QUADRATIC STRUCTURES IN SURGERY THEORY

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- *The chain complex theory offers many advantages ...*
  a simple and satisfactory algebraic version of the whole setup.
  I hope it can be made to work.

  C.T.C. Wall, *Surgery on Compact Manifolds* (1970)

Past

- The chain complex theory developed in *The algebraic theory of surgery* (R., 1980) expressed the surgery obstruction groups $L_*(A)$ as the cobordism groups of 'quadratic Poincaré complexes', chain complexes C with quadratic Poincaré duality ψ.
- The Wall surgery obstruction of a normal map $(f, b) : M \to X$ from an m-dimensional manifold M to an m-dimensional geometric Poincaré complex X

  $$\sigma_*(f, b) \in L_m(\mathbb{Z}[\pi_1(X)])$$

  was expressed as the cobordism class of a quadratic Poincaré complex (C, ψ) obtained directly from (f, b), without preliminary surgeries below the middle dimension. The homology of C consists of the kernel $\mathbb{Z}[\pi_1(X)]$-modules

  $$H_*(C) \;=\; K_*(M) \;=\; \ker(\widetilde{f}_* : H_*(\widetilde{M}) \to H_*(\widetilde{X})) \;.$$

Advantages and a disadvantage

- The algebraic theory of surgery did indeed offer the advantages predicted by Wall, such as all kinds of exact sequences.
- However, the identification $\sigma_*(f,b) = (C,\psi)$ was not as nice as could have been wished for!
- Specifically, the chain homotopy theoretic treatment of the Wall self-intersection function counting double points

$$\mu(g : S^n \looparrowright M^{2n}) \in \frac{\mathbb{Z}[\pi_1(M)]}{\{x - (-)^n x^{-1} \,|\, x \in \pi_1(M)\}}$$

  was too indirect, making use of Wall's result that for $n \geqslant 3$ $\mu(g) = 0$ if and only if g is regular homotopic to an embedding – proved by the Whitney trick for removing double points.
- Need to count double points of immersions using $\mathbb{Z}_2$-equivariant homotopy theory.

Present

- The 'geometric Hopf invariant' $h(F)$ of Michael Crabb (Aberdeen) provides a satisfactory homotopy-theoretic foundation for algebraic surgery theory.
- Let X, Y be pointed spaces. The geometric Hopf invariant of a stable map $F : \Sigma^\infty X \to \Sigma^\infty Y$ is a stable map

  $$h(F) \; : \; \Sigma^\infty X \to \Sigma^\infty((S^\infty)^+ \wedge_{\mathbb{Z}_2} (Y \wedge Y))$$

  with good naturality properties: if π is a group, X, Y are π-spaces and F is π-equivariant then $h(F)$ is π-equivariant.
- The quadratic structure of a normal map $(f, b) : M \to X$ is the evaluation $\psi = (h(F)/\pi)[X]$ with $\pi = \pi_1(X)$, $F : \Sigma^\infty \widetilde{X}^+ \to \Sigma^\infty \widetilde{M}^+$ a stable π-equivariant map inducing the Umkehr $f^! : C(\widetilde{X}) \to C(\widetilde{M})$ and

  $$h(F)/\pi \; : \; H_m(X) \to H_m(S^\infty \times_{\mathbb{Z}_2} (\widetilde{M} \times_\pi \widetilde{M})) \; .$$

  The resulting quadratic Poincaré complex (C, ψ) has a direct connection with double points of immersions $g : S^n \looparrowright M^m$.

The Umkehr chain map

- The Umkehr of a map $f : N \to M$ of geometric Poincaré complexes is the 'wrong-way' $\mathbb{Z}[\pi_1(M)]$-module chain map

$$f^! \ : \ C(\widetilde{M}) \simeq C(\widetilde{M})^{m-*} \xrightarrow{\widetilde{f}^*} C(\widetilde{N})^{m-*} \simeq C(\widetilde{N})_{*-m+n}$$

with $\widetilde{M}$ the universal cover of M, $\widetilde{N} = f^*\widetilde{M}$ the pullback cover of N, $m = \dim M$, $n = \dim N$ and

$$C(\widetilde{M})^* \ = \ \mathrm{Hom}_{\mathbb{Z}[\pi_1(M)]}(C(\widetilde{M}), \mathbb{Z}[\pi_1(M)]) \ .$$

- In the cases of interest $f^!$ is induced by a stable map F, and the geometric Hopf invariant $h(F)$ captures the double point class of an immersion, and the quadratic structure of a normal map.

