CIRCLE-VALUED MORSE THEORY AND NOVIKOV HOMOLOGY

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- Traditional Morse theory deals with differentiable real-valued functions $f : M \to \mathbb{R}$ and ordinary homology $H_*(M)$.

- Circle-valued Morse theory deals with differentiable circle-valued functions $f : M \to S^1$ and Novikov homology $H_*^{Nov}(M)$. The circle-valued theory is newer and harder!

- The circle-valued theory has applications to the structure theory of non-simply-connected manifolds, dynamical systems, symplectic topology, Floer theory, Seiberg-Witten theory etc.

Novikov

- S.P.Novikov (1938 –), one of the founding fathers of surgery theory.
- Proved the topological invariance of rational Pontrjagin classes for differentiable manifolds (1965), for which he was awarded the Fields Medal in 1970.
- Last paper in surgery theory (1969) formulated the Novikov conjecture.
- Introduced circle-valued Morse theory in 1981, motivated by physical problems in electromagnetism and fluid mechanics.
- Author of "Topology" (Volume 12 of Encyclopedia of Mathematical Sciences, Springer, 1996) – the best introduction to high-dimensional manifold topology!

The programme

- The geometrically defined Morse-Smale chain complex $C^{MS}(f)$ of a real-valued Morse function $f : M \to \mathbb{R}$ is well-understood. The geometrically defined Novikov chain complex $C^{Nov}(f)$ of a circle-valued Morse function $f : M \to S^1$ is not so well-understood.

- Objective: make the Novikov complex as well-understood as the Morse-Smale complex! Feed algebra back into topology.

- The strategy: lift $f : M \to S^1$ to infinite cyclic covers $\overline{f} : \overline{M} \to \mathbb{R}$ and compare $C^{Nov}(f)$ to $C^{MS}(f_N)$, with

$$f_N = \overline{f}| : M_N = \overline{f}^{-1}[0,1] \to [0,1]$$

- The general theory works for arbitrary $\pi_1(M)$. Will concentrate on the 'simply-connected' special case $\pi_1(M) = \mathbb{Z}$, $\pi_1(\overline{M}) = \{1\}$.

Real-valued Morse functions

- A critical point of a differentiable function $f : M \to \mathbb{R}$ is a zero $p \in M$ of $\nabla f : \tau_M \to \tau_{\mathbb{R}}$.

- A critical point $p \in M$ is nondegenerate if

$$f(p+(x_1,x_2,\ldots,x_m)) \\ = f(p) - \sum_{j=1}^{i}(x_j)^2 + \sum_{j=i+1}^{m}(x_j)^2 \text{ near } p$$

  with i the index of f. Write $\text{Crit}_i(f)$ for the set of index i critical points of f.

- A function $f : M \to \mathbb{R}$ is Morse if every critical point is nondegenerate. If M is compact and non-empty then a Morse $f : M \to \mathbb{R}$ has a finite number

$$c_i(f) = |\text{Crit}_i(f)| \geq 0$$

  of critical points with index i. Note that $c_0(f) > 0$, $c_m(f) > 0$ (minimax principle).

Where do real-valued Morse functions come from?

- Nature (= geometry)

- Morse functions $f : M \to \mathbb{R}$ are dense in the space of all differentiable functions on M.

- Morse theory investigates the relationship between the algebraic topology of M and the Morse functions on M. Typical problem: given M, what are the minimum number of critical points of a Morse function $f : M \to \mathbb{R}$? As usual, it is easier to find answer for $\dim(M) \geq 5$.

Gradient flow

- A vector field $v : M \to \tau_M$ is gradient-like for a Morse function $f : M \to \mathbb{R}$ if there exists a Riemannian metric $\langle\ ,\ \rangle$ on M with

$$\langle v, w \rangle \;=\; \nabla f(w) \in \mathbb{R} \quad (w \in \tau_M)\ .$$

- A downward v-gradient flow line $\gamma : \mathbb{R} \to M$ satisfies

$$\gamma'(t) \;=\; -\,v(\gamma(t)) \in \tau_M(\gamma(t)) \quad (t \in \mathbb{R})\ .$$

  A v-gradient flow line starts at a critical point of index i

$$\lim_{t \to -\infty} \;=\; p \in \mathrm{Crit}_i(f)$$

  and ends at a critical point of index $i-1$

$$\lim_{t \to \infty} \;=\; q \in \mathrm{Crit}_{i-1}(f)\ .$$

Morse theory and surgery

- A <u>critical value</u> of Morse $f : M \to \mathbb{R}$ is $f(p) \in \mathbb{R}$ for critical point $p \in M$. Can assume the critical values are distinct, and that index(p) $\leq$ index(p') if $f(p) < f(p')$.

