NILPOTENCE = TORSION

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Nilpotent endomorphisms

- Let A be an associative ring with 1.
- An endomorphism $\nu : P \to P$ of an A-module P is **nilpotent** if $\nu^N = 0 : P \to P$ for some $N \geqslant 0$.
- If ν is nilpotent then $1 + \nu : P \to P$ is an isomorphism with
$$(1+\nu)^{-1} \;=\; 1 - \nu + \nu^2 - \cdots + (-)^{N-1}\nu^{N-1} \;:\; P \to P \;.$$
- For an indeterminate z let $A[z]$ be the polynomial extension, and let $A[[z]]$ be the ring of formal power series.
- **Proposition 1** Let $f, g : P \to Q$ be morphisms of f.g. projective A-modules. The $A[z]$-module morphism $f + gz : P[z] \to Q[z]$ is an isomorphism if and only if $f : P \to Q$ is an isomorphism and $f^{-1}g : P \to P$ is nilpotent.
- **Remark 1** Proposition 1 is false if P is not f.g., for example if
$$f \;=\; 1 \;,\; g \;=\; y \;:\; P \;=\; A[[y]] \to P \;=\; A[[y]]$$
with $(f + gz)^{-1} = \sum\limits_{j=0}^{\infty}(-)^j g^j z^j : P[z] \to P[z]$.

Near-projections

- Let $A[z,z^{-1}]$ be the Laurent polynomial extension of A.
- An endomorphism $\rho : P \to P$ of an A-module P is a **near-projection** if $\rho(1-\rho) : P \to P$ is nilpotent.
- **Example 1** If ν is nilpotent then ν is a near-projection.
- **Example 2** If ν is nilpotent then $1-\nu$ is a near-projection.
- **Proposition 2** Let $f,g : P \to Q$ be morphisms of f.g. projective A-modules. The $A[z,z^{-1}]$-module morphism $f+gz : P[z,z^{-1}] \to Q[z,z^{-1}]$ is an isomorphism if and only if $f+g : P \to Q$ is an isomorphism and $(f+g)^{-1}g : P \to P$ is a near-projection.
- **Remark 2** Proposition 2 is false if P is not f.g. – same counterexample as in Remark 1.

Why is $1-\rho+\rho z$ an isomorphism for a near-projection ρ?

- Given a near-projection $\rho : P \to P$ let $\nu = \rho(1-\rho) : P \to P$, so that $\nu^N = 0$ for some $N \geqslant 0$. Define the projection

$$\begin{aligned}\pi &= (\rho^N + (1-\rho)^N)^{-1}\rho^N \\ &= \rho + (1/2)(2\rho-1)((1-4\nu)^{-1/2}-1) \\ &= \rho + (2\rho-1)(\nu + 3\nu^2 + 10\nu^3 + \dots) \ : \ P \to P\end{aligned}$$

- The near-projection splits as

$$\rho \ = \ \rho_+ \oplus \rho_- \ : \ P \ = \ P_+ \oplus P_- \to P \ = \ P_+ \oplus P_-$$

with $P_+ = (1-\pi)(P)$, $P_- = \pi(P)$ and the endomorphisms

$$\rho_+ \ = \ \rho| \ : \ P_+ \to P_+ \ , \ 1-\rho_- \ = \ (1-\rho)| \ : \ P_- \to P_-$$

nilpotent.

- The endomorphism of $(P_+ \oplus P_-)[z, z^{-1}]$

$$1-\rho+\rho z \ = \ (1+\rho_+(z-1)) \oplus z(1+(1-\rho_-)(z^{-1}-1))$$

is an isomorphism, by a double application of Proposition 1.

Algebraic K-theory

- The **algebraic K-groups** of A are the algebraic K-groups of the exact category Proj(A) of f.g. projective A-modules

$$K_*(A) \;=\; K_*(\mathrm{Proj}(A)) \;.$$

- The **nilpotent K-groups** of A are the algebraic K-groups of the exact category Nil(A) of f.g. projective A-modules P with a nilpotent endomorphism $\nu : P \to P$

$$\mathrm{Nil}_*(A) \;=\; K_*(\mathrm{Nil}(A)) \;=\; K_*(A) \oplus \widetilde{\mathrm{Nil}}_*(A) \;.$$