The stable Umkehr of an immersion

- An immersion $f : N^n \looparrowright M^m$ has a normal bundle $\nu_f : N \to BO(m-n)$ with $f^*\tau_M = \tau_N \oplus \nu_f$.
- For some $k \geqslant 0$ (e.g. if $k \geqslant 2n-m+1$) can approximate f by an embedding $(e,f) : N \hookrightarrow \mathbb{R}^k \times M$, with $e : N \to \mathbb{R}^k$ and
$$\nu_{(e,f)} \;=\; \nu_f \oplus \epsilon^k \;:\; N \to BO(m-n+k)\ .$$
- Let $\widetilde{M}$ be the universal cover of M. The Pontrjagin-Thom construction applied to the $\pi_1(M)$-equivariant embedding
$$(\widetilde{e},\widetilde{f}) \;:\; \widetilde{N} \;=\; f^*\widetilde{M} \hookrightarrow \mathbb{R}^k \times \widetilde{M}$$
is a $\pi_1(M)$-equivariant stable Umkehr map to the Thom space
$$F \;:\; \Sigma^k \widetilde{M}^+ \to T(\nu_{(\widetilde{e},\widetilde{f})}) \;=\; \Sigma^k T(\nu_{\widetilde{f}})$$
inducing
$$F : \dot{C}(\Sigma^k \widetilde{M}^+) \simeq C(\widetilde{M})_{*-k} \xrightarrow{\;f^!\;} \dot{C}(T(\nu_{(\widetilde{e},\widetilde{f})})) \simeq C(\widetilde{N})_{*-m+n-k}\ .$$
- If f is an embedding can take $k=0$, and F is unstable.

The stable Umkehr of a normal map

- The algebraic mapping cone $C = \mathcal{C}(f^!)$ of the Umkehr $f^! : C(\widetilde{X}) \to C(\widetilde{M})$ of a degree 1 map $f : M \to X$ of m-dimensional geometric Poincaré complexes is such that
$$H_*(C) \;=\; K_*(M) \;=\; \ker(\widetilde{f}_* : H_*(\widetilde{M}) \to H_*(\widetilde{X}))$$
with $\widetilde{f}_*$ a surjection split by $f^!$.
- For a manifold M and a normal map $(f,b) : M \to X$ $f^!$ is induced by a $\pi_1(X)$-equivariant S-dual $F : \Sigma^k \widetilde{X}^+ \to \Sigma^k \widetilde{M}^+$ of the map $T(\widetilde{b}) : T(\nu_{\widetilde{M}}) \to T(\nu_{\widetilde{X}})$ of Thom spaces.
- F can also be constructed geometrically: apply Wall's π-π theorem to obtain a homotopy equivalence
$$(X \times D^k, X \times S^{k-1}) \;\simeq\; (W, \partial W)\ (k \geqslant 3)$$
with $(W, \partial W)$ an $(m+k)$-dimensional manifold with boundary. For $k \geqslant 2n - m + 1$ approximate (f,b) by a framed embedding $M \hookrightarrow W$ and apply the Pontrjagin-Thom construction to $\widetilde{M} \hookrightarrow \widetilde{W}$.

