset X \times S^1$ iff

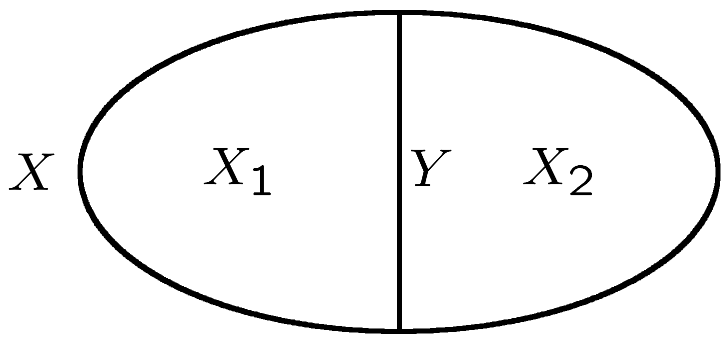
$$\tau(h) \in \text{im}(Wh(\pi_1(X)) \to Wh(\pi_1(X \times S^1)))\ .$$

- Algebraic proof used algebraic transversality: every quadratic complex (C,ψ) over $A[z,z^{-1}]$ has fundamental domain $D \subset C$, and if (C,ψ) is simple Poincaré can arrange for $(D; D \cap z^{-1}D, zD \cap D)$ to be finite n-dimensional quadratic Poincaré cobordism.

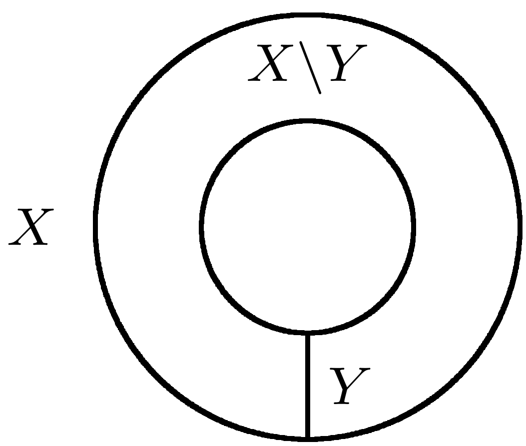
Amalgamated free products and HNN extensions

- X, Y connected, $Y \times \mathbb{R} \subset X$.

- Case A: The complement $X \backslash Y = X_1 \cup X_2$ is disconnected.



- Case B: The complement $X \backslash Y$ is connected.



The Mayer–Vietoris exact sequence

- The homology of X is determined by the homologies of Y, $X\backslash Y$ by the Mayer–Vietoris exact sequence.
  - Case A: $X = X_1 \cup_Y X_2$

$$\cdots \to H_n(Y) \to H_n(X_1) \oplus H_n(X_2)$$
$$\to H_n(X) \xrightarrow{\partial} H_{n-1}(Y) \to \ldots \ .$$

  - Case B:

$$\cdots \to H_n(Y) \xrightarrow{i_1 - i_2} H_n(X\backslash Y)$$
$$\to H_n(X) \xrightarrow{\partial} H_{n-1}(Y) \to \ldots$$

    with $i_1, i_2 : Y \to X\backslash Y$ the two inclusions.

- Proved by codimension 1 transversality on cycle level.

The Seifert–van Kampen Theorem

- The fundamental group of codimension 1 $(X, Y \subset X)$ with injective $\pi_1(Y) \to \pi_1(X)$ given by generalized free products of the fundamental groups of Y, $X \backslash Y$, amalgamated along injections.

- Case A: <u>amalgamated free product</u>

$$\pi_1(X) \;=\; \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2)$$

- Case B: <u>HNN extension</u>

$$\pi_1(X) \;=\; \pi_1(X \backslash Y) *_{\pi_1(Y)} \{t\}$$

with $i_1, i_2 : \pi_1(Y) \to \pi_1(X \backslash Y)$, $i_1 t = t i_2$.

- Amalgamated free products and HNN extensions are the groups which act on trees with quotient I and S^1 (Bass-Serre).

Waldhausen's theorem

- <u>Theorem</u> (1976) The higher Whitehead groups $Wh_*(X) = Wh_*(\pi_1(X))$ for codimension 1 pair $(X, Y \subset X)$ with $\pi_1(Y) \to \pi_1(X)$ injective fit into MV-type exact sequences :

  - Case A: $X \backslash Y$ disconnected $X = X_1 \cup_Y X_2$

    $$\cdots \to \mathrm{Wh}_n(Y) \to \mathrm{Wh}_n(X_1) \oplus \mathrm{Wh}_n(X_2)$$

    $$\to \mathrm{Wh}_n(X) \xrightarrow{\partial} \mathrm{Wh}_{n-1}(Y) \oplus \widetilde{\mathrm{Nil}}_n \to \ldots$$

  - Case B: $X \backslash Y$ connected

    $$\cdots \to \mathrm{Wh}_n(Y) \xrightarrow{i_1 - i_2} \mathrm{Wh}_n(X \backslash Y)$$

    $$\to \mathrm{Wh}_n(X) \xrightarrow{\partial} \mathrm{Wh}_{n-1}(Y) \oplus \widetilde{\mathrm{Nil}}_n \to \ldots$$

- <u>Corollary</u> If π is the fundamental group of a Haken 3-manifold then $\mathrm{Wh}_*(\pi) = 0$. Motivated by absence of Whitehead torsion in rigidity theorem for Haken 3-manifolds.

