Lecture notes of Jacob Lurie's 2009 Harvard course
"Topics in geometric topology"
http://www.math.harvard.edu/~lurie/937.html

Introduction (Lecture 1)

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One of the basic problems of manifold topology is to give a classification for manifolds (of some fixed dimension n) up to diffeomorphism. In the best of all possible worlds, a solution to this problem would provide the following:

(i) A list of n-manifolds $\{M_\alpha\}$, containing one representative from each diffeomorphism class.

(ii) A procedure which determines, for each n-manifold M, the unique index α such that $M \simeq M_\alpha$.

In the case $n = 2$, it is possible to address these problems completely: a connected oriented surface Σ is classified up to homeomorphism by a single integer g, called the *genus* of Σ. For each $g \geq 0$, there is precisely one connected surface Σ_g of genus g up to diffeomorphism, which provides a solution to (i). Given an arbitrary connected oriented surface Σ, we can determine its genus simply by computing its Euler characteristic $\chi(\Sigma)$, which is given by the formula $\chi(\Sigma) = 2 - 2g$: this provides the procedure required by (ii).

Given a solution to the classification problem satisfying the demands of (i) and (ii), we can extract an algorithm for determining whether two n-manifolds M and N are diffeomorphic. Namely, we apply the procedure (ii) to extract indices α and β such that $M \simeq M_\alpha$ and $N \simeq M_\beta$: then $M \simeq N$ if and only if $\alpha = \beta$. For example, suppose that $n = 2$ and that M and N are connected oriented surfaces with the same Euler characteristic. Then the classification of surfaces tells us that there is a diffeomorphism ϕ from M to N. In practice, we might want to apply this information by using ϕ to make some other construction. In this case, it is important to observe that ϕ is not unique: there are generally many different diffeomorphisms from M to N.

Example 1. Let M be a compact oriented 3-manifold, and suppose we are given a submersion $M \to S^1$. Fix a base point $* \in S^1$. The fiber $M \times_{S^1} *$ is a compact oriented surface which we will denote by Σ. Write S^1 as a quotient $[0,1]/\{0,1\}$, so that M is obtained from the pullback $M' = M \times_{S^1} [0,1]$ by gluing together the fibers $M'_0 \simeq \Sigma$ and M'_1. Since the interval $[0,1]$ is contractible, we can write M' as a product $\Sigma \times [0,1]$. In order to recover M from M', we need to supply a diffeomorphism of $M' \simeq \Sigma$ to $M'_1 \simeq \Sigma$: in other words, we need to supply a diffeomorphism ϕ of Σ with itself. The diffeomorphism ϕ depends on a choice of identification $M' \simeq \Sigma \times [0,1]$. If we assume that this diffeomorphism is normalized to be the identity on M'_0, then we see that ϕ is well-defined up to *isotopy* (recall that two diffeomorphisms $\gamma_0, \gamma_1 : \Sigma \to \Sigma$ are *isotopic* if there is a continuous family $\{\gamma_t : \Sigma \to \Sigma\}_{t \in [0,1]}$ of diffeomorphisms which interpolates between γ_0 and γ_1).

Motivated by this example, it is natural to refine our original classification problem: given two n-manifolds M and N, we would like to know not only whether M and N are diffeomorphic, but to have a classification of all diffeomorphisms from M to N, at least up to isotopy. Note that the collection $\mathrm{Diff}(M,N)$ of diffeomorphisms from M to N carries a natural topology, and the isotopy classes of diffeomorphisms from M to N can be identified with elements of the set $\pi_0 \mathrm{Diff}(M,N)$ of path components of $\mathrm{Diff}(M,N)$. Our goal in this class is to address the following more refined question:

Problem 2. *Given a pair of n-manifolds M and N, determine the homotopy type of the space* $\mathrm{Diff}(M,N)$.

Remark 3. The space $\mathrm{Diff}(M, N)$ is nonempty if and only if M and N are diffeomorphic. If $\mathrm{Diff}(M, N)$ is nonempty, then $M \simeq N$ so we can identify $\mathrm{Diff}(M, N)$ with the group $\mathrm{Diff}(M) = \mathrm{Diff}(M, M)$ of diffeomorphisms from M to itself.

Remark 4. One might ask why Problem 2 is addressing the right question. For example, why do we want to understand the homotopy type of $\mathrm{Diff}(M, N)$ as opposed to some more precise invariant (like the topological space $\mathrm{Diff}(M, N)$ itself) or less precise invariant (like the set $\pi_0 \mathrm{Diff}(M, N)$)?

One answer is that the exact topological space $\mathrm{Diff}(M, N)$ depends on exactly what we mean by a diffeomorphism. For example, should we work with diffeomorphisms that are merely differentiable, or should they be infinitely differentiable? The exact topological space $\mathrm{Diff}(M, N)$ will depend on how we answer this question. But, as we will see later, the homotopy type of $\mathrm{Diff}(M, N)$ does not.

To motivate why we would like to understand the entire homotopy type of $\mathrm{Diff}(M, N)$, rather than just its set of path components, we remark that Example 1 can be generalized as follows: given a pair of compact manifolds M and B, the collection of isomorphism classes of smooth fiber bundles $E \to B$ with fiber M can be identified with the collection of homotopy classes of maps from B into the classifying space $B\,\mathrm{Diff}(M)$. In other words, understanding the homotopy types of the groups $\mathrm{Diff}(M)$ is equivalent to understanding the classification of *families* of manifolds.

Problem 2 is very difficult in general. To address it, it is useful to divide manifolds into two different "regimes":

- If $n \geq 5$, then we are in the world of high-dimensional topology. In this case, it is possible to obtain partial information about the homotopy type of $\mathrm{Diff}(M)$ (for example, a description of its rational homotopy groups in a range of degrees) using the techniques of surgery theory. The techniques for obtaining this information are generally algebraic in nature (involving Waldhausen K-theory and L-theory).
- If $n \leq 4$, then we are in the world of low-dimensional topology. In this case, it is customary to approach Problem 2 using geometric and combinatorial techniques. The success of these method is highly dependent on n.

Our goal in this course is to study Problem 2 in the low-dimensional regime. When $n = 4$, very little is known about Problem 2: for example, little is known about the homotopy type of the diffeomorphism group $\mathrm{Diff}(S^4)$. We will therefore restrict our attention to manifolds of dimension n for $1 \leq n \leq 3$. We will begin in this lecture by studying the case $n = 1$. In this case, there is only one connected closed 1-manifold up to diffeomorphism: the circle S^1. However, we can study S^1 from many different points of view:

- Geometry: We can regard the circle S^1 as a Riemannian manifold, and study its isometry group $\mathrm{Isom}(S^1)$.
- Differential topology: We can regard the circle S^1 as a smooth manifold, and study its diffeomorphism group $\mathrm{Diff}(S^1)$.
- Point-Set Topology: We can regard the circle S^1 as a topological manifold, and study the group $\mathrm{Homeo}(S^1)$ of homeomorphisms of S^1 with itself.
- Homotopy Theory: We can ignore the actual topology of S^1 in favor of its homotopy type, and study the monoid $\mathrm{Self}(S^1)$ of homotopy equivalences $S^1 \to S^1$.

We have evident inclusions

$$\mathrm{Isom}(S^1) \subseteq \mathrm{Diff}(S^1) \subseteq \mathrm{Homeo}(S^1) \subseteq \mathrm{Self}(S^1).$$

Theorem 5. *Each of the above inclusions is a homotopy equivalence.*

Proof. Each of the spaces above can be decomposed into two pieces, depending on whether or not the underlying map preserves or reverses orientations. Consider the induced sequence

$$\mathrm{Isom}^+(S^1) \subseteq \mathrm{Diff}^+(S^1) \subseteq \mathrm{Homeo}^+(S^1) \subseteq \mathrm{Self}^+(S^1)$$

where the superscript indicates that we restrict our attention to orientation-preserving maps. The group $\mathrm{Isom}^+(S^1)$ is homeomorphic to the circle S^1 itself: an orientation-preserving isometry from S^1 to itself is just given by a rotation. The other groups admit decompositions

$$\mathrm{Diff}^+(S^1) = \mathrm{Diff}^+_0(S^1)\,\mathrm{Isom}^+(S^1)$$

$$\mathrm{Homeo}^+(S^1) = \mathrm{Homeo}^+_0(S^1)\,\mathrm{Isom}^+(S^1)$$

$$\mathrm{Self}^+(S^1) = \mathrm{Self}^+_0(S^1)\,\mathrm{Isom}^+(S^1),$$

where the subscript 0 indicates that we consider maps from S^1 to itself which fix a base point $* \in S^1$. To complete the proof, it will suffice to show that the spaces $\mathrm{Diff}^+_0(S^1)$, $\mathrm{Homeo}^+_0(S^1)$, and $\mathrm{Self}^+_0(S^1)$ are contractible.

We first treat the case of $\mathrm{Self}^+_0(S^1)$. We note that the circle S^1 can be identified with the quotient $\mathbb{R}/\mathbf{Z}$. If f is a map from the circle S^1 to itself which preserves the base point (the image of $0 \in \mathbb{R}$), then we can lift f to a base-point preserving map $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ satisfying $\widetilde{f}(x+1) = \widetilde{f}(x) + d$, where d is the degree of the map $f : S^1 \to S^1$. Conversely, any map $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ satisfying this condition descends to give a map $f : S^1 \to S^1$ of degree d. We observe that f is a homotopy equivalence if and only if $d = \pm 1$, and that f is an orientation-preserving homotopy equivalence if and only if $d = 1$. We may therefore identify $\mathrm{Self}^+_0(S^1)$ with the space

$$V = \{\widetilde{f} : \mathbb{R} \to \mathbb{R} : \widetilde{f}(0) = 0 \wedge \widetilde{f}(x+1) = \widetilde{f}(x) + 1\}$$

We wish to prove that V is contractible. In fact, for any element $\widetilde{f} \in V$, there is a canonical path from $\widetilde{f} = \widetilde{f}_0$ to the identity map $\mathrm{id}_{\mathbb{R}} = \widetilde{f}_1$, given by the formula

$$\widetilde{f}_t(x) = (1-t)\widetilde{f}(x) + tx.$$

We can use the identification $\mathrm{Self}^+_0(S^1) \simeq V$ to identify $\mathrm{Homeo}^+_0(S^1)$ and $\mathrm{Diff}^+_0(S^1)$ with subsets of V: the former can be identified with the collection of all strictly increasing functions $\widetilde{f} \in V$, and the latter with the collection of all maps $\widetilde{f} \in V$ which are smooth and have nowhere vanishing derivative. Exactly the same contracting homotopy shows that these spaces are contractible as well. □

We can summarize Theorem 5 as follows:

(1) There is essentially no difference between smooth 1-manifolds and topological 1-manifolds.

