THE RISE, FALL AND RISE OF SIMPLICIAL COMPLEXES

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Simplicial complexes

- A **simplicial complex** is a combinatorial scheme K for building a topological space $\|K\|$ from simplices - such spaces are called **polyhedra**.
- Combinatorial topology: polyhedra are the easiest spaces to construct. On the syllabus of every introductory course of algebraic topology, including Warwick!
- Piecewise linear topology of polyhedra, especially PL manifolds.

Some of the branches of topology which feature simplicial complexes

- Homotopy theory: simplicial complexes are special cases of simplicial sets.
- Algebraic topology of finite sets.
- Algebraic topology of groups and categories.
- Topological data analysis: simplicial complexes in persistence homology, arising as the nerves of covers of clouds of point data in Euclidean space.
- Surgery theory of topological manifolds: the homotopy types of spaces with Poincaré duality.

A combinatorial proof of the Poincaré duality theorem

- The theorem: for any n-dimensional homology manifold K with fundamental class $[K] \in H_n(K)$ the cap products

$$[K] \cap - : \; H^{n-*}(K) \to H_*(K)$$

  are $\mathbb{Z}$-module isomorphisms.
- There have been many proofs of the duality theorem, but none as combinatorial as the one involving the "$(\mathbb{Z}, K)$-module" category.

The first converse of Poincaré duality

- **Converse 1** "When is a topological space a manifold?"
  A simplicial complex K is an n-dimensional homology manifold with fundamental class $[K] \in H_n(K)$ if and only if the "$(\mathbb{Z}, K)$-module" chain map

  $$[K] \cap - \ : \ C(K)^{n-*} \to C(K')$$

  is a chain equivalence, with $K' =$ barycentric subdivision of K.
- Proof is purely combinatorial, and is relatively straightforward.
- If an n-dimensional topological manifold is homeomorphic to a polyhedron $\|K\|$ then K is an n-dimensional homology manifold.
- For $n \geqslant 5$ an n-dimensional topological manifold is homotopy equivalent but not in general homeomorphic to the polyhedron $\|K\|$ of an n-dimensional homology manifold K.

The second converse of Poincaré duality

- **Converse 2** "When is a topological space homotopy equivalent to a manifold?"
  For $n \geqslant 5$ a polyhedron $\|K\|$ is homotopy equivalent to an n-dimensional topological manifold if and only if it has just the right amount of "$(\mathbb{Z}, K)$-module" Poincaré duality.
- Proof requires both all of the geometric surgery theory of Browder-Novikov-Sullivan-Wall, the topological manifold structure theory of Kirby and Siebenmann, as well as my algebraic theory of surgery on chain complexes with Poincaré duality.

Simplicial complexes

- A **simplicial complex** K consists of an ordered set $K^{(0)}$ and a collection $\{K^{(m)} \mid m \geqslant 0\}$ of m-element subsets $\sigma \leqslant K^{(0)}$, such that if $\sigma \in K$ and $\tau \leqslant \sigma$ is non-empty then $\tau \in K$.
- Call $\tau \leqslant \sigma$ a **face** of σ. Also written $\sigma \geqslant \tau$.
- For any $m \geqslant 0$ the **standard** m**-simplex** Δ^m, the simplicial complex consisting of all the non-empty subsets

$$\sigma \leqslant (\Delta^m)^{(0)} = \{0, 1, \ldots, m\} \ .$$

- The **polyhedron** (or **realization**) of a simplicial complex K is the topological space

$$\|K\| \ = \ (\coprod_{m=0}^{\infty} K^{(m)} \times \|\Delta^m\|)/\sim$$

with $\|\Delta^m\|$ the convex hull of the $(m+1)$ unit vectors $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^{m+1}$.

Combinatorial sheaf theory: the $(\mathbb{Z}, K)$-category I.

- In 1990, R.+Weiss introduced an additive category $\mathbb{A}(\mathbb{Z}, K)$ of "$(\mathbb{Z}, K)$-modules".
- Convenient for the "local K-controlled algebraic topology" of spaces X with a map $X \to \|K\|$.
- A $(\mathbb{Z}, K)$-**module** is a f.g. free $\mathbb{Z}$-module M with a direct sum decomposition
$$M \;=\; \sum_{\sigma \in K} M(\sigma)\ .$$

Combinatorial sheaf theory: the $(\mathbb{Z}, K)$-category II.