- Write $N_a = f^{-1}(a) \subset M$ for any regular (= non-critical) value $a \in \mathbb{R}$.

- <u>Theorem</u> (Thom, 1949) (i) If $f : M \to [a,b]$ has no critical values then

$$(M; N_a, N_b) \;\cong\; N_a \times ([0,1]; \{0\}, \{1\}) \ .$$

(ii) If $f : M \to [a,b]$ has only one critical value $c \in [a,b]$, of index i, then $(M; N_a, N_b)$ is the trace of surgery on $S^{i-1} \times D^{m-i} \subset N_a$ with

$$\begin{aligned} N_b &= (N_a \backslash S^{i-1} \times D^{m-i}) \cup D^i \times S^{m-i-1} \ , \\ M &= N_a \times [0,1] \cup D^i \times D^{m-i} \ . \end{aligned}$$

- A Morse function $f : M \to \mathbb{R}$ determines a handlebody decomposition of M

$$M \;=\; \bigcup_{i=0}^{m} \bigcup_{c_i(f)} D^i \times D^{m-i} \;.$$

The Morse-Smale transversality condition

- Theorem (Smale, 1962) For every Morse $f : M \to \mathbb{R}$ there is a class $\mathcal{GT}(f)$ of gradient-like vector fields v for f such that there is only a finite number $n(p,q)$ of v-gradient flow lines from p to q whenever

$$\text{index}(q) \;=\; \text{index}(p) - 1 \;.$$

  $\mathcal{GT}(f)$ is dense in the space of all gradient-like vector fields on M.

The Morse-Smale complex

- The Morse-Smale complex $C = C^{MS}(M, f, v)$ for Morse $f : M \to \mathbb{R}$ and $v \in \mathcal{GT}(f)$ is a based f.g. free $\mathbb{Z}$-module chain complex with $C_i = \mathbb{Z}[\mathrm{Crit}_i(f)]$.

- The differentials are given by the signed numbers of v-gradient flow lines

$$d \ : \ C_i \to C_{i-1} \ ; \ p \mapsto \sum_{q \in \mathrm{Crit}_{i-1}(f)} n(p,q)q \ .$$

- The Morse-Smale complex is the cellular chain complex of the CW structure on M with one i-cell for each critical point of f of index i, $C^{MS}(M, f, v) = C(M)$, so

$$H_*(C^{MS}(M, f, v)) \ = \ H_*(M) \ .$$

- Can also define C^{MS} for Morse $f : (M; N, N') \to ([0,1]; \{0\}, \{1\})$, with $C^{MS}(M, f, v) = C(M, N)$.

The Morse inequalities

- The Betti numbers of a finite CW complex M are defined by

  $$b_i(M) = \text{dim}_{\mathbb{Z}}(H_i(M)/T_i(M)) \ ,$$

  $$q_i(M) = \text{minimum no. generators of } T_i(M)$$

  with

  $$T_i(M) = \{x \in H_i(M) \,|\, nx = 0 \text{ for some } n \neq 0 \in \mathbb{Z}\}$$

  the torsion subgroup of $H_i(M)$.

- Theorem (Morse, 1927) The number $c_i(f)$ of index i critical points of a Morse function $f : M \to \mathbb{R}$ is bounded below by

  $$c_i(f) \geq b_i(M) + q_i(M) + q_{i-1}(M) \ .$$

  Proof A f.g. free $\mathbb{Z}$-module chain complex C with $H_*(C) = H_*(M)$ must have

  $$\text{dim}_{\mathbb{Z}}(C_i) \geq b_i(M) + q_i(M) + q_{i-1}(M) \ .$$

  In particular, this applies to $C = C^{MS}(M, f, v)$.