(2) Every smooth 1-manifold M admits a Riemannian metric which accurately reflects its topology, in the sense that every diffeomorphism of M can be canonically deformed to an isometry.

(3) A 1-manifold M is determined, up to canonical homeomorphism, by its homotopy type.

In this course, we will study to what extent these assertions can be generalized to manifolds of dimensions 2 and 3. Here is a loose outline of the material we might cover in this class:

- In large dimensions, there is an appreciable difference between working with smooth and topological manifolds. A famous example is Milnor's discovery that there exist nondiffeomorphic smooth structures on the sphere S^7. In fact, these differences are apparent already in lower dimensions: Milnor's example comes from the fact that there exist diffeomorphisms of the standard sphere S^6 which are topologically isotopic but not smoothly isotopic, and a similarly the inclusion $\mathrm{Diff}(S^5) \to \mathrm{Homeo}(S^5)$ fails to be a homotopy equivalence. Even more dramatic failures occur in dimension 4: the topological space $\mathbb{R}^4$ can be endowed with uncountably many nondiffeomorphic smooth structures. However, in dimensions

≤ 3 these difficulties do not occur. Namely, one can show that the classification of manifolds (including information about the homotopy types of automorphism groups) of dimension ≤ 3 is the same in the smooth, topological, and piecewise linear categories. The first part of this course will be devoted to making this statement more precise and sketching how it can be proved.

- If Σ is a closed oriented surface of genus $g > 0$, then Σ is aspherical: the homotopy groups $\pi_i(\Sigma)$ vanish for $i > 1$. It follows that the homotopy type of Σ is determined by its fundamental group. In this case, we will also see that the diffeomorphism group $\mathrm{Diff}(\Sigma)$ is homotopy equivalent to the monoid of self-diffeomorphisms $\mathrm{Self}(\Sigma)$, so that $\mathrm{Diff}(\Sigma)$ can be described in an entirely combinatorial way in terms of the fundamental group $\pi_1\Sigma$.

  For 3-manifolds, the situation is a bit more complicated. A general 3-manifold M need not be aspherical: the group $\pi_2(M)$ usually does not vanish. However, via somewhat elaborate geometric arguments one can use the nonvanishing of $\pi_2(M)$ to construct embedded spheres in M which cut M into aspherical pieces (except in a few exceptional cases). The homotopy type of an aspherical manifold M is again determined by the fundamental group $\pi_1(M)$. In many cases, one can show that M is determined up to diffeomorphism by $\pi_1(M)$: this is true whenever M is a *Haken manifold*. We will study the theory of Haken manifolds near the end of this course. (Another case in which M can be recovered from the fundamental group $\pi_1 M$ occurs when M is a hyperbolic 3-manifold: this is the content of *Mostow's rigidity theorem*.)

- Manifolds of dimension 2 and 3 can be fruitfully studied by endowing them with additional structure. For example, we can gain a lot of information about surfaces by choosing conformal structures and then applying the methods of complex analysis. Using the uniformization theorem, one can show that every 2-manifold admits a Riemannian metric of constant curvature: this curvature is positive for the case of a 2-sphere, zero for a torus, and otherwise negative. In dimension 3, Thurston's geometrization conjecture provides a much more complicated but somewhat analogous picture: every 3-manifold can be broken into pieces which admit "geometric structures". If time allows, we will discuss this near the end of the course.

Piecewise Linear Topology (Lecture 2)

February 8, 2009

Our main goal for the first half of this course is to discuss the relationship between smooth manifolds and piecewise linear manifolds. In this lecture, we will set the stage by introducing the essential definitions.

Definition 1. Throughout these lectures, we will use the term *manifold* to refer to a paracompact Hausdorff space M with the property that each point $x \in M$ has an open neighborhood homeomorphic to $\mathbb{R}^n$, for some fixed integer $n \geq 0$; we refer to n as the *dimension* of M.

The study of manifold topology becomes substantially easier if we assume that our manifolds are endowed with additional structures, such as a smooth structure.

Definition 2. Let M be a manifold. We let $\mathcal{O}_M^{\text{Top}}$ denote the sheaf of continuous real-valued functions on M, so that for each open set $U \subseteq M$ we have $\mathcal{O}_M^{\text{Top}}(U) = \{f : U \to \mathbb{R} : f \text{ is continuous}\}$.

A *smooth structure* on M consists of a subsheaf $\mathcal{O}_M^{\text{sm}} \subseteq \mathcal{O}_M^{\text{Top}}$ with the following property: for every point $x \in M$, there exists an open embedding $f : \mathbb{R}^n \to M$ whose image contains f, such that $f^* \mathcal{O}_M^{\text{sm}} \subseteq \mathcal{O}_{\mathbb{R}^n}^{\text{Top}}$ can be identified with the sheaf of smooth (in other words, infinitely differentiable) functions on $\mathbb{R}^n$. In this case, we will refer to f as a *smooth chart* on M.

A *smooth manifold* is a manifold M equipped with a smooth structure. If $f : M \to N$ is a continuous map between smooth manifolds, we will say that f is *smooth* if the map $f^* \mathcal{O}_N^{\text{sm}} \to \mathcal{O}_M^{\text{Top}}$ factors through $\mathcal{O}_M^{\text{sm}}$: in other words, if and only if composition with f carries smooth functions on N to smooth functions on M.

We now introduce the (perhaps less familiar) notion of a *piecewise linear*, or *combinatorial* manifold.

Definition 3. Let K be a subset of a Euclidean space $\mathbb{R}^n$. We will say that K is a *linear simplex* if it can be written as the convex hull of a finite subset $\{x_1, \ldots, x_k\} \subset \mathbb{R}^n$ which are independent in the sense that if $\sum c_i x_i = 0 \in \mathbb{R}^n$ and $\sum c_i = 0 \in \mathbb{R}$, then each c_i vanishes.

We will say that K is a *polyhedron* if, for every point $x \in K$, there exists a finite number of linear simplices $\sigma_i \subseteq K$ such that the union $\bigcup_i \sigma_i$ contains a neighborhood of X.

Remark 4. Any open subset of a polyhedron in $\mathbb{R}^n$ is again a polyhedron.

Remark 5. Every polyhedron $K \subseteq \mathbb{R}^n$ admits a *triangulation*: that is, we can find a collection of linear simplices $S = \{\sigma_i \subseteq K\}$ with the following properties:

(1) Any face of a simplex belonging to S also belongs to S.

(2) Any nonempty intersection of any two simplices of S is a face of each.

(3) The union of the simplices σ_i is K.

Definition 6. Let $K \subseteq \mathbb{R}^n$ be a polyhedron. We will say that a map $f : K \to \mathbb{R}^m$ is *linear* if it is the restriction of an affine map from $\mathbb{R}^n$ to $\mathbb{R}^m$. We will say that f is *piecewise linear* (PL) if there exists a triangulation $\{\sigma_i \subseteq K\}$ such that each of the restrictions $f|\sigma_i$ is linear.

If $K \subseteq \mathbb{R}^n$ and $L \subseteq \mathbb{R}^m$ are polyhedra, we say that a map $f : K \to L$ is piecewise linear if the underlying map $f : K \to \mathbb{R}^m$ is piecewise linear.

Remark 7. Let $f : K \to L$ be a piecewise linear homeomorphism between polyhedra. Then the inverse map $f^{-1} : L \to K$ is again piecewise linear. To see this, choose any triangulation of K such that the restriction of f to each simplex of the triangulation is linear. Taking the image under f, we obtain a triangulation of L such that the restriction of f^{-1} to each simplex is linear.

Remark 8. The collection of all polyhedra can be organized into a category, where the morphisms are given by piecewise linear maps. This allows us to think about polyhedra *abstractly*, without reference to an embedding into a Euclidean space: a pair of polyhedra $K \subseteq \mathbb{R}^n$ and $L \subseteq \mathbb{R}^m$ can be isomorphic even if $n \neq m$.

Definition 9. Let M be a polyhedron. We will say that M is a *piecewise linear manifold* (of dimension n) if, for every point $x \in M$, there exists an open neighborhood $U \subseteq M$ containing x and a piecewise linear homeomorphism $U \simeq \mathbb{R}^n$.

Remark 10. Definition 9 can be rephrased so as to better resemble Definition 2. Namely, let M be a topological manifold. We define a *combinatorial structure* on M to be a subsheaf $\mathcal{O}_M^{PL} \subseteq \mathcal{O}_M$ with the following property: for every point $x \in X$, there exists an open embedding $f : \mathbb{R}^n \to M$ whose image contains f, such that $f^* \mathcal{O}_M^{PL} \subseteq \mathcal{O}_{\mathbb{R}^n}$ can be identified with the sheaf whose value on an open subset $U \subseteq \mathbb{R}^n$ consists of piecewise linear maps from U to $\mathbb{R}$.

Every piecewise linear manifold M comes equipped with a combinatorial structure, where we define $\mathcal{O}_M^{PL}$ to be the sheaf of piecewise linear maps on M with values in $\mathbb{R}$. Conversely, if M is a topological manifold endowed with a combinatorial structure, then by choosing sufficiently many sections $f_1, \ldots, f_m \in \mathcal{O}_M^{PL}(M)$ we obtain an embedding $M \to \mathbb{R}^m$ whose image is a polyhedron $\mathbb{R}^m$ (which is a piecewise linear manifold). We can therefore regard the data of a piecewise linear manifold as equivalent to the data of a topological manifold with a combinatorial structure.

Let K be a polyhedron containing a vertex x, and choose a triangulation of K containing x as a vertex of the triangulation. The *star* of x is the union of those simplices of the triangulation which contain x. The *link* of x consists of those simplices belonging to the star of x which do not contain x. We denote the link of x by $\mathrm{lk}(x)$.

As a subset of K, the link $\mathrm{lk}(x)$ of x depends on the choice of triangulation of K. However, one can show that as an abstract polyhedron, $\mathrm{lk}(x)$ is independent of the triangulation up to piecewise linear homeomorphism. Moreover, $\mathrm{lk}(x)$ depends only on a neighborhood of x in K.