- A $(\mathbb{Z}, K)$-**module morphism** $f : M \to N$ is a collection of $\mathbb{Z}$-module morphisms $f(\sigma, \tau) : M(\sigma) \to N(\tau)$ $(\sigma \leqslant \tau)$, i.e. a $\mathbb{Z}$-module morphism f such that

$$f(M(\sigma)) \subseteq \sum_{\tau \geqslant \sigma} N(\tau) \ .$$

- In terms of matrices, f is upper triangular.
- f is a $(\mathbb{Z}, K)$-module isomorphism if and only if each $f(\sigma, \sigma) : M(\sigma) \to N(\sigma)$ $(\sigma \in K)$ is a $\mathbb{Z}$-module isomorphism.

$C(K)$ is not a $(\mathbb{Z}, K)$-module chain complex I.

- The $\mathbb{Z}$-module chain complex $C(K)$ is defined using the ordering of $K^{(0)}$, with

$$d \ : \ C(K)_m \ = \ \mathbb{Z}[K^{(m)}] \to C(K)_{m-1} \ = \ \mathbb{Z}[K^{(m-1)}] \ ;$$
$$(v_0 v_1 \dots v_m) \mapsto \sum_{i=0}^{m} (-)^{i+1} (v_0 v_1 \dots v_{i-1} v_{i+1} \dots v_m)$$
$$\big(v_0 < v_1 < \dots < v_m \in K^{(0)}\big) \ .$$

$C(K)$ **is not a** $(\mathbb{Z}, K)$**-module chain complex II.**

- $C(K)$ used to define the homology $\mathbb{Z}$-modules of K and $\|K\|$

$$\begin{aligned} H_m(K) &= H_m(\|K\|) \\ &= \frac{\ker(d : C(K)_m \to C(K)_{m-1})}{\operatorname{im}(d : C(K)_{m+1} \to C(K)_m)} \end{aligned}$$

- $C(K)$ is not a $(\mathbb{Z}, K)$-module chain complex, since

$$(v_0 v_1 \dots v_m) \text{ is not a face of } (v_0 v_1 \dots v_{i-1} v_{i+1} \dots v_m) \ .$$

$C(K)^*$ is a $(\mathbb{Z},K)$-module cochain complex I.

- The $\mathbb{Z}$-module cochain complex

$$C(K)^* \ = \ \mathrm{Hom}_{\mathbb{Z}}(C(K),\mathbb{Z})$$

dual to $C(K)$ has

$$d^* \ : \ C(K)^m \ = \ \mathrm{Hom}_{\mathbb{Z}}(C(K)_m,\mathbb{Z}) \ = \ \mathbb{Z}[K^{(m)}] \\ \to C(K)^{m+1} \ = \ \mathbb{Z}[K^{(m+1)}] \ ; \ f \mapsto fd$$

with

$$d^*(\sigma) \ = \ \sum \pm\tau \ (\sigma \in K^{(m)}, \tau \in K^{(m+1)}, \sigma < \tau) \ .$$

$C(K)^*$ is a $(\mathbb{Z}, K)$-module cochain complex II.

- $C(K)^*$ used to define the cohomology $\mathbb{Z}$-modules of K and $\|K\|$

$$\begin{aligned} H^m(K) &= H^m(\|K\|) \\ &= \frac{\ker(d^* : C(K)^m \to C(K)^{m+1})}{\operatorname{im}(d^* : C(K)^{m-1} \to C(K)^m)} \end{aligned}$$

- $C(K)^*$ is a $(\mathbb{Z}, K)$-module cochain complex.

The barycentric subdivision K'

- The **barycentric subdivision** of a simplicial complex K is the simplicial complex K' with one 0-simplex $\widehat{\sigma} \in (K')^0 = K$ for each simplex $\sigma \in K$ and one m-simplex $\widehat{\sigma}_0\widehat{\sigma}_1 \dots \widehat{\sigma}_m \in (K')^{(m)}$ for each $(m+1)$ term sequence $\sigma_0 < \sigma_1 < \dots < \sigma_m \in K$ of proper faces in K.
- Homeomorphism $\|K'\| \to \|K\|$ sending $\widehat{\sigma} \in K'^{(0)}$ of $\sigma \in K^{(m)}$ to the barycentre of $\|\sigma\|$.