The Morse inequalities are sharp for $\pi_1(M) = \{1\}$

- Theorem (Smale, 1962) An m-dimensional manifold M with $m \geq 5$ and $\pi_1(M) = \{1\}$ admits a Morse function $f : M \to \mathbb{R}$ with

$$c_i(f) \ = \ b_i(M) + q_i(M) + q_{i-1}(M) \ .$$

- Proved by handle cancellation.

- The situation is much more complicated for $\pi_1(M) \neq \{1\}$. Need algebraic K-theory of the $\mathbb{Z}[\pi_1(M)]$-module version of $C^{MS}(M, f, v)$ to give sharp bounds on minimum number of critical points of Morse $f : M \to \mathbb{R}$ (Sharko).

Circle-valued Morse functions

- A critical point of a differentiable function $f : M \to S^1$ is zero $p \in M$ of $\nabla f : \tau_M \to \tau_{S^1}$.

- A critical point $p \in M$ is nondegenerate if

$$f(p + (x_1, x_2, \ldots, x_m)) \\ = f(p) - \sum_{j=1}^{i} (x_j)^2 + \sum_{j=i+1}^{m} (x_j)^2 \text{ near } p$$

with i the index of f. A function f is Morse if every critical point is nondegenerate.

- If M is compact and non-empty then a Morse $f : M \to S^1$ has a finite number $c_i(f) \geq 0$ of critical points with index i.

- Can define gradient-like $v : M \to \tau_M$, $\mathcal{GT}(f)$ etc., as for the real-valued case.

Where do circle-valued Morse functions come from?

- Nature, cohomology, and knot theory.

- Morse functions $f : M \to S^1$ are dense in the space of all differentiable functions on M representing fixed $c \in H^1(M) = [M, S^1]$.

- Typical problem: given $c \in H^1(M)$ what are the minimum numbers $c_i(f)$ of critical points of a Morse function $f : M \to S^1$ with $f^*(1) = c \in H^1(M)$?

- For $m \geq 6$ can apply the cancellation method of real-valued Morse theory, but the algebraic book-keeping is much harder.

- Circle-valued Morse theory extends to the Morse theory of closed 1-forms, representing classes $c \in H^1(M; \mathbb{R})$.

Fibre bundles over S^1

- The mapping torus of a map $h : N \to N$ is

  $$T(h) \ = \ N \times [0,1]/\{(x,0) \sim (h(x),1)\} \ ,$$

  with canonical projection

  $$p : T(h) \to S^1 = [0,1]/(0 \sim 1) \ ; \ [x,t] \mapsto [t] \ .$$

- If $h : N \to N$ is a diffeomorphism of a closed $(m-1)$-dimensional manifold then $T(h)$ is a closed m-dimensional manifold. The projection $p : T(h) \to S^1$ is a fibre bundle, such that $p^{-1}(a) \cong N$ for each $a \in S^1$. The infinite cyclic cover of $T(h)$

  $$p^*\mathbb{R} \ = \ \overline{T(h)} = N \times \mathbb{R}$$

  is homotopy equivalent to N.

- A fibre bundle $f : M \to S^1$ is a Morse map with $c_*(f) = 0$.

Fibering obstruction theory

- If $f: M \to S^1$ is homotopic to fibre bundle then $\overline{M} = f^*\mathbb{R}$ is homotopy equivalent to a finite CW complex (= fibre). Stallings (1962): partial converse for 3-manifolds M.

- Browder-Levine (1965) : for $m \geq 6$ a function $f : M^m \to S^1$ with $f_* : \pi_1(M) \cong \mathbb{Z}$ is homotopic to fibre bundle if and only if $\overline{M} = f^*\mathbb{R}$ is homotopy equivalent to a finite CW complex.

- Farrell (1967) and Siebenmann (1970) : for $m \geq 6$ a function $f : M \to S^1$ is homotopic to the projection of a fibre bundle if and only if $\overline{M}$ finitely dominated and a Whitehead group obstruction $\Phi(M) \in Wh(\pi_1(M))$ is $\Phi(M) = 0$.

- Proved by handle cancellation and exchanges.