If $K = \mathbb{R}^n$ and $x \in K$ is the origin, then the link $\mathrm{lk}(x)$ can be identified with the sphere S^{n-1} (which can be regarded as a polyhedron via the realization $S^{n-1} \simeq \partial\,\Delta^n$). It follows that if K is any piecewise linear n-manifold, then the link $\mathrm{lk}(x)$ is equivalent to S^{n-1} for every point $x \in K$. Conversely, if K is any polyhedron such that every link in K is an $(n-1)$-sphere, then K is a piecewise linear n-manifold. To see this, we observe that for each $x \in K$, if we choose a triangulation of K containing x as a vertex, then the star of x can be identified with the cone on $\mathrm{lk}(x)$. If $\mathrm{lk}(x) \simeq S^{n-1}$, then the star of x is a piecewise linear (closed) disk, so that x has a neighborhood which admits a piecewise linear homeomorphism to the open disk in $\mathbb{R}^n$.

We have proven the following:

Proposition 11. *Let K be a polyhedron. The following conditions are equivalent:*

(i) For each $x \in K$, the link $\mathrm{lk}(x)$ is a piecewise linear $(n-1)$-sphere.

(ii) K is a piecewise linear n-manifold.

Remark 12. Very roughly speaking, we can think of a piecewise linear manifold M as a topological manifold equipped with a triangulation. However, this is not quite accurate, since a polyhedron does not come equipped with a particular triangulation. Instead, we should think of M as equipped with a distinguished class of triangulations, which is stable under passing to finer and finer subdivisions.

Warning 13. Let K be a polyhedron whose underlying topological space is an n-manifold. Then K need not be a piecewise linear n-manifold: it is generally not possible to choose local charts for K in a piecewise linear fashion.

To get a feel for the sort of problems which might arise, consider the criterion of Proposition 11. To prove that K is a piecewise linear n-manifold, we need to show that for each $x \in K$, the link $\mathrm{lk}(x)$ is a (piecewise-linear) n-sphere. Using the fact that K is a topological manifold, we deduce that $\mathrm{H}_*(K, K - \{x\}; \mathbf{Z})$ is isomorphic to $\mathbf{Z}$ in degree n and zero elsewhere; this is equivalent to the assertion that $\mathrm{lk}(x)$ has the homology of an $(n-1)$-sphere. Of course, this does not imply that $\mathrm{lk}(x)$ is itself a sphere. A famous counterexample is due to Poincare: if we let I denote the binary icosahedral group, regarded as a subgroup of $\mathrm{SU}(2) \simeq S^3$, then the quotient $P = \mathrm{SU}(2)/I$ is a homology sphere which is not a sphere (since it is not simply connected).

The suspension ΣP is a 4-dimensional polyhedron whose link is isomorphic to P at precisely two points, which we will denote by x and y. However, ΣP is not a topological manifold. To see this, we note that the point x does not contain arbitrarily small neighborhoods U such that $U - \{x\}$ is simply connected. In other words, the failure of ΣP to be a manifold can be detected by computing the *local fundamental group of* $P - \{x\}$ *near* x (which turns out to be isomorphic to the fundamental group of P). However, if we apply the suspension functor again, the same considerations do not apply: the space ΣP is simply connected (by van Kampen's theorem). Surprisingly enough, it turns out to be a manifold:

Theorem 14 (Cannon-Edwards)**.** *Let P be a topological n-manifold which is a homology sphere. Then the double suspension $\Sigma^2 P$ is homeomorphic to an $(n+2)$-sphere.*

In particular, if we take P to be the Poincare homology sphere, then there is a homeomorphism $\Sigma^2 P \simeq S^5$. However, $\Sigma^2 P$ is not a piecewise linear manifold: it contains two points whose links are given by ΣP, which is not even a topological 4-manifold (let alone a piecewise linear 4-sphere).

The upshot of Warning 13 is that a topological manifold M (such as the 5-sphere) admits triangulations which are badly behaved, in the sense that the underlying polyhedron is not locally equivalent to Euclidean space. The situation is different if we require our triangulations to be compatible with a smooth structure on M. We will take this point up in the next lecture.

Whitehead Triangulations (Lecture 3)

February 13, 2009

In the last lecture, we cited the theorem of Cannon-Edwards which shows that the 5-sphere S^5 admits "bad" triangulations: that is, S^5 can be realized as the underlying topological space of polyhedra which are not piecewise linear manifolds. In this lecture, we will see that such a triangulation is necessarily "wild" in the sense that the simplices are not smoothly embedded in S^5. To be more precise, we need to introduce some terminology.

Definition 1. Let K be a polyhedron and M a smooth manifold. We say that a map $f : K \to M$ *piecewise differentiable* (PD) if there exists a triangulation of K such that the restriction of f to each simplex is smooth. We will say that f is a *PD homeomorphism* if f is piecewise differentiable, a homeomorphism, and the restriction of f to each simplex has injective differential at each point.

The problems of smoothing and triangulating manifolds can now be formulated as follows:

(i) Given a smooth manifold M, does there exist a piecewise linear manifold N and a PD homeomorphism $N \to M$?

(ii) Given a piecewise linear manifold N, does there exist a smooth manifold M and a PD homeomorphism $N \to M$?

Question (i) is much easier, and was addressed by Whitehead in the first half of the last century. More precisely, Whitehead proved the following:

(1) Given a smooth manifold M, there exists a polyhedron K and a PD homeomorphism $K \to M$.

(2) Any such polyhedron K is automatically a piecewise linear manifold.

(3) The polyhedron K is unique up to PL homeomorphism.

Remark 2. Whitehead actually worked in the context of C^1 maps, rather than the infinitely differentiable maps considered here. The distinction will not be important. However, the difference between C^1 maps and continuous maps is vital: as the Cannon-Edwards theorem shows, assertions (2) and (3) fail if we do not assume that our triangulations have some degree of smoothness.

Question (ii) is more difficult, and does not always have an affirmative answer. It is true provided that N has dimension ≤ 7, but false in general. Moreover, if N has dimension 7 then M need not be unique (Milnor's exotic 7-spheres provide examples). Our eventual goal is to show that if N has dimension ≤ 3, then M is unique in a very strong homotopy-theoretic sense.

Our goal for this week is to prove Whitehead's theorems. We will begin with part (2), which asserts that the existence of a piecewise differentiable homeomorphism $f : K \to M$ implies that K is a PL manifold. This question is local, so we may replace M by a Euclidean space $\mathbb{R}^n$. To prove that K is a piecewise linear manifold, it will suffice to show that near every point $x \in K$, we can choose a PD map $f' : K \to \mathbb{R}^n$ which is piecewise linear in a neighborhood of x. We will prove this in two steps:

Proposition 3. *Let K be a polyhedron and let $f : K \to \mathbb{R}^n$ be a PD map. Let K_0 be a finite subpolyhedron. Then there exists another map $f' : K \to \mathbb{R}^n$ with the following properties:*

(1) *The map f' is an arbitrarily good approximation to f in the C^1-sense: that is, we may assume that there is a triangulation S of K such that for each simplex σ of S, both $f|\sigma$ and $f'|\sigma$ are smooth, and $(f - f')|\sigma$ can be chosen to have arbitrarily small values and arbitrarily small first derivatives.*

(2) *The restriction $f'|K_0$ is piecewise linear.*

(3) *The maps f and f' coincide outside of a compact subset of K.*

Proposition 4. *Let $f, f' : K \to \mathbb{R}^n$ be PD maps. Suppose that f is a PD homeomorphism and that f' is a sufficiently good approximation to f in the C^1-sense. Then f is a homeomorphism onto an open subset of $\mathbb{R}^n$.*

We begin with the proof of Proposition 4, since it is easier. The proof is based on the following classical result from point-set topology:

Theorem 5 (Brouwer)**.** *Let $g : M \to N$ be a continuous injective map between topological n-manifolds. Then g is a homeomorphism from M onto some open subset of N.*

Proof of Proposition 4. Since f is a homeomorphism, K is a topological manifold. Consequently, by Theorem 5, it will suffice to show that f' is injective. This is equivalent to the assertion that the map $g = f' \circ f^{-1}$ is injective. Choose a triangulation S of K such that f and f' are smooth on each simplex of S. For each simplex σ of S, the map g is smooth when restricted to the simplex $\sigma' = f(\sigma)$. We will assume that f' is a sufficiently good approximation to f that for each $x \in \sigma'$, the derivative $D_x(g|\sigma) = \mathrm{id}_{\mathbb{R}^n} + A_{x,\sigma}$ for some linear map $A_{x,\sigma} : \mathbb{R}^n \to \mathbb{R}^n$ having operator norm $\leq \frac{1}{2}$. To show that g is injective, it will suffice to prove the following estimate:

$$(g(x) - g(y), x - y) \geq \frac{(x - y, x - y)}{2}.$$

The collection of pairs $x, y \in \mathbb{R}^n$ which satisfy this condition is closed. It will therefore suffice to prove that this condition holds for a dense set of pairs $x, y \in \mathbb{R}^n$.

Let us say that a pair of elements $x, y \in \mathbb{R}^n$ is *good* if the closed interval $\overline{xy}$ is transverse to the PD triangulation of $\mathbb{R}^n$ provided by the map f. We note that the collection of pairs (x, y) which are not good is the image in $\mathbb{R}^{2n}$ of a countably many smooth maps whose domains are manifolds of dimension $2n - 1$, and therefore has measure zero (by Sard's theorem). It follows that the collection of good pairs is dense in $\mathbb{R}^{2n}$.

Suppose now that (x, y) is good, and let $h : [0, 1] \to \mathbb{R}$ be the map defined

$$h(t) = (g(x) - g((1 - t)x + ty), x - y).$$

Then $h(t)$ is a piecewise differentiable function of t, and $h(0) = 0$. We wish to prove that $h(1) \geq \frac{(x-y,x-y)}{2}$. It will suffice to show that the derivative h' (which is defined at all but finitely many points) satisfies the inequality

$$h'(t) \geq \frac{(x - y, x - y)}{2}.$$

Choose a simplex σ such that $z = (1 - t)x + ty \in \sigma'$; then we can write

$$h'(t) = (D_z(g)(x - y), x - y) = (x - y, x - y) + (A_{z,\sigma}(x - y), x - y) \geq \frac{(x - y, x - y)}{2}$$

as desired. □

The proof of Proposition 3 is more difficult. First, choose a PL map $\chi : K \to [0, 1]$ supported in a compact subset $K_1 \subseteq K$ such that $\chi(x) = 1$ for $x \in K_0$. If $f'' : K_1 \to \mathbb{R}^n$ is a PL map which closely approximates $f|K_1$, then the map $f' = \chi f'' + (1 - \chi)f$ satisfies the conditions of Proposition 3. It will therefore suffice to prove the following:

Proposition 6. *Let K be a finite polyhedron and let $f : K \to \mathbb{R}^n$ be a PD map. Then there exists a piecewise linear map $f' : K \to \mathbb{R}^n$ which is an arbitrarily good approximation to f (in the C^1-sense).*

To prove Proposition 6, we need a way of producing piecewise linear maps.