Dual cells

- The **dual cell** of $\sigma \in K$ is the subcomplex

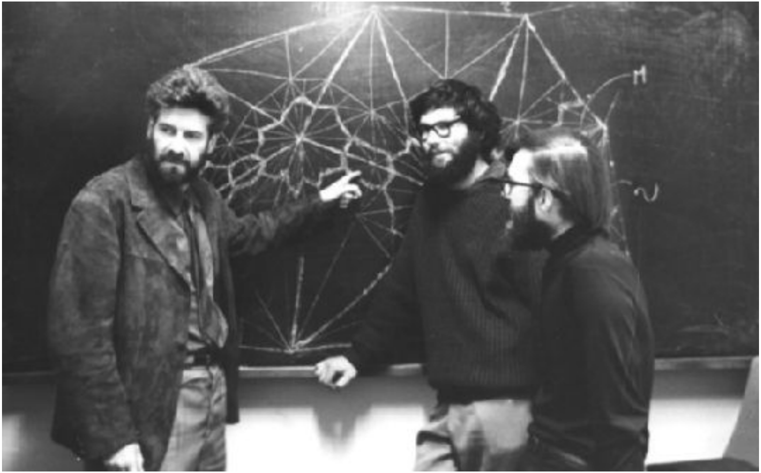
$$D(\sigma, K) \;=\; \{\widehat{\sigma}_0\widehat{\sigma}_1 \dots \widehat{\sigma}_m \,|\, \sigma \leqslant \sigma_0 < \sigma_1 < \cdots < \sigma_m\} \subseteq K' \;.$$

- The **boundary** of the dual cell is the subcomplex

$$\partial D(\sigma, K) \;=\; \{\widehat{\sigma}_0\widehat{\sigma}_1 \dots \widehat{\sigma}_m \,|\, \sigma < \sigma_0 < \sigma_1 < \cdots < \sigma_m\} \subset D(\sigma, K) \;.$$

- **Proposition** $C(K')$ is a $(\mathbb{Z}, K)$-module chain complex which is $\mathbb{Z}$-module chain equivalent to $C(K)$.

The second barycentric subdivision K''



Christopher Zeeman, Colin Rourke and Brian Sanderson, 1965

Homology manifolds

- For any simplicial complex K and m-simplex $\sigma \in K^{(m)}$ there are isomorphisms

$$H_*(D(\sigma, K)) \cong \mathbb{Z} \text{ if } * = 0, \quad = 0 \text{ otherwise },$$
$$H_*(D(\sigma, K), \partial D(\sigma, K)) \cong H_{*+m}(\|K\|, \|K\| - \{\widehat{\sigma}\}) .$$

- K is an n-**dimensional homology manifold** if for each $x \in \|K\|$

$$H_*(\|K\|, \|K\| - \{x\}) \cong \mathbb{Z} \text{ if } * = n, \quad = 0 \text{ otherwise} .$$

- Equivalent to each $\partial D(\sigma, K)$ being a homology $(n - m - 1)$-sphere

$$H_*(\partial D(\sigma, K)) \cong \mathbb{Z} \text{ if } * = 0 \text{ or } = n-m-1, \quad = 0 \text{ otherwise} .$$

The assembly functor $A : (\mathbb{Z}, K)$**-modules** $\to \mathbb{Z}[\pi_1(K)]$**-modules**

- Let K be a connected simplicial complex with universal covering projection $p : \widetilde{K} \to K$.
- The **assembly** of a $(\mathbb{Z}, K)$-module $M = \sum_{\sigma \in K} M(\sigma)$ is the f.g. free $\mathbb{Z}[\pi_1(K)]$-module

$$A(M) \ = \ \sum_{\widetilde{\sigma} \in \widetilde{K}} M(p(\widetilde{\sigma})) \ .$$

- The assembly functor $A : M \mapsto A(M)$ sends K-local to $\mathbb{Z}[\pi_1(K)]$-global algebraic topology.
- The assembly $A(C(K'))$ is $\mathbb{Z}[\pi_1(K)]$-module chain equivalent to $C(\widetilde{K})$.

The chain duality

- The proof of Poincaré duality will now be expressed as an assembly of K-local to $\mathbb{Z}[\pi_1(K)]$-global, using a "chain duality" on $\mathbb{A}(\mathbb{Z}, K)$.
- The **chain dual** of a $(\mathbb{Z}, K)$-module chain complex C is the $(\mathbb{Z}, K)$-module chain complex

$$TC \ = \ \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{(\mathbb{Z},K)}(C(K)^{-*}, C), \mathbb{Z}) \ .$$

- TC is $\mathbb{Z}$-module chain equivalent to $C^{-*} = \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Z})^{-*}$. However, even if C is concentrated in dimension 0, then TC is not concentrated in dimension 0.
- Combinatorial version of Verdier duality in sheaf theory.