The Novikov ring

- The ring $\mathbb{Z}[[z]]$ consists of the power series

$$p(z) \;=\; \sum_{j=0}^{\infty} n_j z^j \;\; (n_j \in \mathbb{Z}) \ .$$

  Note that $p(z) \in \mathbb{Z}[[z]]$ is a unit if and only if $p(0) = n_0 \in \mathbb{Z}$ is a unit $(= \pm 1)$. Example: $1 - z$.

- The <u>Novikov ring</u>

$$\mathbb{Z}((z)) \;=\; \mathbb{Z}[[z]][z^{-1}]$$

  consists of the power series $\sum\limits_{j=-\infty}^{\infty} n_j z^j$ with coefficients $n_j \in \mathbb{Z}$ such that for some $k \in \mathbb{Z}$

$$n_j \;=\; 0 \text{ for } j < k \ .$$

The real-valued lift of a circle-valued Morse function

- Given Morse $f : M \to S^1$, $v \in \mathcal{GT}(f)$ lift to Morse $\overline{f} : \overline{M} \to \mathbb{R}$, $\overline{v} \in \mathcal{GT}(\overline{f})$. Lift each $p \in \text{Crit}_i(f)$ to $\overline{p} \in \text{Crit}_i(\overline{f})$.

- Choose the generating covering translation $z : \overline{M} \to \overline{M}$ to be the one parallel to $v : M \to \tau_M$, $\langle dz, v \rangle > 0$. In the universal example

$$z \ : \ \overline{S}^1 \ = \ \mathbb{R} \to \mathbb{R} \ ; \ t \mapsto t - 1 \ .$$

- For any $p \in \text{Crit}_i(f)$, $q \in \text{Crit}_{i-1}(f)$ let

$$k \ = \ [\overline{f}(\overline{p}) - \overline{f}(\overline{q})] \in \mathbb{Z} \ .$$

The signed numbers $n_j = n(\overline{p}, z^j\overline{q}) \in \mathbb{Z}$ of $\overline{v}$-gradient flow lines are such that

$$n_j \ = \ 0 \text{ for } j < k \ .$$

The Novikov complex

- The Novikov complex $C = C^{Nov}(M,f,v)$ for Morse $f : M \to S^1$ and $v \in \mathcal{GT}(f)$ is defined geometrically to be the based f.g. free $\mathbb{Z}((z))$-module chain complex with

$$C_i = \mathbb{Z}((z))[\mathrm{Crit}_i(f)] .$$

- The differentials are given by the signed numbers of $\overline{v}$-gradient flow lines

$$d : C_i \to C_{i-1} ; \overline{p} \mapsto \sum_{q \in \mathrm{Crit}_{i-1}(f)} n(\overline{p}, z^j\overline{q}) z^j \overline{q} .$$

- Example $C^{Nov}(M,f,v) = 0$ for fibre bundle.

- Exercise Work out $C^{Nov}(S^1,f,v)$ for

$$f : S^1 \to S^1 ; [t] \mapsto [4t - 9t^2 + 6t^3] \ (0 \leq t \leq 1) .$$

Novikov homology

- The <u>Novikov homology</u> of a finite CW complex M with a map $f : M \to S^1$ is defined by

  $$H_*^{Nov}(M,f) \;=\; H_*(\mathbb{Z}((z)) \otimes_{\mathbb{Z}[z,z^{-1}]} C(\overline{M}))$$

  with $\overline{M} = f^*\mathbb{R}$. The Novikov homology depends only on the cohomology class

  $$c \;=\; f^*(1) \in [M,S^1] \;=\; H^1(M)\ .$$

- <u>Theorem</u> For any map $f : M \to S^1$ on a finite CW complex M the Novikov homology is $H_*^{Nov}(M,f) = 0$ if (and for $\pi_1(\overline{M}) = \{1\}$ only if) $\overline{M}$ is homotopy equivalent to a finite CW complex.

- <u>Example</u> If $f : T(2 : S^1 \to S^1) \to S^1$ is the canonical projection then

  $$H_*^{Nov}(T(2),f) \;=\; \mathbb{Z}((z))/(2-z) \;=\; \widehat{\mathbb{Q}}_2 \neq 0\ .$$

The Novikov complex has Novikov homology

- Theorem (Novikov, 1982) The Novikov complex $C^{Nov}(M, f, v)$ of a Morse function $f : M \to S^1$ is chain equivalent to $\mathbb{Z}((z)) \otimes_{\mathbb{Z}[z,z^{-1}]} C(\overline{M})$, so that

$$H_*(C^{Nov}(M, f, v)) \cong H_*^{Nov}(M, f) \ .$$

- The Novikov complex is directly constructed from $f : M \to S^1$.