Definition 7. Let K be a polyhedron equipped with a triangulation $S = \{\sigma_i\}$ and let $f : K \to \mathbb{R}^n$ be a map. We define the map $L^S_f : K \to \mathbb{R}^n$ so that the following conditions are satisfied:

(1) For every point $x \in K$ which is a vertex of the triangulation S, we have $L^S_f(x) = f(x)$.

(2) The restriction of L^S_f to each simplex σ of the triangulation S is a linear map $\sigma \to \mathbb{R}^n$.

It is easy to see that for any map $f : K \to \mathbb{R}^n$, the map L^S_f is well-defined and piecewise linear. To prove Proposition 6, we need to show that we can choose the triangulation S such that L^S_f is a good approximation to f (in the C^1-sense). First, fix a triangulation S_0 of K such that the restriction of f to each simplex of S_0 is smooth. Fix $\epsilon > 0$. Refining the triangulation S_0 if necessary, we may assume that f carries each simplex σ of S_0 into an open ball U_σ of radius ϵ. If S refines S_0, then a convexity argument shows that L^S_f carries σ into U_σ, so that $|L^S_f(x) - f(x)|$ is bounded above by 2ϵ. Thus, L^S_f is a good approximation to f in the C^0-sense for any sufficiently fine triangulation.

To guarantee that L^S_f is a good approximation to f in the C^1-sense, we need to work a bit harder. Let us identify K with a finite polyhedron embedded in Euclidean space $\mathbb{R}^m$. For each simplex σ of K (which we will assume is contained in a simplex of S_0), we define the *diameter* $d(\sigma)$ of σ to be the supremum of the distance between any two points of σ (by a convexity argument, this coincides with the length of the longest side of σ). We define the *radius* $r(\sigma)$ to be the distance from the barycenter of σ to the boundary of σ. We define the *thickness* $t(\sigma)$ to be the ratio $\frac{r(\sigma)}{d(\sigma)}$.

We will need the following fact:

Lemma 8. *Let $K \subseteq \mathbb{R}^m$ be a finite polyhedron equipped with a triangulation S_0. Then there exists a positive constant $\delta \leq 1$ such that K has arbitrarily fine triangulations S (in other words, triangulations such that the each simplex has diameter $\leq \epsilon$, for any $\epsilon > 0$) refining S_0 such that each simplex of S has thickness $\geq \delta$.*

Proof. We first note that the claim is independent of the choice of embedding $K \to \mathbb{R}^m$: an embedding $K \to \mathbb{R}^{m'}$ which is linear on each simplex of S_0 (or any triangulation refining S_0) can change the widths of simplices contained in simplices of S_0 by at most a bounded factor.

Let $\{x_1, \ldots, x_k\}$ be the set of vertices of the triangulation S_0, and let $\{y_1, \ldots, y_k\}$ be a linearly independent set in $\mathbb{R}^k$. Then there exists a unique map $K \to \mathbb{R}^k$ which is linear on each simplex of S_0 and carries each x_i to y_i. This map is a PL embedding and its image is a union of faces of the simplex spanned by $\{y_1, \ldots, y_k\}$. Replacing m by k and K by its image in $\mathbb{R}^k$, we may assume that K is a union of faces of some linearly embedded simplex (with its standard triangulation). Enlarging K if necessary, we may suppose that K is itself a simplex Δ^n, where $n = k - 1$.

The existence of the desired triangulations is now a consequence of the following assertion:

($*$) For each $n \geq 0$, there exists a tesselation of Euclidean space $\mathbb{R}^n$ by n-simplices, all congruent to one another, such that multiplication by any integer gives a map $\mathbb{R}^n \to \mathbb{R}^n$ which is a refinement of tesselations.

This proves that the n-simplex admits arbitrarily fine subdivisions into pieces which are similar to itself, thereby providing fine triangulations of Δ^n whose simplices have their width bounded below.

One way to prove ($*$) is to identify $\mathbb{R}^n$ with the Lie algebra of a maximal torus of any compact simplex Lie group G of rank n, and choose the tesselation of $\mathbb{R}^n$ by Weyl alcoves. □

We now continue to fix $\epsilon > 0$, and let δ be as in Lemma 8. Let σ be a k-simplex of K which is contained in a simplex of S_0. Then σ has a tangent plane which we may identify with a subspace $V_\sigma \subseteq \mathbb{R}^m$ of dimension k. The restriction $f|\sigma$ is smooth, and therefore has a differential $D(f|\sigma) : \sigma \to \mathrm{Hom}(V_\sigma, \mathbb{R}^n)$. Choose a triangulation S of K refining S_0 with the following properties:

(i) The triangulation S satisfies the hypothesis of Lemma 8.

(ii) For each simplex σ of S and each pair of elements $x, y \in \sigma$, we have $|D_x(f|\sigma) - D_y(f|\sigma)| \leq \frac{\epsilon\delta}{4m}$.

(Since the functions $D(f|\sigma)$ are continuous on each simplex σ of S_0, assertion (ii) holds for any sufficiently fine refinement of S_0). We will prove that $|D_x(f|\sigma) - D_x(L_f^S|\sigma)| \leq \epsilon$ for each $x \in \sigma \in S$.

Let σ be a k-dimensional simplex given as the convex hull of a set of points $\{v_0, \ldots, v_k\} \in \mathbb{R}^m$. The proof proceeds in several steps:

(a) Since L_f^S is linear on σ, we have $D_x(L_f^S|\sigma) = D_{v_0}(L_f^S|\sigma)$. It will therefore suffice to prove the inequalities

$$|D_x(f|\sigma) - D_{v_0}(f|\sigma)| \leq \frac{\epsilon}{2}$$

$$|D_{v_0}(f|\sigma) - D_{v_0}(L_f^S|\sigma)| \leq \frac{\epsilon}{2}.$$

The first of these follows immediately from assumption (2).

(b) Let $A = D_{v_0}(f|\sigma) - D_{v_0}(L_f^S|\sigma)$. It will suffice to prove that if $q \in V_\sigma$ is a vector of length $\leq r(\sigma)$, then $|A(q)| \leq \frac{r(\sigma)\epsilon}{2}$.

(c) Since $r(\sigma) \geq d(\sigma)\delta$, it will suffice to show that $|A(q)| \leq \frac{d(\sigma)\delta\epsilon}{2}$.

(d) Let v be the barycenter of σ. Since $|q| \leq r(\sigma)$, we have $v, v+q \in \sigma$, so that $|A(q)| \leq |A(v - v_0)| + A(v + q - v_0)|$. It will therefore suffice to show that if $v_0 + w \in \sigma$, then $|A(w)| \leq \frac{d(\sigma)\delta\epsilon}{4}$.

(e) If $v_0 + w \in \sigma$, then we can write $w = \sum_i c_i(v_i - v_0)$ where $0 \leq c_i \leq 1$. It will therefore suffice to show that $|A(v_i - v_0)| \leq \frac{d(\sigma)\delta\epsilon}{4m}$ (since $k \leq m$).

(f) We have

$$\begin{aligned} A(v_i - v_0) &= D_{v_0}(f|\sigma)(v_i - v_0) - D_{v_0}(L_f^S|\sigma)(v_i - v_0) \\ &= D_{v_0}(f|\sigma)(v_i - v_0) + f(v_0) - f(v_i) \\ &= \int_0^1 (D_{v_0}(f|\sigma) - D_{tv_0+(1-t)v_i}(f|\sigma))(v_i - v_0)dt. \end{aligned}$$

Since $|v_i - v_0| \leq d(\sigma)$, we can apply (ii) to deduce that $|A(v_i - v_0)| \leq \frac{d(\sigma)\delta\epsilon}{4m}$ which completes the proof.

Existence of Triangulations (Lecture 4)

February 10, 2009

In the last lecture, we proved that if M is a smooth manifold, K a polyhedron, and $f : K \to M$ a piecewise differentiable homeomorphism (required to be an immersion on each simplex), then K is a piecewise linear manifold. The proof was based on two basic principles:

Proposition 1. *Let $f : K \to \mathbb{R}^n$ be a PD map and $K_0 \subseteq K$ a finite subpolyhedron. Then there exists another PD map $f' : K \to \mathbb{R}^n$ which is piecewise linear on K_0 and agrees with f outside a compact set. Moreover, we can arrange that f' is arbitrarily good approximation to f (in the C^1-sense).*

Proposition 2. *If $f, f' : K \to \mathbb{R}^n$ are PD maps which are sufficiently close to one another (in the C^1-sense) and f is a PD homeomorphism, then f' is a PD homeomorphism onto an open subset of $\mathbb{R}^n$.*

Our goal in this lecture is to apply these results to show that every smooth manifold M admits a Whitehead compatible triangulation. For simplicity, we will assume that M is compact; the noncompact case can be handled using same methods.

Definition 3. Let K be a finite polyhedron, M a smooth manifold, and $f : K \to M$ a map. We say that f is a *PD embedding* if f is injective and there exists a triangulation of K such that f is a smooth immersion on each simplex.

If $f : K \to M$ is a PD embedding, then we can identify K with its image $f(K)$. Any triangulation of K determines a triangulation of $f(K)$ by smooth embedded simplices in M.

Definition 4. Let $f : K \to M$ and $g : K' \to M$ be PD embeddings. We will say that f and g are *compatible* if the following conditions are satisfied:

(1) Let $X = f(K) \cap g(K') \subseteq M$. Then $f^{-1}(X) \subseteq K$ and $g^{-1}(X) \subseteq K'$ are polyhedral subsets of K and K'.

(2) The identification $f^{-1}(X) \simeq X \simeq g^{-1}(X)$ is a piecewise linear homeomorphism.