The quadratic construction on a space

- The quadratic construction on a space X is

$$Q(X) \;=\; S^\infty \times_{\mathbb{Z}_2} (X \times X)$$

with the generator $T \in \mathbb{Z}_2$ acting by

$$\begin{aligned} T \;:\; S^\infty \;=\; \varinjlim_k S^k \to S^\infty \;;\; s \mapsto -s \;, \\ T \;:\; X \times X \to X \times X \;;\; (x,y) \mapsto (y,x) \;. \end{aligned}$$

- Let $X^+ = X \sqcup \{+\}$, i.e. X with an adjoined base point $+$.
- The reduced quadratic construction on a pointed space Y is

$$\dot{Q}(Y) \;=\; (S^\infty)^+ \wedge_{\mathbb{Z}_2} (Y \wedge Y) \;.$$

In particular

$$\dot{Q}(X^+) \;=\; Q(X)^+ \;.$$

Unstable vs. stable homotopy theory

- Given pointed spaces X, Y let $[X, Y]$ be the set of homotopy classes of maps $X \to Y$.
- The <u>stable homotopy group</u> is

$$\{X; Y\} \;=\; \varinjlim_{k} [\Sigma^k X, \Sigma^k Y] \;=\; [X, \Omega^\infty \Sigma^\infty X]$$

- The stabilization map

$$[X, Y] \to \{X; Y\} \;=\; [X, \Omega^\infty \Sigma^\infty Y]$$

  is in general not an isomorphism!
- The quotient of $Y \hookrightarrow \Omega^\infty \Sigma^\infty Y$ has a filtration, much studied by homotopy theorists. If $f : N^n \looparrowright M^m$ is an immersion with Umkehr stable map $F : \Sigma^\infty M^+ \to \Sigma^\infty T(\nu_f)$, the adjoint $\mathrm{adj}(F) : M^+ \to \Omega^\infty \Sigma^\infty T(\nu_f)$ sends the k-tuple points of M to the k-th filtration.

The James-Hopf map

- (1950's) James decomposition $\Omega\Sigma Y \simeq_s \bigvee_{k=1}^{\infty} (Y \wedge \cdots \wedge Y)$.
- (1970's) Snaith and others constructed a stable homotopy equivalence

$$\Omega^\infty\Sigma^\infty Y \ \simeq_s \ \bigvee_{k=1}^{\infty} E\Sigma_k^+ \wedge_{\Sigma_k} (Y \wedge \cdots \wedge Y)$$

  for connected Y, group completion in general.
- The stable homotopy projection

$$\Sigma^\infty\Omega^\infty\Sigma^\infty Y \to \Sigma^\infty(E\Sigma_2^+ \wedge_{\Sigma_2} (Y \wedge Y)) \ \ (E\Sigma_2 = S^\infty)$$

  is the <u>James-Hopf double point map</u>. However, only defined for connected Y, and not natural in Y.
- In order to get the quadratic structure of a normal map $(f,b) : M \to X$ need to split off the quadratic part of the $\pi_1(X)$-equivariant map $\mathrm{adj}(F) : \widetilde{X}^+ \to \Omega^\infty\Sigma^\infty\widetilde{M}^+$.

The geometric Hopf invariant $h(F)$

- Let X, Y be pointed π-spaces. When is a k-stable π-map $F : \Sigma^k X \to \Sigma^k Y$ homotopic to the k-fold suspension $\Sigma^k F_0$ of an unstable π-map $F_0 : X \to Y$?
- The geometric Hopf invariant of F is the stable $\pi \times \mathbb{Z}_2$-equivariant map

$$h(F) \;=\; (F \wedge F)\Delta_X - \Delta_Y F \;:\; \Sigma^{k,k} X \to \Sigma^{k,k}(Y \wedge Y)$$

with

$$T \;:\; \Sigma^{k,k} X \;=\; S^k \wedge S^k \wedge X \to \Sigma^{k,k} X \;;\; (s,t,x) \mapsto (t,s,x)\;,$$
$$T \;:\; \Sigma^{k,k}(Y \wedge Y) \to \Sigma^{k,k}(Y \wedge Y)\;;\; (s,t,y_1,y_2) \mapsto (t,s,y_2,y_1)\;.$$

- *The stable $\mathbb{Z}_2$-equivariant homotopy class of*

$$h(F)/\pi \;:\; \Sigma^{k,k} X/\pi \to \Sigma^{k,k}(Y \wedge_\pi Y)$$

is the primary obstruction to the k-fold desuspension of F.