- **Proposition 3** Let Near(A) be the exact category of f.g. projective A-modules P with a near-projection $\rho : P \to P$. The equivalence of exact categories

$$\mathrm{Near}(A) \xrightarrow{\approx} \mathrm{Nil}(A)\times\mathrm{Nil}(A)\,;\; (P,\rho) \mapsto (P_+,\rho_+)\times(P_-,1-\rho_-)$$

induces an isomorphism of algebraic K-groups

The Bass-Heller-Swan Theorem

- **Theorem** (B-H-S 1965 for $n \leqslant 1$, Quillen 1972 for $n \geqslant 2$) For any ring A there are natural splittings

$$K_n(A[z]) = K_n(A) \oplus \widetilde{\mathrm{Nil}}_{n-1}(A) ,$$
$$K_n(A[z,z^{-1}]) = K_n(A) \oplus K_{n-1}(A) \oplus \widetilde{\mathrm{Nil}}_{n-1}(A) \oplus \widetilde{\mathrm{Nil}}_{n-1}(A) .$$

- **Original proof** (i) Use Higman linearization to represent every $\tau \in K_1(A[z])$ by a linear invertible $k \times k$ matrix

$$B = B_0 + zB_1 \in GL_k(A[z])$$

with $B_0 \in M_k(A)$ invertible and $(B_0)^{-1}B_1 \in M_k(A)$ nilpotent.
- (ii) Represent every $\tau \in K_1(A[z,z^{-1}])$ by

$$B = B_0 + zB_1 \in GL_k(A[z,z^{-1}])$$

with $B_0 + B_1 \in M_k(A)$ invertible and $(B_0 + B_1)^{-1}B_1 \in M_k(A)$ a near-projection.
- (iii) For $n \in \mathbb{Z}$ apply the algebraic K-theory commutative localization exact sequence for $A[z] \to \{z\}^{-1}A[z] = A[z,z^{-1}]$.

The Farrell-Hsiang splitting theorem

- **Theorem** (1968)
  A homotopy equivalence $h : M^n \to X^{n-1} \times S^1$ with M an n-dimensional manifold and X an $(n-1)$-dimensional manifold has a **splitting obstruction**

  $$\Phi(h) \in \mathrm{Nil}_0(\mathbb{Z}[\pi_1(X)])/\mathrm{Nil}_0(\mathbb{Z}) \ = \ \widetilde{K}_0(\mathbb{Z}[\pi_1(X)]) \oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi_1(X)])\ .$$

- $\Phi(h) = 0$ if (and for $n \geqslant 6$ only if) h is h-cobordant to a split homotopy equivalence $h : M \to X \times S^1$, with the restriction

  $$h| \ : \ V^{n-1} \ = \ h^{-1}(X \times \{*\}) \to X$$

  also a homotopy equivalence.
- $\Phi(h)$ is a component of the Whitehead torsion

  $$\begin{aligned}&\tau(h) = (-)^{n-1}\tau(h)^* \in \mathrm{Wh}(\pi_1(X) \times \mathbb{Z})\\ &= \mathrm{Wh}(\pi_1(X)) \oplus \widetilde{K}_0(\mathbb{Z}[\pi_1(X)]) \oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi_1(X)]) \oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi_1(X)]).\end{aligned}$$

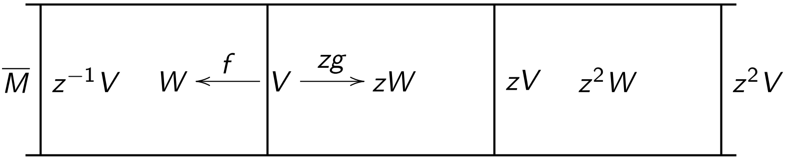
Geometric transversality over S^1

- Given a map $h : M \to X \times S^1$ let $\overline{M} = h^*(X \times \mathbb{R})$ be the pullback infinite cyclic cover of M, with $z : \overline{M} \to \overline{M}$ a generating covering translation.
- Assuming M is an n-dimensional manifold make h transverse regular at $X \times \{*\} \subset X \times S^1$, with

$$V^{n-1} \; = \; h^{-1}(X \times \{*\}) \subset M^n$$

  a 2-sided codimension 1 submanifold. Cutting M at $V \subset M$ there is obtained a fundamental domain $(W; z^{-1}V, V)$ for $\overline{M}$

$$\overline{M} \; = \; \bigcup_{k=-\infty}^{\infty} z^k(W; z^{-1}V, V) \; .$$



Algebraic transversality over S^1

- Let $C(V)$, $C(W)$ denote the cellular finite based f.g. free $\mathbb{Z}[\pi_1(X)]$-module chain complexes of the pullbacks to V, W of the universal cover $\widetilde{X}$ of X.
- Identify $\mathbb{Z}[\pi_1(X \times S^1)] = \mathbb{Z}[\pi_1(X)][z, z^{-1}]$ and let $C(\overline{M})$ denote the cellular finite based f.g. free $\mathbb{Z}[\pi_1(X)][z, z^{-1}]$-module chain complex of the pullback to M of the universal cover $\widetilde{X} \times \mathbb{R}$ of $X \times S^1$.
- The decomposition $\overline{M} = \bigcup\limits_{k=-\infty}^{\infty} z^k W$ determines a