Suppose that f and g are compatible, and let X be as above. Then the coproduct $K \coprod_X K'$ can be endowed with the structure of a polyhedron, and the maps f and g can be amalgamated to give a PD embedding $f \cup g : K \coprod_X K'$ into M. Moreover, $f \cup g$ is compatible with another PD embedding $h : K'' \to M$ if and only if both f and g are compatible with h.

To prove that a compact smooth manifold M admits a Whitehead compatible triangulation, it will suffice to show that there exists a finite collection of PD embeddings $f_i : K_i \to M$ which are pairwise compatible and whose images cover M. (We can then iterate the amalgamation construction described above to produce a PD homeomorphism $K \to M$.)

For each point $x \in M$, choose a neighborhood W_x of x in M and a smooth identification $W_x \simeq \mathbb{R}^n$ which carries x to the origin in $\mathbb{R}^n$. Let $U_x \subseteq W_x$ denote the image of the unit ball in $\mathbb{R}^n$, and let f_x denote the composite map $[-2, 2]^n \hookrightarrow \mathbb{R}^n \hookrightarrow M$. Since M is compact, the covering $\{U_x\}_{x \in M}$ admits a finite subcovering by $\{U_x\}_{x \in \{x_1, \ldots, x_k\}}$. Let $W_i = W_{x_i}$, $U_i = U_{x_i}$, and $f_i = f_{x_i}$ for $1 \leq i \leq k$. The maps $f_i : [-2, 2]^n \to M$ are PD embeddings whose images cover M. However, the f_i are not necessarily pairwise compatible. To prove the existence of a Whitehead compatible triangulation of M, it will suffice to prove the following:

Proposition 5. *There exist PD embeddings $f'_i : [-2,2]^n \to M$ which are pairwise compatible, and can be chosen to be arbitrarily good approximations (in the C^1 sense) to the maps f_i.*

In fact, if f'_i is sufficiently close to f_i, then f'_i will factor through $W_i \simeq \mathbb{R}^n$ and will not carry the boundary of $[-2,2]^n$ into the closure $\overline{U}_i$, so that U_i is contained in the image of f'_i; thus the images of the f'_i will cover M and give us the desired triangulation of M.

To prove Proposition 5, we will prove by induction on $j \leq k$ that we can choose maps $\{f_i^j\}_{1 \leq i \leq j}$ which are pairwise compatible PD embeddings where f_i^j is an arbitrarily close approximation to f_i (in the C^1-sense). The case $j = 1$ is obvious (take $f_1^1 = f_1$) and the case $j = k$ yields a proof of Proposition 5.

For the inductive step, let us suppose that the maps $\{f_i^{j-1}\}_{1 \leq i < j}$ have already been constructed. Since these maps are compatible, they can be amalgamated to produce a single PD embedding $f^{j-1} : K \to M$. We will replace $f^{j-1} : K \to M$ by a close approximation g which is compatible with f_j. We can then complete the proof by defining $f_j^j = f_j$ and f_i^j to be the composition

$$[-2,2]^n \hookrightarrow K \xrightarrow{g} M.$$

To prove the existence of g, we need the following:

Lemma 6. *Let M be a smooth manifold equipped with a smooth chart $\mathbb{R}^n \hookrightarrow M$, and let $f : K \to M$ be a PD embedding (where K is a finite polyhedron). Then there exist arbitrarily close approximations (in the C^1-sense) of f which are compatible with the embedding $[-2,2]^n \subset \mathbb{R}^n \hookrightarrow M$.*

Proof. Let L be the open subset of K corresponding to the inverse image of $\mathbb{R}^n$, and let L_0 be a finite subpolyhedron of L containing the inverse image of $[-3,3]^n$. According to Proposition 1, the map $f|L : L \to \mathbb{R}^n$ admits arbitrarily good approximations $f' : L \to \mathbb{R}^n$ which are piecewise linear on L_0 and which agree with $f|L$ outside a compact set. Provided that the approximation is sufficiently good, the inverse image ${f'}^{-1}[-2,2]^n$ will be contained in L_0. Since f' is piecewise linear on L_0, we deduce that f' is compatible with the embedding $[-2,2]^n \subset \mathbb{R}^n \hookrightarrow M$. Since $f' = f|L$ outside a compact set, the map $g : K \to M$ defined by the formula

$$g(x) = \begin{cases} f'(x) & \text{if } x \in L \\ f(x) & \text{if } x \notin L \end{cases}$$

is a well-defined PD embedding of K into M, which has the desired properties. □

Variant 7. *Suppose that M is a (compact) smooth manifold with boundary. Then we can modify the above proof to show that any PD homeomorphism $f_0 : K_0 \to \partial M$ can be extended to a PD homeomorphism $K \to M$ where K contains K_0 as a subpolyhedron. For example, we can first extend f_0 to a PD embedding $K_0 \times [0,1] \to M$ by choosing a smooth collar of ∂M. Then M can be covered by the image of $K_0 \times [0,1]$ together with finitely PD embeddings $[-2,2]^n \hookrightarrow \mathbb{R}^n \subseteq M$, and we can apply the above argument without essential change to make these embeddings compatible with one another.*

Variant 8. *Suppose that M is noncompact. The existence of Whitehead compatible triangulations of M can be established by adapting the above arguments: we cannot generally assume that the covering $\{U_i\}$ is finite, but we can use a paracompactness argument to guarantee that the covering is locally finite which is sufficient for the above constructions to go through.*

An alternative strategy uses Variant 7. Choose a smooth proper map $\chi : M \to \mathbb{R}$ with isolated critical points (for example, a Morse function). Then the critical values of χ are isolated, so we can choose a sequence of regular values

$$\{\ldots < r_{-1} < r_0 < r_1 < r_2 < \ldots\}$$

tending to infinity in both directions. We first apply the result in the compact case to find Whitehead compatible triangulations of the inverse images $\chi^{-1}\{r_i\}$, and then apply Variant 7 to extend these to Whitehead compatible triangulations of $\chi^{-1}[r_i, r_{i+1}]$; the result is a Whitehead compatible triangulation for the whole of M.

Uniqueness of Triangulations (Lecture 5)

February 13, 2009

Our goal in this lecture is to prove the following result:

Theorem 1. *Let M be a smooth manifold, and suppose we are given a pair of PD homeomorphisms $f : K \to M$ and $g : L \to M$. Then there exist PD homeomorphisms $f' : K \to M$, $g' : L \to M$ which are arbitrarily good approximations to f and g (in the C^1-sense) such that $f'^{-1} \circ g' : L \to K$ is a PL homeomorphism. In particular, there is a PL homeomorphism between L and K.*

For simplicity, we will assume that M is compact (so that the polyhedra K and L are finite). We will need three lemmas, the first of which is a more refined version of the result of Lecture 3:

Lemma 2. *Let $f : K \to \mathbb{R}^n$ be a PD map and $K_0 \subseteq K$ a finite subpolyhedron. Then there exists another PD map $f' : K \to \mathbb{R}^n$ which is piecewise linear on K_0 and agrees with f outside a compact set. Moreover, we can arrange that f' is arbitrarily good approximation to f (in the C^1-sense), and that f' coincides with f on any subpolyhedron $L \subseteq K$ such that $f|L$ is piecewise linear.*

Proof. We apply the same argument as in Lecture 3: choose a PL map $\chi : K \to [0,1]$ such that χ is supported in a compact subpolyhedron $K_1 \subseteq K$ with $K_0 \subseteq \chi^{-1}\{1\}$. Let S_0 be a triangulation of K_1 such that $L \cap K_1$ is a union of simplices of S_0 and $f|K_1$ is smooth on each simplex of S_0. In lecture 3, we saw that for an appropriate subdivision S of S_0, if we define $f'(x) = \begin{cases} f(x) & \text{if } x \notin K_1 \\ \chi(x)L_f^S(x) + (1-\chi(x))f(x) & \text{if } x \in K_1. \end{cases}$ then f' is a good approximation to f which is PL on K_0 and coincides with f outside of K_1. It also coincides with f on $L \cap K_1$, since the linearization construction will not change the values of f on any simplex where f is already linear. □

Lemma 3. *Let K be a finite polyhedron, K_0 a finite subpolyhedron, and let $f : K \to M$ be a PD map. Let $f'_0 : K_0 \to M$ be another map. If f'_0 is sufficiently close to $f|K_0$, then f'_0 can be extended to a PD map $f' : K \to \mathbb{R}^n$. Moreover, we can arrange that f' is an arbitrarily close approximation to f (in the C^1-sense) provided that f'_0 is a sufficiently good approximation to $f|K_0$ (in the C^1-sense).*

Proof. Working simplex by simplex in a sufficiently fine triangulation, we can reduce to the case where $K = \Delta^k$, $K_0 = \partial\,\Delta^k$, and $M = \mathbb{R}^n$. Let $C \subseteq K$ be a piecewise linear collar of the boundary $\partial\,\Delta^k$, so that $C \simeq [0,1] \times \partial\,\Delta^k$. Let $\pi_1 : C \to [0,1]$ and $\pi_2 : C \to \partial\,\Delta^k$ denote the two projection maps. We define f' by the formula

$$f'(x) = \begin{cases} f(x) & \text{if } x \notin C \\ (1-\pi_1(x))(f'_0(\pi_2(x)) - f(\pi_2(x))) + f(x) & \text{if } x \in C. \end{cases}$$

Then f' is a PD extension of f which coincides with f'_0 on K_0 Moreover, the difference $f' - f$ (and its first derivatives) are easily bounded in terms of the difference $f'_0 - f|K_0$ (and its first derivatives). □

Lemma 4. *Let K be a polyhedron, M a smooth manifold, and $f : K \to M$ a PD homeomorphism. Fix a smooth chart $\mathbb{R}^n \hookrightarrow M$, and let $B \subseteq \mathbb{R}^n$ be an open ball. Then there exist arbitrarily close approximations $f' : K \to M$ to f (in the C^1-sense) such that the restriction of f' to $f'^{-1}(B)$ is a PL homeomorphism.*

Proof. Let B' be an open ball in $\mathbb{R}^n$ containing the closure of B, let $L \subseteq K$ be the inverse image of $\mathbb{R}^n \subseteq M$, and let $L_0 \subseteq L$ be a finite polyhedron containing the inverse image $f^{-1}(B')$. Applying Lemma 2, we conclude that there exist arbitrarily close approximations f'_0 to $f|L$ such that $f'_0|L_0$ is PL and f'_0 agrees with f outside a compact subset of L. Provided that f'_0 is sufficiently close to $f|L$, we deduce that ${f'_0}^{-1}(B) \subseteq f^{-1}(B') \subseteq L_0$, so that the restriction of f'_0 to ${f'_0}^{-1}(B)$ is PL. We conclude by defining

$$f'(x) = \begin{cases} f(x) & \text{if } x \notin L \\ f'_0(x) & \text{if } x \in L. \end{cases}$$

□

We now return to the proof of Theorem 1. Since K is compact, there exists a finite collection of closed subpolyhedra $\{K_i \subseteq K\}_{1 \leq i \leq m}$ with the following property: the image $f(K_i)$ is contained in a smooth chart $\mathbb{R}^n \simeq U_i \subseteq M$. We will prove the following claim by induction on i:

$(*)$ There exist arbitrarily good approximations f_i and g_i to f and g, respectively, such that $f_i|(K_1 \cup \ldots \cup K_i)$ is compatible with g_i.