The stable $\mathbb{Z}_2$-equivariant homotopy groups

- Given pointed $\mathbb{Z}_2$-spaces X, Y let $[X, Y]_{\mathbb{Z}_2}$ be the set of $\mathbb{Z}_2$-equivariant homotopy classes of $\mathbb{Z}_2$-equivariant maps $X \to Y$.
- The stable $\mathbb{Z}_2$-equivariant homotopy group is

$$\{X; Y\}_{\mathbb{Z}_2} \ = \ \varinjlim_{k} [\Sigma^{k,k} X, \Sigma^{k,k} Y]_{\mathbb{Z}_2}$$

- Example The $\mathbb{Z}_2$-equivariant Pontrjagin-Thom isomorphism identifies $\{S^0; S^0\}_{\mathbb{Z}_2}$ with the cobordism group of 0-dimensional framed $\mathbb{Z}_2$-manifolds ($=$ finite $\mathbb{Z}_2$-sets). The decomposition of finite $\mathbb{Z}_2$-sets as fixed $\cup$ free determines an isomorphism

$$\{S^0; S^0\}_{\mathbb{Z}_2} \ \cong \ \mathbb{Z} \oplus \mathbb{Z} \ ; \ D = D^{\mathbb{Z}_2} \cup (D - D^{\mathbb{Z}_2}) \mapsto \left(|D^{\mathbb{Z}_2}|, \frac{|D| - |D^{\mathbb{Z}_2}|}{2} \right)$$

$\mathbb{Z}_2$-equivariant stable homotopy theory = fixed-point + fixed-point-free

- Theorem (Crabb+R.) For any pointed π-spaces X, Y there is a naturally split short exact sequence of abelian groups

$$0 \longrightarrow \{X/\pi; \dot{Q}(Y)/\pi\} \longrightarrow \{X/\pi; Y \wedge_\pi Y\}_{\mathbb{Z}_2} \xrightarrow{\rho} \{X/\pi; Y/\pi\} \longrightarrow 0$$

- The surjection ρ is given by the $\mathbb{Z}_2$-fixed points, and is split by

$$\{X/\pi; Y/\pi\} \to \{X/\pi; Y \wedge_\pi Y\}_{\mathbb{Z}_2} \; ; \; F \mapsto \Delta_Y F \; .$$

- The injection is induced by the projection $S^\infty \to \{*\}$

$$\{X/\pi; \dot{Q}(Y)/\pi\} = \{X/\pi; (S^\infty)^+ \wedge Y \wedge_\pi Y\}_{\mathbb{Z}_2} \to \{X/\pi; Y \wedge_\pi Y\}_{\mathbb{Z}_2} \; .$$

Properties of the geometric Hopf invariant $h(F)$

- Proposition (Crabb+R.) The geometric Hopf invariant of a stable π-map $F : \Sigma^\infty X \to \Sigma^\infty Y$

$$\begin{aligned} h(F) &= (F \wedge F)\Delta_X - \Delta_Y F \\ &\in \ker(\rho : \{X/\pi; Y \wedge_\pi Y\}_{\mathbb{Z}_2} \to \{X/\pi; Y/\pi\}) \\ &\qquad = \operatorname{im}(\{X/\pi; \dot{Q}(Y)/\pi\} \hookrightarrow \{X/\pi; Y \wedge_\pi Y\}_{\mathbb{Z}_2}) . \end{aligned}$$

  has the following properties:

  (i) For $F_1, F_2 : \Sigma^\infty X \to \Sigma^\infty Y$

$$h(F_1 + F_2) = h(F_1) + h(F_2) + (F_1 \wedge F_2)\Delta_X .$$

  (ii) For $F : \Sigma^\infty X \to \Sigma^\infty Y$, $G : \Sigma^\infty Y \to \Sigma^\infty Z$

$$h(GF) = (G \wedge G)h(F) + h(G)F .$$

  (iii) If $F \in \operatorname{im}([X, Y]_\pi \to \{X; Y\}_\pi)$ then $h(F) = 0$.