- The Novikov homology uses the structure of M as a CW complex, which in general will have many more cells than there are critical points in f.

The Morse-Novikov inequalities

- The Novikov numbers of a finite CW complex M with $f \in H^1(M)$ are defined by

  $$b_i^{Nov}(M,f) = \dim_{\mathbb{Z}((z))}(H_i^{Nov}(M,f)/T_i^{Nov}(M,f)) \ ,$$
  $$q_i^{Nov}(M,f) = \text{min. no. of generators of } T_i^{Nov}(M,f)$$

  with

  $$T_i^{Nov}(M,f) = \{x \in H_i^{Nov}(M,f) \,|\, nx = 0 \text{ for some } n \neq 0 \in \mathbb{Z}((z))\}$$

  the torsion $\mathbb{Z}((z))$-submodule of $H_i^{Nov}(M,f)$.

- Theorem (Novikov, 1982) The number $c_i(f)$ of index i critical points of a Morse function $f : M \to S^1$ is bounded below by

  $$c_i(f) \geq b_i^{Nov}(M,f) + q_i^{Nov}(M,f) + q_{i-1}^{Nov}(M,f) \ .$$

  Proof Since $\mathbb{Z}((z))$ is a principal ideal domain, a f.g. free $\mathbb{Z}((z))$-module chain complex C with $H_*(C) = H_*^{Nov}(M,f)$ must have

  $$\dim_{\mathbb{Z}((z))}(C_i) \geq b_i(M,f) + q_i(M,f) + q_{i-1}(M,f) \ .$$

The Morse-Novikov inequalities are sharp for $\pi_1(M) = \mathbb{Z}$

- Theorem (Farber, 1985) An m-dimensional manifold M with $m \geq 6$ and $\pi_1(M) = \mathbb{Z}$ admits a Morse function $f : M \to S^1$ with

  $$c_i(f) = b_i^{Nov}(M,f) + q_i^{Nov}(M,f) + q_{i-1}^{Nov}(M,f) .$$

- Proved by handle cancellation and handle exchanges.

- The situation is much more complicated for $\pi_1(M) \neq \mathbb{Z}$. Need algebraic K-theory of the $\widehat{\mathbb{Z}[\pi_1(M)]}$-module version of $C^{Nov}(M,f,v)$ to give sharp bounds on minimum number of critical points of Morse $f : M \to S^1$, with $\widehat{\mathbb{Z}[\pi_1(M)]}$ the Novikov completion of $\mathbb{Z}[\pi_1(M)]$ (Pajitnov).

Geometric fundamental domains

- Given Morse $f : M \to S^1$ and regular value $a \in S^1$ lift to $\overline{a} \in \mathbb{R}$. Cut M along $f^{-1}(a) = N \subset M$ to get fundamental domain

  $(M_N; N, z^{-1}N) = \overline{f}^{-1}([\overline{a}, \overline{a}+1]; \{\overline{a}\}, \{\overline{a}+1\})$

  for the infinite cyclic cover

  $$\overline{M} = f^*\mathbb{R} = \bigcup_{j=-\infty}^{\infty} z^j M_N .$$

- The restriction

  $f_N = \overline{f}| : (M_N; N, z^{-1}N) \to ([\overline{a}, \overline{a}+1]; \{\overline{a}\}, \{\overline{a}+1\})$

  is a real-valued Morse function with the same numbers of critical points as f

  $$c_i(f_N) = c_i(f) .$$

- The Morse theory of circle-valued f is the Morse theory of real-valued f_N <u>for all possible choices of N</u>.

Handle exchanges

- Suppose given a map $f : M \to S^1$ on an m-dimensional manifold M and a fundamental domain $(M_N; N, z^{-1}N)$ for $\overline{M} = f^*\mathbb{R}$, with $N = f^{-1}(a)$ for a regular value $a \in S^1$.