Taking $i = m$, we will be able to deduce that f_m is compatible with g_m and the proof of Theorem 1 will be complete. The base case for the induction is obvious: if $i = 0$, we can take $f_i = f$ and $g_i = g$. It will therefore suffice to carry out the inductive step.

Assume that f_i and g_i have already been constructed. Let $K(i) = K_1 \cup \ldots \cup K_i$. Since $f_i|K(i)$ is compatible with g_i, we deduce that $g_i^{-1} f_i K(i)$ is a subpolyhedron of L, which we will denote by $L(i)$. Moreover, the composition $g_i^{-1} \circ f_i$ is a PL homeomorphism h from $K(i)$ to $L(i)$.

Applying Lemma 4, we can find a map f'_i which approximates f_i such that the f'_i induces a PL homeomorphism between an open neighborhood V of K_{i+1} and an open ball $B \subseteq U_{i+1}$. The composition $f'_i \circ h^{-1} : L(i) \to M$ is a close approximation to $g_i|L(i)$. Applying Lemma 3, we can extend $f'_i \circ h^{-1}$ to a PD map $g'_i : L \to M$, which we can assume is an arbitrarily close approximation to g_i (and therefore a PD homeomorphism). By construction, $f'_i|K(i)$ is compatible with g'_i.

Let $W \subseteq L$ be the inverse image ${g'_i}^{-1}(B)$. Since h is PL and the homeomorphism $V \simeq B$ is PL, we deduce that the homeomorphism $k : W \simeq B$ obtained by restricting g'_i is piecewise linear on $L(i) \cap W$. Let $B' \subset B$ be a slightly smaller ball which still contains the image $f_i(K_{i+1})$. It follows from Lemma 2 that k admits arbitrarily close approximations k' such that k' is PL on ${k'}^{-1}B'$, k' agrees with k outside a compact set, and k' agrees with k on $L(i) \cap W$. We now set $f_{i+1} = f'_i$ and define g_{i+1} by the formula

$$g_{i+1}(x) = \begin{cases} k'(x) & \text{if } x \in W \\ g'_i(x) & \text{if } x \notin W. \end{cases}$$

Since f_{i+1} and g_{i+1} are both PL on the inverse image of B', we deduce that $f_{i+1}|K_{i+1}$ is compatible with g_{i+1}. The compatibility of $f_{i+1}|K(i)$ with g_{i+1} follows from the compatibility of $f_{i+1}|K(i)$ with g'_i (since $g_{i+1} = g'_i$ on $L(i)$). This completes the proof of Theorem 1

The results of Whitehead can be summarized as follows: every smooth manifold M admits a Whitehead compatible triangulation, which yields a piecewise linear manifold K. Moreover, this piecewise linear manifold is unique up to piecewise linear homeomorphism. Our next goal in this course is to obtain a more refined uniqueness result: roughly speaking, we would like to know not only that K is unique up to PL homeomorphism but in some sense up to a contractible space of choices. Another way of articulating this idea is to say that the existence and uniqueness results for Whitehead triangulations are true not only for individual manifolds, but for parametrized families of manifolds. Many of the results of the last few lectures have parametrized analogues, which can be proven using exactly the same arguments. We will conclude this lecture with an example. First, we need to introduce a bit of terminology:

Definition 5. Let $f : K \to L$ be a PL map of polyhedra. We will say that f is a *submersion* (of dimension n) if for every point $x \in K$, there exist open neighborhoods $U \subseteq K$ of x and $V \subseteq L$ of $f(x)$ and a PL homeomorphism $U \simeq V \times \mathbb{R}^n$ (such that f is given by projection onto the first factor).

Example 6. A polyhedron K is a piecewise linear manifold if and only if the unique map $K \to *$ is a submersion.

There is an analogous notion of submersion in the smooth category, which is probably more familiar: a map of smooth manifolds $M \to N$ is a submersion if its differential is surjective at every point. By the implicit function theorem, this is equivalent to the assertion that every point $x \in M$ has a neighborhood diffeomorphic to $V \times \mathbb{R}^n$, where V is an open subset of N.

The main result of lecture 3 admits the following relative version:

Theorem 7. *Suppose given a commutative diagram*

$$\begin{array}{ccc} K & \xrightarrow{f} & M \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ L & \longrightarrow & N \end{array}$$

where K and L are polyhedra, M and N are smooth manifolds, and the horizontal maps are PD homeomorphisms. Assume that p is a submersion of smooth manifolds. Then q is a submersion of PL manifolds.

If $L = N = *$, then the theorem reduces to the assertion that for any Whitehead compatible triangulation of a smooth manifold, the underlying polyhedron is a PL manifold. In the general case, we can use essentially the same argument. The assertion is local, so we can assume that M has the form $N \times \mathbb{R}^n$. We can then apply the "linearization" construction to the composite map

$$K \to M \to \mathbb{R}^n,$$

to approximate f arbitrarily well by maps $K \to L \times \mathbb{R}^n$ which are piecewise linear in a neighborhood of any given point in $x \in K$. Any sufficiently good approxmation will be a PL homeomorphism in a neighborhood of x, so that q is a submersion.

Diffeomorphisms and PL Homeomorphisms (Lecture 6)

February 16, 2009

Let M be a smooth manifold. In the previous lectures, we showed that M admits a Whitehead compatible triangulation, so that we can regard M as having an underlying piecewise linear manifold. Moreover, this piecewise linear manifold is unique up to piecewise linear homeomorphism. Our goal for the next few lectures is to obtain a more precise form of this statement. For example, we would like to show that every diffeomorphism of smooth manifolds determines a PL homeomorphism, every smooth isotopy of diffeomorphisms determines a piecewise linear isotopy, and so forth. We can summarize the situation by saying that there is a classifying space for smooth manifolds which maps to a suitable classifying space for PL manifolds. Our goal in this lecture is to define the relevant classifying spaces and to outline the relationship between them.

We begin with the smooth case. Let M be a compact smooth manifold. We let $C^\infty(M, M)$ denote the set of smooth maps from M to itself, and $\mathrm{Diff}(M)$ the subset consisting of diffeomorphisms. The set $C^\infty(M, M)$ can be endowed with a topology, where a sequence of functions $f_1, f_2, \ldots : M \to M$ converges to a function $f : M \to M$ if all of the derivatives of $\{f_i\}$ converge uniformly to the derivatives of f. With respect to this topology, $C^\infty(M, M)$ is a Frechet manifold, and the collection of diffeomorphisms $\mathrm{Diff}(M)$ is an open subset (hence also a Frechet manifold).

We will generally not be interested in the exact definition of $\mathrm{Diff}(M)$ (such as the analytic details of what constitutes a convergent sequence of diffeomorphisms), but only the underlying homotopy type. It is therefore convenient to discard the topological space $\mathrm{Diff}(M)$ and work instead with its singular complex $\mathrm{Sing}_\bullet(\mathrm{Diff}(M))$. This is a simplicial set whose n-simplices are given by the formula

$$\mathrm{Sing}_n(\mathrm{Diff}(M)) = \mathrm{Hom}(\Delta^n, \mathrm{Diff}(M)).$$

By general nonsense, we can recover a space homotopy equivalent to $\mathrm{Diff}(M)$ by passing to the geometric realization $|\mathrm{Sing}_n \mathrm{Diff}(M)|$.

Unwinding the definitions, we can describe the simplices of $\mathrm{Sing}_\bullet \mathrm{Diff}(M)$ more explicitly as follows: an n-simplex of $\mathrm{Sing}_\bullet \mathrm{Diff}(M)$ is a homeomorphism

$$f : M \times \Delta^n \to M \times \Delta^n$$

with the following properties:

(1) The function f commutes with the projection to Δ^n.

(2) The function f is smooth in the first variable. In other words, if we write f as $f(m, t)$, then f has arbitrarily many derivatives in the first variable, and these derivatives are continuous in both variables.

(3) For every $t \in \Delta^n$, the induced map $f_t : M \to M$ (which is smooth, by virtue of (2)) is a diffeomorphism.

The advantage of this description is that it does away with some analysis. It tends to be easier to describe what we mean by a continuous map $K \to \mathrm{Diff}(M)$ when K is a simplex (which is equivalent to describing the simplicial set $\mathrm{Sing}_\bullet \mathrm{Diff}(M)$) than in the case where K is a general space (which is equivalent to describing the topological space $\mathrm{Diff}(M)$).

It is even easier to describe the class of *smooth* maps from a simplex into $\mathrm{Diff}(M)$. These can be organized into another simplicial set $\mathrm{Sing}^{\mathrm{sm}}_\bullet \mathrm{Diff}(M)$, whose n-simplices are diffeomorphisms $f : M \times \Delta^n \to M \times \Delta^n$ which commute with the projection to Δ^n. There is no harm in restricting our attention to such simplices, by virtue of the following:

Proposition 1. *The inclusion* $\mathrm{Sing}^{\mathrm{sm}}_\bullet \mathrm{Diff}(M) \subseteq \mathrm{Sing}_\bullet \mathrm{Diff}(M)$ *is a homotopy equivalence of Kan complexes.*

Proof. By general nonsense, it suffices to show the following: given a map $f_0 : \partial \Delta^k \to \mathrm{Sing}^{\mathrm{sm}}_\bullet \mathrm{Diff}(M)$ and an extension of f_0 to $f : \Delta^k \to \mathrm{Sing}_\bullet \mathrm{Diff}(M)$, there exists another extension $f' : \Delta^k \to \mathrm{Sing}^{\mathrm{sm}}_\bullet \mathrm{Diff}(M)$ which is homotopic to f via a homotopy fixed on f_0.