  Mayer-Vietoris presentation of $C(\overline{M})$

  $$0 \longrightarrow C(V)[z, z^{-1}] \xrightarrow{f - zg} C(W)[z, z^{-1}] \longrightarrow C(\overline{M}) \longrightarrow 0$$

  with $f, g : C(V) \to C(W)$ the left and right inclusions.
- For any ring A every finite f.g. free $A[z, z^{-1}]$-module chain complex C has a Mayer-Vietoris presentation.

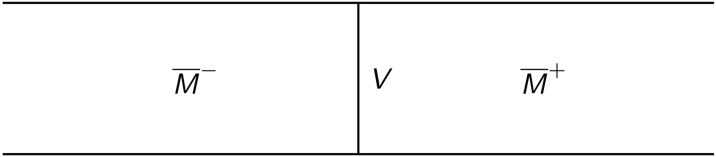
The two ends of $\overline{M}$

- *Everything has an end, except a sausage which has two!*
- The infinite cyclic cover of M is a union

$$\overline{M} \;=\; \overline{M}^{+} \cup_{V} \overline{M}^{-}$$

with

$$\overline{M}^{+} \;=\; \bigcup_{k=1}^{\infty} z^{k} W \;,\; \overline{M}^{-} \;=\; \bigcup_{k=-\infty}^{0} z^{k} W \;.$$



Chain homotopy nilpotence

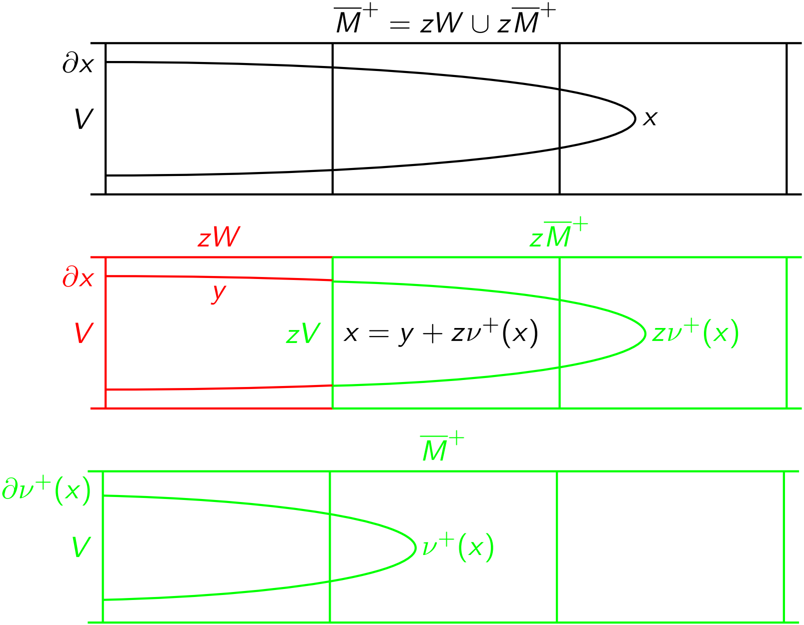
- An A-module chain complex C is **finitely dominated** if it is chain equivalent to a finite f.g. projective A-module chain complex.
- An A-module chain map $\nu : C \to C$ is **chain homotopy nilpotent** if $\nu^N \simeq 0 : C \to C$ for some $N \geqslant 0$.
- If $h : M^n \to X \times S^1$ is a homotopy equivalence then

$$C(\overline{M}^+, V) \oplus C(\overline{M}^-, V) \to \mathcal{C}(V \to X)$$

  is a chain equivalence with $\mathcal{C}(V \to X)$ a finite f.g. free $\mathbb{Z}[\pi_1(X)]$-module chain complex.
- The free $\mathbb{Z}[\pi_1(X)]$-module chain complex $C(\overline{M}^+, V)$ is finitely dominated.
- The $\mathbb{Z}[\pi_1(X)]$-module chain map

$$\nu^+ \ : \ C(\overline{M}^+, V) \to C(\overline{M}^+, zW) \cong C(z\overline{M}^+, zV) \cong C(\overline{M}^+, V)$$

  is chain homotopy nilpotent.