  (iv) The function

$$\{X; Y\} \to \{X; \dot{Q}(Y)\} ; \; F \mapsto h(F) \;\; (\pi = \{1\})$$

  is the James-Hopf double point map.

Double point sets

- The double point set of a map $f : N \to M$ is the $\mathbb{Z}_2$-set

$$D(f,f) \;=\; \{(x,y) \in N \times N \,|\, f(x) = f(y) \in M\} \;.$$

- The ordered double point set of f is the free $\mathbb{Z}_2$-set

$$\overline{D}(f) \;=\; \{(x,y) \in N \times N \,|\, x \neq y \in N,\; f(x) = f(y) \in M\} \;.$$

- The unordered double point set is

$$D(f) \;=\; \overline{D}(f)/\mathbb{Z}_2 \;,$$

  so that the projection $\overline{D}(f) \to D(f)$ is a double covering.
- f is an embedding if and only if $D(f) = \emptyset$.

Immersions

- For $n < m$ the double point set of a self-transverse immersion $f : N^n \looparrowright M^m$ is a stratified set
$$D(f,f) \;=\; \Delta_N \cup \overline{D}(f) \cup (\leqslant 3n-2m)\text{-dimensional strata}$$
with Δ_N n-dimensional, $\overline{D}(f)$ $(2n-m)$-dimensional.
- The normal bundle of the immersion
$$g \;:\; \overline{D}(f) \looparrowright M \;;\; (x,y) \mapsto f(x) = f(y)$$
is
$$\nu_g \;:\; \overline{D}(f) \xrightarrow{h\,=\,incl.} N \times N \xrightarrow{\nu_f \times \nu_f} BO(2(m-n)) \;.$$
- If M, N are oriented then $\overline{D}(f)$ is oriented, with a fundamental class
$$[\overline{D}(f)] \in H_{2n-m}(\overline{D}(f)) \;.$$
- In general, $D(f)$ is not oriented: $T[\overline{D}(f)] = (-)^{m-n}[\overline{D}(f)]$, so $D(f)$ has a $(-)^{m-n}$-twisted fundamental class
$$[D(f)] \in H_{2n-m}(D(f);\mathbb{Z}^{(-)^{m-n}}) \;.$$

The double point class

- Given an immersion $f : N^n \looparrowright M^m$ lift an approximating embedding $(e,f) : N \hookrightarrow \mathbb{R}^k \times M$ to a π-equivariant embedding $(\widetilde{e},\widetilde{f}) : \widetilde{N} \hookrightarrow \mathbb{R}^k \times \widetilde{M}$ with $\pi = \pi_1(M)$, $\widetilde{M}$ the universal cover of M, and $\widetilde{N} = f^*\widetilde{M}$. The map

$$\overline{d} \ : \ \overline{D}(\widetilde{f}) \to S^{k-1} \times (\widetilde{N} \times \widetilde{N}) \ ; \ (x,y) \mapsto \left(\frac{\widetilde{e}(x) - \widetilde{e}(y)}{\|\widetilde{e}(x) - \widetilde{e}(y)\|}, x, y \right)$$

is $\mathbb{Z}_2 \times \pi$-equivariant, and so determines a map

$$d \ : \ D(f) \to S^{k-1} \times_{\mathbb{Z}_2} (\widetilde{N} \times_\pi \widetilde{N}) \subset Q(\widetilde{N})/\pi \ .$$