Homology = $(\mathbb{Z}, K)$-module chain maps

- For any simplicial complex K the $(\mathbb{Z}, K)$-module chain complexes $C(K')$, $C(K)^{-*}$ are chain dual, with chain equivalences

$$TC(K') \simeq C(K)^{-*} \, , \; TC(K)^{-*} \simeq C(K') \, .$$

- The $\mathbb{Z}$-module morphism

$$H_n(K) \to H_0(\mathrm{Hom}_{(\mathbb{Z},K)}(C(K)^{n-*}, C(K'))) \, ; \; [K] \mapsto [K] \cap -$$

  sending a homology class $[K] \in H_n(K)$ to the chain homotopy classes of the $(\mathbb{Z}, K)$-module chain map $[K] \cap - : C(K)^{n-*} \to C(K')$ is an isomorphism.

$(\mathbb{Z}, K)$-proof of Poincaré duality

- **Theorem** A simplicial complex K is an n-dimensional homology manifold if and only if there exists a homology class

$$[K] \in H_n(K) \ = \ H_0(\mathrm{Hom}_{(\mathbb{Z},K)}(C(K)^{n-*}, C(K')))$$

which is a $(\mathbb{Z}, K)$-module chain equivalence.

- **Proof** For any homology class $[K] \in H_n(K)$ the $\mathbb{Z}$-module chain maps

$$([K] \cap -)(\sigma, \sigma) \ : \ C(K)^{n-*}(\sigma) \ \simeq \ C(D(\sigma, K))^{n-m-*}$$
$$\to C(K')(\sigma) \ \simeq \ C(D(\sigma, K), \partial D(\sigma, K)) \ (\sigma \in K^{(m)})$$

are chain equivalences if and only if each $\partial D(\sigma, K)$ is a homology $(n - m - 1)$-sphere.

Poincaré complexes

- The assembly of a $(\mathbb{Z}, K)$-module chain map $[K] \cap - : C(K)^{n-*} \to C(K')$ is a $\mathbb{Z}[\pi_1(K)]$-module chain map

$$\begin{aligned} A([K] \cap -) \ : \ A(C(K)^{n-*}) \ &= \ \mathrm{Hom}_{\mathbb{Z}[\pi_1(K)]}(C(\widetilde{K}), \mathbb{Z}[\pi_1(K)])^{n-*} \\ &\to A(C(K')) \ = \ C(\widetilde{K}') \ . \end{aligned}$$

- An n-**dimensional Poincaré complex** K is a simplicial complex with a homology class $[K] \in H_n(K)$ such that $A([K] \cap -)$ is a $\mathbb{Z}[\pi_1(K)]$-module chain equivalence.
- If the polyhedron $\|K\|$ is an n-dimensional topological manifold then K is an n-dimensional Poincaré complex.
- For each $n \geqslant 4$ there exist n-dimensional topological manifolds which are not polyhedra of simplicial complexes.

Homotopy types of topological manifolds

- If $\|K\|$ is homotopy equivalent to an n-dimensional topological manifold then K is an n-dimensional Poincaré complex.
- The **total surgery obstruction** $s(K) \in \mathbb{S}_n(K)$ of an n-dimensional Poincaré complex K is a homotopy invariant taking value in an abelian group, such that $s(K) = 0$ if (and for $n \geqslant 5$ only if) $\|K\|$ is homotopy equivalent to an n-dimensional topological manifold.
- The **total rigidity obstruction** $s(f) \in \mathbb{S}_{n+1}(f)$ of a homotopy equivalence $f : M \to N$ of n-dimensional topological manifolds is a homotopy invariant such that $s(f) = 0$ if (and for $n \geqslant 5$ only if) f is homotopic to a homeomorphism.

The algebraic surgery exact sequence

- The $\mathbb{S}$-groups of a simplicial complex K are defined to fit into long exact sequence of abelian groups, cobordism groups of chain complexes C with Poincaré duality (generalized Witt groups)

$$\ldots \longrightarrow H_n(K;\mathbb{L}_\bullet(\mathbb{Z})) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(K)]) \longrightarrow \mathbb{S}_n(K) \longrightarrow H_{n-1}(K;\mathbb{L}_\bullet(\mathbb{Z})) \longrightarrow \ldots .$$

- The total surgery obstruction $s(K) \in \mathbb{S}_n(K)$ is the cobordism class of the $\mathbb{Z}[\pi_1(K)]$-module contractible $(\mathbb{Z}, K)$-module chain complex

$$C = \mathcal{C}([K] \cap - : C(K)^{n-*} \to C(K'))_{*+1}$$

with $(n-1)$-dimensional quadratic Poincaré duality.
- Underlying homotopy theory developed in book with Michael Crabb.

The future

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Dedicatee of *The geometric Hopf invariant and surgery theory* (2018)

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