$\overline{M}^+ = zW \cup z\overline{M}^+$
∂x
V
x
zW
$z\overline{M}^+$
∂x
y
V
zV
$x = y + z\nu^+(x)$
$z\nu^+(x)$
$\overline{M}^+$
$\partial\nu^+(x)$
V
$\nu^+(x)$

The F-H splitting obstruction from the chain complex point of view

- For a homotopy equivalence $h : M^n \to X \times S^1$ the contractible finite based f.g. free $\mathbb{Z}[\pi_1(X)][z, z^{-1}]$-module chain complex $\mathcal{C}(\overline{h} : \overline{M} \to X \times \mathbb{R})$ fits into a short exact sequence

$$0 \to C(V, X)[z, z^{-1}] \xrightarrow{f - zg} C(W, X \times I)[z, z^{-1}] \to \mathcal{C}(\overline{h}) \to 0$$

- The splitting obstruction of h is the nilpotent class

$$\Phi(h) = (C(\overline{M}^+, V), \nu^+) \in \mathrm{Nil}_0(\mathbb{Z}[\pi_1(X)])/\mathrm{Nil}_0(\mathbb{Z})$$

  where
  $C(\overline{M}^+, V) = \mathrm{coker}(f - zg : zC(V, X)[z] \to C(W, X \times I)[z])$.
- $\Phi(h) = 0$ if and only if $(C(\overline{M}^+, V), \nu^+)$ is equivalent to 0 by a finite sequence of algebraic handle exchanges.
- For $n \geqslant 6$ can realize algebraic handle exchanges by geometric handle exchanges.

Universal localization

- (P.M.Cohn, 1971) Given a ring R and a set Σ of morphisms $\sigma : P \to Q$ of f.g. projective R-modules there exists a **universal localization** $\Sigma^{-1}R$, a ring with a morphism $R \to \Sigma^{-1}R$ universally inverting each σ
- **Universal property** For any ring morphism $R \to S$ such that $1 \otimes \sigma : S \otimes_R P \to S \otimes_R Q$ is an S-module isomorphism for each $\sigma \in \Sigma$ there is a unique factorization $R \to \Sigma^{-1}R \to S$.
- **Warning 1** $R \to \Sigma^{-1}R$ need not be injective.
- **Warning 2** $\Sigma^{-1}R$ could be 0.
- **Gerasimov-Malcolmson normal form** An element $q\sigma^{-1}p \in \Sigma^{-1}R$ is an equivalence class of triples

$$((\sigma : P \to Q) \in \Sigma, p \in P, q \in Q^* = \mathrm{Hom}_R(Q, R)) \ .$$

The algebraic K-theory localization exact sequence

- Assume $R \to \Sigma^{-1}R$ is injective.
- An (R,Σ)-**torsion module** is an R-module T such that

$$0 \longrightarrow P_1 \xrightarrow{d} P_0 \longrightarrow T \longrightarrow 0$$

  with P_0, P_1 f.g. projective R-modules and
  $1 \otimes d : \Sigma^{-1}P_1 \to \Sigma^{-1}P_0$ a $\Sigma^{-1}R$-module isomorphism.
- **Theorem** (Neeman+R., 2004) For an injective universal localization $R \to \Sigma^{-1}R$ such that

$$\mathrm{Tor}_*^R(\Sigma^{-1}R, \Sigma^{-1}R) \;=\; 0 \text{ } (\textbf{stable flatness})$$

  there is a long exact sequence of algebraic K-groups

$$\cdots \to K_n(R) \to K_n(\Sigma^{-1}R) \to K_{n-1}(H(R,\Sigma)) \to K_{n-1}(R) \to \ldots$$

  with $H(R,\Sigma)$ the exact category of (R,Σ)-torsion modules.

Triangular matrix rings

- Given rings R_1, R_2 and an (R_2, R_1)-bimodule Q define the triangular matrix ring

$$R = \begin{pmatrix} R_1 & 0 \\ Q & R_2 \end{pmatrix} .$$

- **Proposition 4** (i) The category of R-modules is equivalent to the category of triples

$$M = (M_1, M_2, \mu : Q \otimes_{R_1} M_1 \to M_2)$$

  with M_i R_i-modules $(i = 1, 2)$, μ an R_2-module morphism.
- (ii) An R-module M is f.g. projective if and only if M_1 is a f.g. projective R_1-module, μ is injective, and coker(μ) is a f.g. projective R_2-module.
- (iii) $K_*(R) = K_*(R_1) \oplus K_*(R_2)$.