- The <u>double point class</u> of f is

$$d_*[D(f)] \in H_{2n-m}(Q(\widetilde{N})/\pi; \mathbb{Z}^{(-)^{m-n}}) \ .$$

The Double Point Theorem

- Theorem (Crabb+R.) If $f : N^n \looparrowright M^m$ is an immersion with stable Umkehr map $F : \Sigma^k \widetilde{M}^+ \to \Sigma^k T(\nu_{\widetilde{f}})$ then

$$\begin{aligned} h(F) = HG &\in \ker\big(\rho : \{M^+; T(\nu_{\widetilde{f}}) \wedge_\pi T(\nu_{\widetilde{f}})\}_{\mathbb{Z}_2} \to \{M^+; T(\nu_f)\}\big) \\ &= \operatorname{im}\big(\{M^+; \dot{Q}(T(\nu_{\widetilde{f}}))/\pi\} \hookrightarrow \{M^+; T(\nu_{\widetilde{f}}) \wedge_\pi T(\nu_{\widetilde{f}})\}_{\mathbb{Z}_2}\big) \end{aligned}$$

with $G : \Sigma^{k,k} M^+ \to \Sigma^{k,k} T(\nu_g)$ the $\mathbb{Z}_2$-equivariant Umkehr map of $(e, e, g) : \overline{D}(f) \hookrightarrow \mathbb{R}^k \times \mathbb{R}^k \times M$, with $\mathbb{Z}_2$-fixed points

$$\rho(G) \ : \ \Sigma^k M^+ \to \Sigma^k \{*\} \ = \ \{*\} \ .$$

$\pi = \pi_1(M)$ and $H : T(\nu_g) \hookrightarrow T(\nu_{\widetilde{f}}) \wedge_\pi T(\nu_{\widetilde{f}})$ is induced by the $\mathbb{Z}_2$-equivariant embedding

$$h \ : \ \overline{D}(f) = \overline{D}(\widetilde{f})/\pi \hookrightarrow \widetilde{N} \times_\pi \widetilde{N} \ .$$

The double point class = the evaluation of the geometric Hopf invariant

- Corollary The double point class of $f : N^n \looparrowright M^m$ is the evaluation on the fundamental class $[M] \in H_m(M)$ of the geometric Hopf invariant $h(F)$ of the $\pi_1(M)$-equivariant stable Umkehr map $F : \Sigma^\infty \widetilde{M}^+ \to \Sigma^\infty T(\nu_{\widetilde{f}})$

$$\begin{aligned} d_*[D(f)] \ &= \ h(F)[M] \\ &\in \dot{H}_m(\dot{Q}(T(\nu_{\widetilde{f}}))/\pi_1(M)) \ = \ H_{2n-m}(Q(\widetilde{N})/\pi_1(M);\mathbb{Z}^{(-)^{m-n}}) \ , \end{aligned}$$

identifying $\dot{Q}(T(\nu_{\widetilde{f}})) = T(e_2(\widetilde{\nu_f}))$, the Thom space of the 2nd extended power bundle $e_2(\widetilde{\nu_f}) : Q(\widetilde{N}) \to BO(2(m-n))$.
- Example The Wall self-intersection μ of $f : N^n \looparrowright M^{2n}$ is

$$\begin{aligned} \mu(f) &= d_*[D(f)] = h(F)[M] \in \dot{H}_{2n}(\dot{Q}(T(\nu_{\widetilde{f}}))/\pi_1(M)) \\ &= H_0(Q(\widetilde{N})/\pi_1(M);\mathbb{Z}^{(-)^n}) = \frac{\mathbb{Z}[\pi_1(M)]}{\{x-(-)^n x^{-1} | x \in \pi_1(M)\}} \ . \end{aligned}$$

Symmetric and quadratic structures on chain complexes I.

- Let A be a ring with involution $A \to A; a \mapsto \bar{a}$.
- Given an A-module chain complex C let $C \otimes_A C$ be the $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex

$$\begin{aligned} C \otimes_A C &= C \otimes_{\mathbb{Z}} C/\{x \otimes ay - \bar{a}x \otimes y \mid a \in A,\, x, y \in C\}\ , \\ T &: C_p \otimes_A C_q \to C_q \otimes_A C_p\ ;\ x \otimes y \mapsto (-)^{pq} y \otimes x\ . \end{aligned}$$

- Use the standard free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$

$$W\ :\ \dots \longrightarrow \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2]$$

  to define the $\mathbb{Z}$-module chain complexes

$$\mathrm{Sym}(C) = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_A C)\ ,\ \mathrm{Quad}(C) = W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes_A C)\ .$$

- Write $Q^m(C) = H_m(\mathrm{Sym}(C))$, $Q_m(C) = H_m(\mathrm{Quad}(C))$.