Unwinding the definitions, we can view f_0 as a smooth map $M \times \partial \Delta^k \to M$ and f as an extension $M \times \Delta^k \to M$. Identify M with a smooth submanifold of $\mathbb{R}^n$ for $n \gg 0$, and let N be a tubular neighborhood of M in $\mathbb{R}^n$ equipped with a smooth projection $\pi : N \to M$.

Since f_0 is smooth, it can be extended to a smooth map $f_1 : M \times U_0 \to M$ where U_0 is an open neighborhood of $\partial \Delta^k$. Shrinking U_0 if necessary, we may assume that $f_1 | M \times \{t\}$ is a diffeomorphism for each $t \in U_0$. Choose an open covering of $\Delta^k - U_0$ by small open subsets $\{U_i \subseteq \Delta^k\}_{1 \leq i \leq n}$, choose a point t_i in each U_i, and let $f_i : M \times \Delta^k \to M$ be given by the formula $f_i(m,t) = f(m, t_i)$. Let $\{\phi_i : \Delta^k \to [0,1]\}_{0 \leq i \leq n}$ be a smooth partition of unity subordinate to the covering $\{U_i\}_{0 \leq i \leq n}$. We now define f' by the formula $f'(m,t) = \pi(\sum_{0 \leq i \leq n} \phi_i(t) f_i(m,t))$. If the open covering is fine enough, then f' will be a smooth extension of f_0 which is a diffeomorphism for each $t \in \Delta^k$, and the functions

$$h_s(m,t) = \pi(s f(m,t) + (1-s) f'(m,t))$$

will give a homotopy from f to f' which is fixed on $M \times \partial \Delta^k$. □

Remark 2. The definitions of $\mathrm{Sing}_\bullet \mathrm{Diff}(M)$ and $\mathrm{Sing}^{\mathrm{sm}}_\bullet \mathrm{Diff}(M)$ extend easily to the case when M is not compact. In this case, one can also define a topology on $\mathrm{Diff}(M)$, but the discussion becomes more technical.

It is convenient to study topological groups G by means of their classifying spaces. In our context, there is a convenient model for these classifying spaces.

Notation 3. Let V be a finite dimensional real vector space, and M a smooth m-manifold. We let $\mathrm{Emb}_{\mathrm{sm}}(M,V)$ denote the simplicial set of *embeddings* of M into V: that is, the simplicial set whose n-simplices are smooth embeddings $M \times \Delta^n \to V \times \Delta^n$ which commute with the projection to n. We let $\mathrm{Sub}^m_{\mathrm{sm}}(V)$ denote the simplicial set of *submanifolds* of V, whose n-simplices are given by smooth submanifolds $X \subseteq V \times \Delta^n$ such that the projection $X \to \Delta^n$ is a smooth fiber bundle of relative dimension m.

If V is infinite dimensional, we let $\mathrm{Emb}_{\mathrm{sm}}(M,V)$ and $\mathrm{Sub}^m_{\mathrm{sm}}(V)$ denote the direct limits of $\mathrm{Emb}_{\mathrm{sm}}(M,V_0)$ and $\mathrm{Sub}_{\mathrm{sm}}(V_0)$, as V_0 ranges over all finite dimensional subspaces of V.

Remark 4. There is a canonical (free) action of $\mathrm{Sing}^{\mathrm{sm}}_\bullet \mathrm{Diff}_{\mathrm{sm}}(M)$ on $\mathrm{Emb}_{\mathrm{sm}}(M,V)$, and the quotient $\mathrm{Emb}_{\mathrm{sm}}(M,V)/\mathrm{Sing}^{\mathrm{sm}}_\bullet \mathrm{Diff}(M)$ can be identified with the union of those components of $\mathrm{Sub}^m_{\mathrm{sm}}(V)$ spanned by submanifolds of V which are diffeomorphic to M.

Remark 5. If V is infinite dimensional, then the simplicial set $\mathrm{Emb}_{\mathrm{sm}}(M,V)$ is a contractible Kan complex. In other words, every smooth embedding $M \times \partial \Delta^n \to V_0 \times \partial \Delta^n$ can be extended to a smooth embedding $M \times \Delta^n \to V_1 \times \Delta^n$ for some $V_0 \subseteq V_1$. This follows from general position arguments.

Combining these remarks, we obtain the following:

Proposition 6. *Let* V *be infinite dimensional. Then the simplicial set* $\mathrm{Sub}^m_{\mathrm{sm}}(V)$ *is homotopy equivalent to a disjoint union* $\coprod_M B(\mathrm{Sing}^{\mathrm{sm}}_\bullet \mathrm{Diff}(M))$, *where* M *ranges over all diffeomorphism types of smooth* m*-manifolds.*

We note that all of the above constructions make sense also in the piecewise linear category. Namely, we have the following definitions:

(1) If M is a piecewise linear m-manifold, we can define a simplicial group $\mathrm{Homeo}_{PL}(M)_\bullet$, whose n-simplices are PL homeomorphisms from $M \times \Delta^n$ to itself that commute with the projection to Δ^n.

(2) If V is a finite dimensional vector space, we let $\mathrm{Emb}_{PL}(M, V)$ be the simplicial set whose n-simplices are PL embeddings $M \times \Delta^n \to V \times \Delta^n$ which commute with the projection to Δ^n. These simplicial sets are acted on freely by $\mathrm{Homeo}_{PL}(M)_\bullet$.

(3) If V is infinite dimensional, we set $\mathrm{Emb}_{PL}(M, V) = \varinjlim_{V_0} \mathrm{Emb}_{PL}(M, V)$, where the colimit is taken over all finite dimensional subspaces $V_0 \subseteq V$. As before, general position arguments guarantee that $\mathrm{Emb}_{PL}(M, V)$ is a contractible Kan complex, so that the quotient $\mathrm{Emb}_{PL}(M, V)/\mathrm{Homeo}_{PL}(M)_\bullet$ is a classifying space for $\mathrm{Homeo}_{PL}(M)_\bullet$.

(4) If V is a finite dimensional vector space, we let $\mathrm{Sub}^m_{PL}(V)$ denote the simplicial set whose n-simplices are subpolyhedra $X \subseteq V \times \Delta^n$ which are PL homeomorphic to $M \times \Delta^n$, for some PL m-manifold M. If V is infinite dimensional set $\mathrm{Sub}^m_{PL}(V) = \varinjlim_{V_0 \subset V} \mathrm{Sub}^m_{PL}(V_0)$.

We have the following analogue of Proposition 7:

Proposition 7. *Let V be infinite dimensional. Then the simplicial set $\mathrm{Sub}^m_{PL}(V)$ is homotopy equivalent to a disjoint union $\coprod_M B(\mathrm{Homeo}_{PL}(M))$, where M ranges over all diffeomorphism types of PL m-manifolds.*

Fix an infinite dimensional vector space V. We define $\mathrm{Man}^m_{\mathrm{sm}} = \mathrm{Sub}^m_{\mathrm{sm}}(V)$, and $\mathrm{Man}^m_{PL} = \mathrm{Sub}^m_{PL}(V)$. We can think of $\mathrm{Man}^m_{\mathrm{sm}}$ and Man^m_{PL} as *classifying spaces* for smooth and PL m-manifolds, respectively. We wish to compare these classifying spaces. To this end, we introduce the following definition:

Definition 8. We define a simplicial set Man^m_{PD} as follows. The n-simplices of Man^m_{PD} are triples (K, M, f) where $K \subseteq V \times \Delta^n$ is an n-simplex of Man^m_{PL}, $M \subseteq V \times \Delta^n$ is an n-simplex of $\mathrm{Man}^m_{\mathrm{sm}}$, and $f : K \to M$ is a PD homeomorphism which commutes with the projection to Δ^n.

By construction, we have forgetful maps

$$\mathrm{Man}^m_{PL} \xleftarrow{\theta'} \mathrm{Man}^m_{PD} \xrightarrow{\theta} \mathrm{Man}^m_{\mathrm{sm}}.$$

In the next lecture, we will sketch the following more refined version of Whitehead's results on the existence and uniqueness of triangulations:

Theorem 9. *The map θ is a trivial Kan fibration.*

It follows that θ admits a section s. Composing s with θ', we obtain a map of classifying spaces $\mathrm{Man}^m_{\mathrm{sm}} \to \mathrm{Man}^m_{PL}$: this is a fancy way of saying that every family of smooth manifolds admits a family of triangulations.

We will eventually sketch the proof of the following "converse":

Theorem 10. *If $m \leq 3$, then the map θ' is a trivial Kan fibration. In particular, the Kan complexes Man^m_{PL} and $\mathrm{Man}^m_{\mathrm{sm}}$ are homotopy equivalent to one another.*

Triangulation in Families (Lecture 7)

February 17, 2009

In the last lecture, we introduced the diagram of simplicial sets

$$\mathrm{Man}^m_{PL} \xleftarrow{\phi'} \mathrm{Man}^m_{PD} \xrightarrow{\phi} \mathrm{Man}^m_{\mathrm{sm}}.$$

Our goal in this lecture is to prove that the map ϕ is a trivial Kan fibration. In other words, we wish to show that every diagram

$$\begin{array}{ccc} \partial\,\Delta^n & \longrightarrow & \mathrm{Man}^m_{PD} \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & \mathrm{Man}^m_{\mathrm{sm}} \end{array}$$

can be completed by adding a suitable dotted arrow.

Let V be an infinite dimensional real vector space. Unwinding the definitions, we are given smooth submanifold $X \subseteq V \times \Delta^n$ which is a fiber bundle over Δ^n, a subpolyhedron $K_0 \subseteq V \times \partial\,\Delta^n$, and a PD homeomorphism $K_0 \to X \times_{\Delta^n} \partial\,\Delta^n$ such that the following diagram commutes:

$$\begin{array}{ccc} K_0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ \partial\,\Delta^n & \longrightarrow & \Delta^n. \end{array}$$

We wish to find the following data:

(i) A polyhedron K equipped with a PL map $\pi : K \to \Delta^n$ and a homeomorphism $K_0 \simeq Y \times_{\Delta^n} \partial\,\Delta^n$.

(ii) A PD homeomorphism $K \to X$ which commutes with the projection to Δ^n.

(iii) A lifting of π to a PL embedding $K \to V \times \Delta^n$ which extends the embedding already given on K_0.