Full matrix rings

- Let $R = \begin{pmatrix} R_1 & 0 \\ Q & R_2 \end{pmatrix}$, $P_1 = \begin{pmatrix} R_1 \\ Q \end{pmatrix}$, $P_2 = \begin{pmatrix} 0 \\ R_2 \end{pmatrix}$.
  The R-modules P_1, P_2 are f.g. projective, since $P_1 \oplus P_2 = R$.
- If $R \to S$ is a ring morphism with $S \otimes_R P_1 \cong S \otimes_R P_2$ then

$$S \;=\; M_2(T)$$

  with $T = \mathrm{End}_S(S \otimes_R P_1) = \mathrm{End}_S(S \otimes_R P_2)$.
- Morita equivalence

$$\{S\text{-modules}\} \xrightarrow{\approx} \{T\text{-modules}\}\ ;\ N \mapsto (T\ \ T) \otimes_S N\ .$$

- The induced functor

$$\{R\text{-modules}\} \to \{S\text{-modules}\} \xrightarrow{\approx} \{T\text{-modules}\}\ ;$$
$$M = (M_1, M_2, \mu : Q \otimes_{R_1} M_1 \to M_2) \mapsto$$
$$(T\ \ T) \otimes_R M = \mathrm{coker}(T \otimes_{R_2} Q \otimes_{R_1} M_1 \to T \otimes_{R_1} M_1 \oplus T \otimes_{R_2} M_2)$$

  is an **assembly** map, i.e. local-to-global.

(R,Σ)-torsion modules

- **Proposition 5** The universal localization of

$$R \;=\; \begin{pmatrix} R_1 & 0 \\ Q & R_2 \end{pmatrix} = P_1 \oplus P_2$$

inverting a set Σ of R-module morphisms $\sigma : P_2 \to P_1$ is $\Sigma^{-1}R = M_2(T)$ with $T = \mathrm{End}_{\Sigma^{-1}R}(\Sigma^{-1}P_1)$.

- **Proposition 6** Assume that $R \to \Sigma^{-1}R = M_2(T)$ is injective, and that Q is a flat right R_1-module.
An R-module $M = (M_1, M_2, \mu)$ is (R,Σ)-torsion if and only if
  - (i) $\cdots \longrightarrow 0 \longrightarrow Q \otimes_{R_1} M_1 \xrightarrow{\mu} M_2$ is homology equivalent to a 1-dimensional f.g.projective R_1-module chain complex,
  - (ii) M_2 is an h.d. 1 R_2-module,
  - (iii) the assembly

$$T \otimes_{R_2} Q \otimes_{R_1} M_1 \to T \otimes_{R_1} M_1 \oplus T \otimes_{R_2} M_2$$

is a T-module isomorphism.

Polynomial extensions as universal localizations

- For any ring A let

$$R = \begin{pmatrix} A & 0 \\ A \oplus A & A \end{pmatrix} , \; P_1 = \begin{pmatrix} A \\ A \oplus A \end{pmatrix} , \; P_2 = \begin{pmatrix} 0 \\ A \end{pmatrix}$$

  and let $\sigma_+, \sigma_- : P_2 \to P_1$ be the two inclusions.
- **Proposition 7** (Schofield, 1985)
  (i) The universal localization of R inverting $\Sigma_+ = \{\sigma_+\}$ is

$$\Sigma_+^{-1} R = M_2(A[z]) .$$

  (ii) The universal localization of R inverting $\Sigma = \{\sigma_+, \sigma_-\}$ is

$$\Sigma^{-1} R = M_2(A[z, z^{-1}]) .$$

Torsion = nilpotence

- Let $R = \begin{pmatrix} A & 0 \\ A \oplus A & A \end{pmatrix}$. An R-module $M = (P, Q, f, g)$ is defined by A-modules P, Q and A-module morphisms $f, g : P \to Q$.
- **Proposition 8** (i) The assembly of $M = (P, Q, f, g)$ with respect to $\Sigma_+^{-1}R = M_2(A[z])$ is the $A[z]$-module