Symmetric and quadratic structures on chain complexes II.

- An <u>m-dimensional symmetric complex</u> (C, ϕ) is an A-module chain complex C together with $\phi \in Q^m(C)$, represented by a collection $\{\phi_s \in (C \otimes_A C)_{m+s} | s \geqslant 0\}$ such that

$$d(\phi_s) \;=\; \phi_{s-1} + (-)^s T\phi_{s-1} \;\; (s \geqslant 0,\, \phi_{-1} = 0)\;.$$

- An <u>m-dimensional quadratic complex</u> (C, ψ) is an A-module chain complex C together with $\psi \in Q_m(C)$, represented by a collection $\{\psi_s \in (C \otimes_A C)_{m-s} | s \geqslant 0\}$ such that

$$d(\psi_s) \;=\; \psi_{s+1} + (-)^{s+1} T\psi_{s+1} \;\; (s \geqslant 0)\;.$$

- The <u>symmetrization</u> chain map

$$1 + T \;:\; \mathrm{Quad}(C) \to \mathrm{Sym}(C)\;;\; \psi \mapsto (1+T)\psi$$

is defined by

$$((1+T)\psi)_s \;=\; \begin{cases} (1+T)\psi_0 & \text{for } s = 0 \\ 0 & \text{for } s \geqslant 1\;. \end{cases}$$

The symmetric construction

- The symmetric construction on a pointed π-space X is the natural chain map

$$\Delta = \{\Delta_s | s \geqslant 0\} \ : \ \dot{C}(X/\pi) \to \mathrm{Sym}(\dot{C}(X))$$

be an Alexander-Whitney-Steenrod diagonal chain approximation, with

$$\Delta_0 \ : \ \dot{C}(X/\pi) \to \dot{C}(X) \otimes_{\mathbb{Z}[\pi]} \dot{C}(X)$$

a chain map, $\Delta_1 : \Delta_0 \simeq T\Delta_0$ a chain homotopy, etc.

- For $\pi = \{1\}$ Δ gives the Steenrod squares of X

$$Sq^i \ : \ H^r(X;\mathbb{Z}_2) \to H^{r+i}(X;\mathbb{Z}_2) = \mathrm{Hom}(H_{r+i}(X;\mathbb{Z}_2);\mathbb{Z}_2) \ ;$$
$$x \mapsto (y \mapsto \langle x \otimes x, \Delta_{r-i}(y) \rangle) \ .$$

Symmetric Poincaré complexes

- An m-dimensional symmetric Poincaré complex (C, ϕ) over A is an m-dimensional f.g. free A-module chain complex
$$C \ : \ C_m \to C_{m-1} \to \cdots \to C_1 \to C_0$$
with $\phi \in Q^m(C)$ such that
$$\phi_0 \ : \ C^{m-*} = \mathrm{Hom}_A(C, A)_{*-m} \to C$$
is an A-module chain equivalence.
- Mishchenko (1974) defined the cobordism group $L^m(A)$ of m-dimensional symmetric Poincaré complexes over A.
- The symmetric signature of an m-dimensional geometric Poincaré complex X is the cobordism class
$$\sigma^*(X) \ = \ (C(\widetilde{X}), \Delta_X[X]) \in L^m(\mathbb{Z}[\pi_1(X)]) \ .$$
Homotopy invariant, generalizing the ordinary signature (= the special case $\sigma^*(X) \in L^{4k}(\mathbb{Z}) = \mathbb{Z}$).

