SMSTC Geometry and Topology 2013-2014

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SMSTC Symposium

Perth, 9th October, 2013

http://www.smstc.ac.uk

http://www.maths.ed.ac.uk/~aar/smstc/gt34info.pdf

http://www.maths.ed.ac.uk/~aar/smstc/gt34symp.pdf

Aims

- An introduction to algebraic topology (with a little bit of algebraic geometry)
- An introduction to differential geometry
- An introduction to differential topology

Backgrounds

- The students taking the Geometry/Topology SMSTC module come from a wide range of mathematical backgrounds. Inevitably:
- some of the material will be known and too easy for some of the students,
- some of the material will be unknown and too hard for some of the students.
- I hope the course strikes a balance between known/easy and unknown/hard material.

Prerequisites

- A course in metric spaces or topological spaces (or both). Important concepts: Open sets and neighbourhoods in metric spaces.
- Standard calculus courses. Some knowledge of vector calculus (e.g. div, grad, curl and Green's theorem) would be useful.
- One or two basic courses in linear algebra. Important concepts: Abstract vector space, quotient vector spaces.
- A course in group theory, including group actions, generators and relations, the structure theorem for finitely generated abelian groups.

Regular lecturers

- Andrew Ranicki, Edinburgh, Stream Leader
- Brendan Owens, Glasgow
- Richard Hepworth, Aberdeen
- Vanya Cheltsov, Edinburgh
- Bernd Schroers, Heriot Watt

Guest lecturers

- Jeremy Gray, Open University.
- Etienne Ghys, ENS Lyon.

Timetable

- Lectures on Thursdays, 1-3PM.
- 10 lectures in Semester 1: October 17 - December 19.
- Guest lecture on "Poincaré and topology" (Jeremy Gray): December 19.
- 10 lectures in Semester 2: January 16 - March 20
- Guest lecture on "Poincaré and geodesy" (Etienne Ghys): March 20
- 4 written assessments due on Fridays December 6, January 17, February 21, March 21

Themes of the Geometry/Topology course

- Topological spaces, compact, connected.
- Continuous functions, homeomorphisms, homotopy.
- The homology groups $H_*(X)$. The Euler characteristic $\chi(X)$.
- Manifolds in general, surfaces in particular.
- The fundamental group $\pi_1(X)$.
- The differential geometry of curves and surfaces in $\mathbb{R}^3$.
- Theorema Egregium.
- The Gauss-Bonnet theorem, expressing $\chi(M)$ of a surface M as an integral of the curvature.
- Vector calculus on smooth manifolds and deRham cohomology.
- Applications of vector calculus to intersections and linking in manifolds, and to homotopy theory.

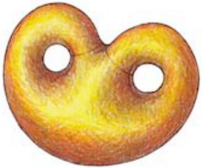
Smooth manifolds

- Manifolds are the topological spaces of greatest interest!
- **Definition** An n-dimensional manifold is a topological space M which at each point $x \in M$ has an open neighbourhood $U \subseteq M$ homeomorphic to $\mathbb{R}^n$.
- **Example** The Euclidean space $\mathbb{R}^n$ is an n-dimensional manifold.
- Roughly speaking, an n-dimensional manifold is a space which can be obtained by glueing together copies of $\mathbb{R}^n$ using homeomorphisms.
- If the homeomorphisms are differentiable then the manifold is smooth (or differentiable).
- Multivariable calculus extends to calculus on smooth manifolds.

Surfaces

- A surface is a 2-dimensional manifold M.
- The genus g of M is the number of holes it has – not a proper mathematical definition as it stands!
- Here are the surfaces M_g with $g = 0, 1, 2$





$M_0 = S^2 =$ sphere $M_1 = S^1 \times S^1 =$ torus $M_2 =$ pretzel

- $\chi(M_g) = 2 - 2g$.
- The classification theorem for surfaces is that two surfaces are homeomorphic if and only if they have the same genus.

Examples of manifolds

- The space $M = \{x \in \mathbb{R}^n \,|\, f(x) = 0 \in \mathbb{R}^m\}$ of the solutions of a set of m simultaneous differential equations in n variables

$$f_i(x_1, x_2, \ldots, x_n) \;=\; 0 \in \mathbb{R} \; (1 \leqslant i \leqslant m)$$

  is an $(n - m)$-dimensional smooth manifold, provided that $n \geqslant m$ and that for each $x \in M$ the Jacobian $m \times n$ matrix $(\partial f_i / \partial x_j)$ has the maximal rank m.
- The unit sphere S^n in $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ is a compact n-dimensional smooth manifold: apply the previous example with

$$f \;:\; \mathbb{R}^{n+1} \to \mathbb{R} \; ; \; x \mapsto \|x\| - 1$$

  and $S^n = f^{-1}(0) \subset \mathbb{R}^{n+1}$.

Homotopy theory

- Idea: Before we can distinguish topological spaces, we must learn to distinguish continuous functions.
- Let X and Y be topological spaces.
  Definition. Two continuous functions $f : X \to Y$ and $g : X \to Y$ are homotopic if there exist continuous functions $h_t : X \to Y$ for $0 \leqslant t \leqslant 1$ such that

$$h_0 = f \, , \; h_1 = g \; : \; X \to Y$$

  and $h_t(x)$ depends continuously on t and x. Regard $\{h_t\}$ as a 'film' which starts at f and ends at g.
- 'Homotopic" is an equivalence relation.
- The determination of the set $[X, Y]$ of equivalence classes of continuous functions $f : X \to Y$, for fixed X and Y, can often be reduced to algebra.