$$(A[z]\ A[z]) \otimes_R M \ = \ \text{coker}(f + gz : P[z] \to Q[z])\ .$$

  M is an (R, Σ_+)-module if and only if P, Q are f.g. projective A-modules and $f + gz$ is an $A[z]$-module isomorphism. Thus

$$\text{Nil}(A) \to H(A[z], \Sigma_+)\ ;\ (P, \nu) \mapsto (P, P, 1, \nu)$$

  is an equivalence of exact categories, by Proposition 1.
- (ii) Likewise for $\Sigma^{-1}A[z] = M_2(A[z, z^{-1}])$, with

$$\text{Near}(A) \to H(A[z], \Sigma)\ ;\ (P, \rho) \mapsto (P, P, \rho, 1 - \rho)$$

  an equivalence of exact categories by Proposition 2.

Universal localization proof of B-H-S theorem

- Apply the universal localization exact sequence

  $$\cdots \to K_n(R) \to K_n(\Sigma^{-1}R) \to K_{n-1}(H(R,\Sigma)) \to K_{n-1}(R) \to \ldots$$

  to the stably flat universal localizations of $R = \begin{pmatrix} A & 0 \\ A \oplus A & A \end{pmatrix}$

  $$\Sigma_+^{-1}R \;=\; M_2(A[z]) \;,\; \Sigma^{-1}R \;=\; M_2(A[z,z^{-1}]) \;.$$

- Identify

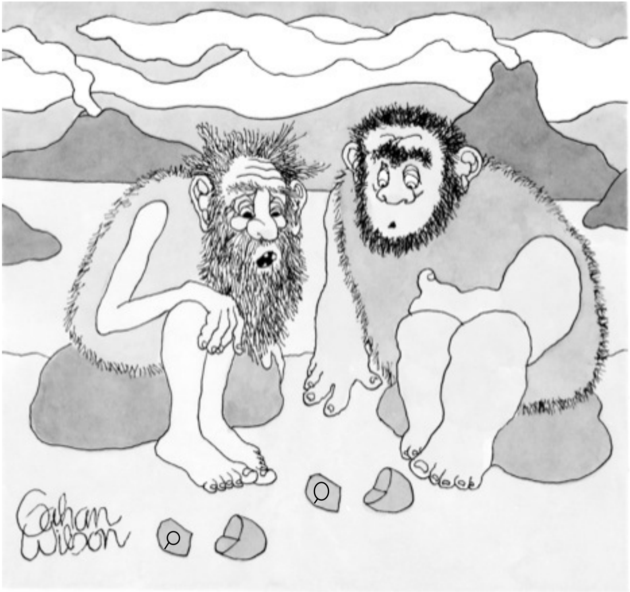
  $$\begin{aligned} K_*(R) \;&=\; K_*(A) \oplus K_*(A) \;, \\ K_*(\Sigma_+^{-1}R) \;&=\; K_*(A[z]) \;,\; H(R,\Sigma_+) \;=\; \mathrm{Nil}(A) \;, \\ K_*(\Sigma^{-1}R) \;&=\; K_*(A[z,z^{-1}]) \;, \\ H(R,\Sigma) \;&=\; \mathrm{Near}(A) \;=\; \mathrm{Nil}(A) \times \mathrm{Nil}(A) \end{aligned}$$

  to recover

  $$\begin{aligned} K_n(A[z]) \;&=\; K_n(A) \oplus \widetilde{\mathrm{Nil}}_{n-1}(A) \;, \\ K_n(A[z,z^{-1}]) \;&=\; K_n(A) \oplus K_{n-1}(A) \oplus \widetilde{\mathrm{Nil}}_{n-1}(A) \oplus \widetilde{\mathrm{Nil}}_{n-1}(A) \;. \end{aligned}$$

Generalized free products

- A group π is a **generalized free product** if it is
  - either amalgamated free product $\pi = \pi_1 *_\rho \pi_2$,
  - or an HNN extension $\pi = \pi_1 *_\rho \{t\}$.
- (Bass-Serre, 1970) A group π is a generalized free product if and only if π acts on a tree T with $T/\pi = [0,1]$ or S^1.
- Article in proceedings of **Noncommutative localization in algebra and topology**, LMS Lecture Notes 330 (2006) includes an outline of the proof of the Waldhausen (1976) algebraic K-theory splitting theorems of generalized free products via noncommutative localization, using T-based Mayer-Vietoris presentations.
- Nilpotence = torsion also in the generalized free product case.
- Also in algebraic L-theory, with the Cappell (1974) UNil-groups.



"There -- now I've taught you everything I know about codimension 1 splitting"