- A handle exchange uses an embedding

  $$(D^i \times D^{m-i}, S^{i-1} \times D^{m-i}) \subset (M_N \backslash z^{-1}N, N)$$

  to obtain another fundamental domain $(M_{N'}; N', z^{-1}N')$ for $\overline{M}$ by

  $$N' = (N \backslash S^{i-1} \times D^{m-i}) \cup D^i \times S^{m-i-1} ,$$
  $$M_{N'} = (M_N \backslash D^i \times D^{m-i}) \cup z^{-1}(D^i \times D^{m-i}) .$$

  Any two fundamental domains for $\overline{M}$ are related by a sequence of handle exchanges.

Handle cancellation

- Given $f : M \to S^1$ and a choice of fundamental domain $(M_N; N, z^{-1}N)$ can try to cancel as many handle pairs in $f_N : M_N \to \mathbb{R}$ as possible. Handle cancellations correspond to homotopies $f \simeq f'$ to another Morse function $f' : M \to S^1$ with fewer critical points, keeping $N = f^{-1}(a) \subset M$ fixed.

- In order to decide if there exists a homotopy $f \simeq f'$ to a Morse f' with fewer critical points need to have algebraic description of all possible choices of N.

- The algebraic theory of surgery has a department dealing with the algebraic theory of handle exchanges.

The algebraic construction of the Novikov complex (I)

- The Novikov complex can be constructed algebraically from the Morse-Smale complex of a fundamental domain.

- Given Morse $f : M \to S^1$, $v \in \mathcal{GT}(f)$, a regular value $a \in S^1$, let $N = f^{-1}(a) \subset M$. Let $(M_N; N, z^{-1}N)$ be the corresponding fundamental domain for $\overline{M} = f^*\mathbb{R}$ with Morse $f_N = \overline{f}| : M_N \to \mathbb{R}$, $v_N = \overline{v}| \in \mathcal{GT}(f_N)$.

- The handlebody structure

$$M_N \;=\; N \times [0,1] \cup \bigcup_{i=0}^{m} \bigcup_{c_i(f)} D^i \times D^{m-i}$$

  gives (M_N, N) the structure of a relative CW pair with $c_i(f)$ i-cells.

The algebraic construction of the Novikov complex (II)

- Given CW structure on N with $c_i(N)$ i-cells obtain CW structures on M_N with
$$c_i(M_N) \ = \ c_i(N) + c_i(f) \ i\text{-cells}$$
and a CW structure on M with
$$c_i(M) \ = \ c_i(N) + c_{i-1}(N) + c_i(f) \ i\text{-cells.}$$

- Let $g : C(N) \to C(M_N)$ be the inclusion of chain complexes induced by $N \subset M_N$ which is the inclusion of a subcomplex. Let $h : C(z^{-1}N) \to C(M_N)$ be the chain map induced by the inclusion $z^{-1}N \subset M_N$ which is <u>not</u> the inclusion of a subcomplex.

- The cellular chain complex of $\overline{M}$ is the algebraic mapping cone $C(\overline{M}) = C(\phi)$ of the $\mathbb{Z}[z, z^{-1}]$-module chain map
$$\phi \ = \ g - zh \ : \ C(N)[z, z^{-1}] \to C(M_N)[z, z^{-1}]$$

The algebraic construction of the Novikov complex (III)

- The $\mathbb{Z}[z,z^{-1}]$-module chain map ϕ induces a $\mathbb{Z}((z))$-module chain map

  $$\widehat{\phi} \;=\; g - zh \;:\; C(N)((z)) \to C(M_N)((z))$$

  which is a split injection in each degree, with contractible kernel (= algebraic model for closed v-gradient flow lines in M).

- <u>Theorem</u> The Novikov complex of a Morse $f : M \to S^1$ for appropriate $v \in \mathcal{GT}(f)$ is

  $$C^{Nov}(M,f,v) \;=\; \text{coker}(\widehat{\phi}) \;.$$

  The projection

  $$C\Big(\overline{M};\mathbb{Z}((z))\Big) \;=\; C(\widehat{\phi}) \to C^{Nov}(M,f,v) \;=\; \text{coker}(\widehat{\phi})$$

  is a chain equivalence: the $\overline{v}$-gradient flow lines in $\overline{M}$ are pieced together from the way they cross $z^j M_N \subset \overline{M}$ $(j \in \mathbb{Z})$.