Quadratic Poincaré complexes

- An m-dimensional quadratic Poincaré complex (C, ψ) over A is an m-dimensional f.g. free A-module chain complex C with $\psi \in Q_m(C)$ such that $(1+T)\psi_0 : C^{m-*} \to C$ is an A-module chain equivalence.
- Proposition (R., 1980) The Wall surgery obstruction group $L_m(A)$ is the cobordism group of m-dimensional quadratic Poincaré complexes over A.
- Proof Every quadratic Poincaré complex (C, ψ) is cobordant to a highly-connected complex, i.e. one with $H_r(C) = 0$ for $2r < n$. The cobordism group of highly-connected m-dimensional quadratic Poincaré complexes is essentially the same as the original group $L_m(A)$ ($m \pmod 4$).

The quadratic construction

- The quadratic construction (R., 1980) on a stable π-map $F : \Sigma^\infty X \to \Sigma^\infty Y$ is a natural chain map

$$\psi_F \ : \ \dot{C}(X/\pi) \to \dot{C}(\dot{Q}(Y)/\pi) \ = \ \mathrm{Quad}(\dot{C}(Y)) \ .$$

such that

$$(1+T)\psi_F \ = \ (F \otimes F)\Delta_X - \Delta_Y F \ : \ \dot{C}(X/\pi) \to \mathrm{Sym}(\dot{C}(Y)) \ .$$

- ψ_F was obtained using a natural (but implicit) chain homotopy

$$\Delta_{\Sigma X} \simeq \{\Delta_{s-1} | s \geqslant 0\} : \dot{C}(\Sigma X/\pi) \simeq \dot{C}(X/\pi)_{*-1} \to \mathrm{Sym}(\dot{C}(\Sigma X))$$

with $\Delta_{-1} = 0$ - cup products vanish in suspensions!

- For $\pi = \{1\}$ ψ_F gives the functional Steenrod squares of F.
- Proposition The quadratic construction ψ_F is induced by the geometric Hopf invariant $h(F) : \Sigma^\infty X/\pi \to \Sigma^\infty \dot{Q}(Y)/\pi$.

Surgery obstruction = quadratic Poincaré cobordism class

- Proposition (R., 1980) The surgery obstruction of a normal map $(f, b) : M \to X$ is a quadratic Poincaré cobordism class

$$\sigma_*(f, b) \;=\; (\mathcal{C}(f^!), \psi) \in L_m(\mathbb{Z}[\pi_1(X)])$$

  with $\mathcal{C}(f^!)$ the algebraic mapping cone of the Umkehr $\mathbb{Z}[\pi_1(X)]$-module chain map $f^! : C(\widetilde{X}) \to C(\widetilde{M})$, such that $H_*(\mathcal{C}(f^!)) = K_*(M)$.
- In the original construction $\psi = i_\% \psi_F[X] \in Q_m(\mathcal{C}(f^!))$ was the evaluation on $[X] \in H_m(X)$ of the composite

$$i_\%\psi_F \;:\; H_m(X) \xrightarrow{\psi_F} Q_m(C(\widetilde{M})) \xrightarrow{i_\%} Q_m(\mathcal{C}(f^!))$$

  with $F : \Sigma^\infty \widetilde{X}^+ \to \Sigma^\infty \widetilde{M}^+$ a stable $\pi_1(X)$-equivariant Umkehr map and $i_\%$ induced by $i : C(\widetilde{M}) \hookrightarrow \mathcal{C}(f^!)$.
- Can now identify $\psi = i_\% h(F)[X]$.

Future

- The interpretation of the geometric Hopf invariant of $F : \Sigma^\infty \widetilde{X}^+ \to \Sigma^\infty \widetilde{M}^+$

$$h(F) \in \{X^+; Q(\widetilde{M})/\pi_1(M)\}$$

as 'universal double points' of normal map $(f, b) : M \to X$, using all of ΩM not just $H_0(\Omega M) = \mathbb{Z}[\pi_1(M)]$.
- Quadratic Poincaré kernels for bounded/controlled normal maps, without preliminary surgeries below the middle dimension.
- Homotopy-theoretic total surgery obstruction $s(X) \in \mathcal{S}_m(X)$ of an m-dimensional geometric Poincaré complex X using an X-local geometric Hopf invariant.
- Homotopy-theoretic surgery on Poincaré complexes.
- Quadratic Poincaré sheaf/intersection homology theory.