Cappell's theorem

- Theorem (1976) The algebraic L-groups $L_*(X) = L_*(\pi_1(X))$ for codimension 1 pair $(X, Y \subset X)$ with $\pi_1(Y) \to \pi_1(X)$ injective fit into MV type exact sequences :
  - Case A: $X \backslash Y$ disconnected $X = X_1 \cup_Y X_2$

$$\cdots \to L_n(Y) \to L_n(X_1) \oplus L_n(X_2)$$
$$\to L_n(X) \xrightarrow{\partial} L_{n-1}(Y) \oplus \mathrm{UNil}_n \to \dots$$

  - Case B: $X \backslash Y$ connected

$$\cdots \to L_n(Y) \xrightarrow{i_1 - i_2} L_n(X \backslash Y)$$
$$\to L_n(X) \xrightarrow{\partial} L_{n-1}(Y) \oplus \mathrm{UNil}_n \to \dots$$

- Originally proved by geometry - still need algebraic proof!

- Nowadays also a version for the $\mathbb{S}$-groups.

Cappell's Theorem (contd.)

- Corollary 1 (C., 1976) For $n \geq 6$ simple homotopy equivalence $f : M^n \to X$ splits along $Y \subset X$ if and only if

$$s_Y(f) \ = \ 0 \in \mathsf{UNil}_{n+1}(f) \oplus \widehat{H}^n(\mathbb{Z}_2; K) \ .$$

- Corollary 2 (C., 1976) The UNil-groups are 2-primary. The Novikov conjecture holds for the class of finitely presented groups π obtained from $\{1\}$ by amalgamated free products and HNN extensions along injections.

- Much progress on the Novikov-Borel Conjectures since then.

- Corollary 3 (Roushon, 2000) If M is a Haken 3-manifold with $\partial M \neq \emptyset$ then $\mathsf{UNil}_* = 0$ and $A : H_n(M; \mathbb{L}_\bullet) \to L_n(\mathbb{Z}[\pi_1(M)])$ is an isomorphism for $n \geq 3$.

Algebraic transversality for generalized free products

- Chain complexes C over the group ring $\mathbb{Z}[\pi]$ of a group π which is an amalgamated free product $\pi_1 *_\rho \pi_2$ or an HNN extension $\pi_1 *_\rho \{t\}$ have the transversality properties of manifolds with these fundamental groups. C has fundamental domains $D \subset C$ and

$$Q_n(C) = \varinjlim_D Q_n(D)$$

  $[C, \psi] \in \text{UNil}_n$ is the quadratic Poincaré splitting obstruction.

- On the Novikov conjecture (Proc. 1993 Oberwolfach meeting on Novikov conjectures, LMS Lecture Notes 226 (1995)) includes survey of codimension 1 splitting theorems.

- Can $\mathbb{S}_*(B\pi)$ for torsion-free π be expressed entirely in terms of UNil_*?

Algebraic transversality over simplicial complex X

- Algebraic surgery exact sequence

$$\cdots \to H_n(X;\mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \xrightarrow{s} \mathbb{S}_n(X) \to H_{n-1}(X;\mathbb{L}_\bullet) \to \dots$$

- Theorem (R., 1992) Every quadratic Poincaré complex (C,ψ) over $\mathbb{Z}[\pi_1(X)]$ is cobordant to assembly $A(B,\theta)$ of quadratic complex (B,θ) over $(\mathbb{Z},X)$.

- Algebraic generalization of Wall π-π theorem and the realization theorems of algebraic L-groups as geometric surgery obstruction groups.

- Warning: (B,θ) need not be Poincaré over $(\mathbb{Z},X)$: the image $s(C,\psi) \in \mathbb{S}_n(X)$ is the algebraic Poincaré transversality obstruction.

Past, present and future

- The Novikov-Borel Conjectures have been verified for what is still a relatively small class of infinite groups, using a wonderful mixture of controlled algebra and differential geometry (Farrell-Jones).

- The algebraic surgery exact sequence gives purely algebraic formulation of NBC. Verification for $\pi = \mathbb{Z}^n$ by pure algebra. To what extent can NBC be verified for other groups π by pure algebra?

- Gromov believes that there are many groups out there, and that there is no reason to believe the conjectures to be true in general. In any case, we shall need powerful combination of <u>algebra and topology</u> to decide NBC in general, if this is at all possible.