Homology

- The homology groups of a topological space X are a sequence of abelian groups $H_n(X)$ for $n = 0, 1, 2, \ldots$. Roughly speaking, $H_n(X)$ measures the number of "n-dimensional holes" in X.
- $H_n(S^m) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \text{ or } m \\ 0 & \text{if } m \neq n. \end{cases}$
- A continuous function $f : X \to Y$ induces morphisms $f_* : H_n(X) \to H_n(Y)$ for $n \geqslant 0$, which depend only on the homotopy class $f \in [X, Y]$
- If $f : X \to Y$ is a homeomorphism then each f_* is an isomorphism.
- If $f_* : H_n(X) \to H_n(Y)$ is not an isomorphism then f is not a homeomorphism, not even homotopic to one.
- For $m \neq n$ $H_n(S^m) = 0$ is not isomorphic to $H_n(S^n) = \mathbb{Z}$. It follows that S^n is not homeomorphic to S^m, and hence that $S^n \backslash \{\text{point}\} = \mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$.

The fundamental group

- The **fundamental group** of a space X is the group of homotopy classes of continuous functions $\omega : S^1 \to X$

$$\pi_1(X) \;=\; [S^1, X] \;.$$

- The **degree** of ω is the number of times it winds around 0, the unique $d \in \mathbb{Z}$ such that ω is homotopic to $z \mapsto z^d$. The degree of analytic ω can be computed by Cauchy's theorem

$$d \;=\; \frac{1}{2\pi i}\int_\omega \frac{dz}{z} \;.$$

- **Theorem** The degree defines an isomorphism of groups

$$\pi_1(S^1) \to \mathbb{Z} \;;\; \omega \mapsto d \;.$$

Vector calculus and homotopy types of manifolds

- In low dimensions, standard vector calculus gives some information about homotopy types.
- Let U be a nonempty open set in $\mathbb{R}^3$. Let A be the vector space of all smooth functions from U to $\mathbb{R}$. Let B be the vector space of all smooth vector fields on U.
- Vector calculus provides linear maps

  $$A \xrightarrow{\text{grad}} B \xrightarrow{\text{curl}} B \xrightarrow{\text{div}} A$$

  such that any two consecutive ones compose to zero.
- Therefore im(grad) $\subset$ ker(curl) and im(curl) $\subset$ ker(div). If $U = \mathbb{R}^3$, these inclusions are equalities, but in general they are not!

  The dimensions of the vector spaces

  $$\text{ker(curl)/im(grad)}\ , \qquad \text{ker(div)/im(curl)}$$

  are invariants of the homotopy type of U.

deRham cohomology

- In our course, vector calculus will be generalised to be applicable to arbitrary smooth n-manifolds M. The above sequence of grad, div and curl generalises to a sequence of vector spaces and linear maps

$$\Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} \Omega^n(M)$$

  where $d_i \circ d_{i-1} = 0$, so that $\mathrm{im}(d_{i-1}) \subseteq \ker(d_i)$.
- The dimensions of the vector spaces

$$H^i(M) = \ker(d_i)/\mathrm{im}(d_{i-1})$$

  are equal to the ranks of the homology groups $H_i(M)$.
- If M is compact, the dimensions are finite.

Riemannian manifolds

- A smooth manifold M becomes a Riemannian manifold through a choice of a Riemannian metric on M. This structure makes it possible to assign a length to any smooth curve segment in M. Following Gauss, Riemann and others, we shall isolate the intrinsic aspects of curvature in terms of length measurements.
- Curvature properties of a Riemannian manifold are often related to the homotopy type of the manifold. Examples in 2 dimensions:
  - For any Riemannian metric on S^2, there will be points where the curvature is positive.
  - For any Riemannian metric on the pretzel, there will be points where the curvature is negative.
- These statements follow from the Gauss-Bonnet theorem, since $\chi(S^2) = 2$, $\chi(\text{pretzel}) = -2$.

Some applications of Geometry and Topology

- Original application of topology: celestial dynamics.
- Algebraic geometry borrows techniques from differential geometry and algebraic topology to investigate algebraic varieties.
- Theoretical physics: gauge theory and general relativity, topological solitons, string theory, ...
- Computational topology. The advent of computers has allowed topology to be applied to pattern recognition for large data sets.

Relations with other SMSTC courses

- **Algebra** Groups, both commutative and non-commutative, are ever-present in the Geometry/Topology course.
  Rings. If you are a commutative ring enthusiast, you might be pleased to know that many essential geometric constructions in our course can be reformulated in terms of commutative rings and their modules.
- **Pure Analysis** Although we don't need Lebesgue integration theory, as developed in the Pure Analysis stream, integrals are of some importance in the Geometry/Topology course.
- **Applied Analysis and PDEs** The third quarter of the Geometry/Topology course, on differential geometry, is somewhat related to the first quarter of the Applied Analysis course, on dynamical systems. An important class of dynamical systems (geodesic flows) comes from differential geometry.