Moreover, this data must satisfy the following condition:

(iv) The projection map $K \to \Delta^n$ exhibits K as a fiber bundle in the PL category (automatically trivial, since Δ^n is contractible). In other words, there is a PL homeomorphism $Y \simeq \Delta^n \times N$, for some PL m-manifold N.

As we saw last time, the data of (iii) comes essentially for free, using general position arguments. We will focus on conditions (i) and (ii) for the time being, and return to (iv) at the end of the discussion.

Since $X \to \Delta^n$ is a fiber bundle in the smooth setting, we can identify X with a product $M \times \Delta^n$ for some smooth manifold M. The PD homeomorphism $K_0 \to X \times_{\Delta^n} \partial\,\Delta^n$ can be viewed as a providing a Whitehead compatible triangulation of $M \times \partial\,\Delta^n$ which is compatible with the polyhedron structure on $\partial\,\Delta^n$ (in other words, a PD homeomorphism $K_0 \to M \times \partial\,\Delta^n$ such that the composite map $K_0 \to \partial\,\Delta^n$ is PL. We wish to extend this to a Whitehead compatible triangulation of $M \times \Delta^n$ which is compatible with the projection $M \times \Delta^n \to \Delta^n$.

In the case $n = 0$, this reduces to the problem we solved in Lecture 4: namely, proving that every smooth manifold M admits a Whitehead compatible triangulation. For our present needs, we will require a more refined version of the same result:

Theorem 1. *Let M be a smooth manifold and let $f_0 : K_0 \to M \times \partial \Delta^n$ be a PD homeomorphism such that the composite map $K_0 \to \partial \Delta^n$ is PL. Then f can be extended to a PD homeomorphism $K \to M \times \Delta^n$, where the projection $K \to \Delta^n$ is PL.*

Proof. Write Δ^n as a union of two closed subpolyhedra L and L' whose interiors cover Δ^n, where L contains a neighborhood of $\partial \Delta^n$, $L' \cap \partial \Delta^n = \emptyset$, and there is a retraction $r : L \to \partial \Delta^n$. Let $\overline{K}_0 = K_0 \times_{\partial \Delta^n} L$. Then f_0 evidently extends to a PD embedding $\overline{f}_0 : \overline{K}_0 \to M \times \Delta^n$ with image $M \times L$.

For each $x \in M$, choose a smooth chart $i_x : \mathbb{R}^n \to M$ carrying 0 to x, and let U_x denote the image of the open ball. Since M is compact, we can choose a finite collection $\{x_1, \ldots, x_k\}$ such that the open balls $U_i = U_{x_i}$ cover M. We then have PD maps $\overline{f}_i : [-2, 2]^n \times L' \to M \times \Delta^n$ whose images cover $M \times L'$. We observe that each of the projections $\pi \circ \overline{f}_i$ is PL, where $\pi : M \times \Delta^n \to \Delta^n$ denotes the projection.

To produce the desired map $K \to M \times \Delta^n$, it will suffice to show that we can approximate the PD maps $\{\overline{f}_0, \ldots, \overline{f}_n\}$ by maps $\{\overline{f}'_i\}$ which are pairwise compatible, where $\overline{f}_0|K_0 = \overline{f}'_0|K_0$ and $\pi \circ \overline{f}_i = \pi \circ \overline{f}'_i$. We proceed as in Lecture 4 to define sequences of approximations $\{\overline{f}^j_0, \ldots, \overline{f}^j_j\}$ to $\{\overline{f}_0, \ldots, \overline{f}_j\}$ using induction on j. When $j = 0$, we set $\overline{f}^j_0 = \overline{f}_0$.

Suppose that we have already defined a sequence of pairwise compatible maps $\overline{f}^j_0, \ldots, \overline{f}^j_j$ which are close approximations to $\overline{f}_0, \ldots, \overline{f}_j$. These maps can therefore be amalgamated to product a single PD map $F : \overline{K}_j \to M \times \Delta^n$, where $\overline{K}_j$ is a polyhedron containing K_0 such that $F|K_0 = f_0$ and $\pi \circ F$ is PL. To complete the proof, it will suffice to show that we can choose close approximations F' to F with the same properties, so that F' is compatible with $\overline{f}_{j+1}$. To prove this, let $P \subseteq \overline{K}_j$ denote the inverse image of $\mathbb{R}^n \times (\Delta^n - \partial \Delta^n) \subseteq M \times \Delta^n$, and let $g : P \to \mathbb{R}^n$ denote the composition of F with the projection $\mathbb{R}^n \times \Delta^n \to \mathbb{R}^n$.

Let $P_0 \subseteq P$ be a compact subpolyhedron containing the inverse image of $[-3, 3]^n \times C$, where C is a closed neighborhood of L' in $\Delta^n - \partial \Delta^n$. Applying the main lemma from the last lecture, we can approximate g arbitrarily well by a map $g' : P \to \mathbb{R}^n$ whose restriction to X_0 is PL and which agrees with g outside a compact set. We can then define a map $F' : \overline{K}_j \to M \times \Delta^n$ by the formula

$$F'(x) = \begin{cases} F(x) & \text{if } x \notin P \\ (g'(x), \pi F(x)) & \text{if } x \in P. \end{cases}$$

If g' is a sufficiently good approximation to g, then $F'^{-1}[-2, 2]^n \times L' \subseteq P_0$, so that F' is PL on $F'^{-1}[-2, 2]^n \times L'$ and therefore compatible with $\overline{f}_{j+1}$. It is readily verified that F' has the desired properties. □

This completes the construction of a polyhedron K containing K_0 and a PD homeomorphism $f : K \to M \times \Delta^n$ such that $\pi \circ f$ is piecewise linear. To complete the proof, we need to verify (iv): that is, we need to show that $\pi \circ f$ exhibits K as a PL fiber bundle over the simplex Δ^n. We first establish a *local* version of this statement.

First, we need to introduce a bit of terminology:

Definition 2. Let $f : K \to L$ be a PL map of polyhedra. We will say that f is a *submersion* (of relative dimension n) if for every point $x \in K$, there exist open neighborhoods $U \subseteq K$ of x and $V \subseteq L$ of $f(x)$ and a PL homeomorphism $U \simeq V \times \mathbb{R}^n$ (such that f is given by projection onto the first factor).

Example 3. A polyhedron K is a piecewise linear manifold if and only if the unique map $K \to *$ is a submersion.

There is an analogous notion of submersion in the smooth category, which is probably more familiar: a map of smooth manifolds $M \to N$ is a submersion if its differential is surjective at every point. By the implicit function theorem, this is equivalent to the assertion that every point $x \in M$ has a neighborhood diffeomorphic to $V \times \mathbb{R}^n$, where V is an open subset of N.

The main result of lecture 3 admits the following relative version:

Theorem 4. *Suppose given a commutative diagram*

$$\begin{array}{ccc} K & \xrightarrow{f} & M \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ L & \longrightarrow & N \end{array}$$

where K and L are polyhedra, M and N are smooth manifolds, and the horizontal maps are PD homeomorphisms. Assume that p is a submersion of smooth manifolds. Then q is a submersion of PL manifolds.

If $L = N = *$, then the theorem reduces to the assertion that for any Whitehead compatible triangulation of a smooth manifold, the underlying polyhedron is a PL manifold. In the general case, we can use essentially the same argument. The assertion is local, so we can assume that M has the form $N \times \mathbb{R}^n$. We can then apply the "linearization" construction to the composite map

$$K \to M \to \mathbb{R}^n,$$

to approximate f arbitrarily well by maps $K \to L \times \mathbb{R}^n$ which are piecewise linear in a neighborhood of any given point in $x \in K$. Any sufficiently good approxmation will be a PL homeomorphism in a neighborhood of x, so that q is a submersion.

Of course, the condition of being a submersion is generally weaker than the condition of being a fiber bundle. To complete the verification of (*iv*) we will need the following technical result, whose proof will occupy our attention during the next few lectures:

Theorem 5. *Suppose given a commutative diagram*

$$\begin{array}{ccc} K & \xrightarrow{f} & M \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ L & \longrightarrow & N \end{array}$$

where K and L are polyhedra, M and N are smooth manifolds, and the horizontal maps are PD homeomorphisms. Assume that p is a submersion of smooth manifolds (so that q is a submersion of PL manifolds). Then p is a smooth fiber bundle if and only if q is a PL fiber bundle.

Remark 6. If the fiber dimensions are not equal to 4, then Theorem 5 can be deduced from the following result.

Theorem 7. *Let $p : M \to N$ be a submersion of smooth (PL) manifolds of relative dimension $\neq 4$, and assume that p is a fiber bundle in the category of topological manifolds. Then p is a fiber bundle in the category of smooth (PL) manifolds.*

Theorem 7 is false in relative dimension 4, but Theorem 5 is true in every dimension.

PL vs. Smooth Fiber Bundles (Lecture 8)

March 16, 2009

Our goal in this lecture is to begin to prove the following result:

Theorem 1. *Suppose given a commutative diagram*

$$\begin{array}{ccc} K & \xrightarrow{f} & M \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ L & \longrightarrow & N \end{array}$$

where K and L are polyhedra, M and N are smooth manifolds, and the horizontal maps are PD homeomorphisms. Assume that p is a submersion of smooth manifolds (so that q is a submersion of PL manifolds). Then p is a smooth fiber bundle if and only if q is a PL fiber bundle.

Since the horizontal maps are homeomorphisms, the morphisms p and q can be identified as continuous maps between topological spaces. It follows that p is proper if and only if q is proper. In the proper case, Theorem 1 is a consequence of the following:

Proposition 2. *Let $p : M \rightarrow N$ be a proper submersion between smooth (PL) manifolds. Then p is a smooth (PL) fiber bundle.*

In the smooth case, this result is elementary. We may assume without loss of generality that $N = \mathbb{R}^k$. Choose a Riemannian metric on M, which determines a splitting of the tangent bundle T_M into vertical and horizontal components $T_M \simeq T_M^v \oplus T_M^h$. Using the fact that p is proper, we deduce that for each $x \in M$ and each smooth path $h : p(x) \rightarrow y$, there exists a unique smooth path $\overline{h} : x \rightarrow \overline{y}$ lifting h such that the derivative of $\overline{h}$ lies in the horizontal tangent bundle T_M^h at every point. In particular, if we choose x to lie in the fiber $X_0 = p^{-1}\{0\}$ and h to be a straight line from $p(x) = 0$ to a point $y \in N$, then we can write $\overline{y}$ as a function $f(x, y)$. The function $f : M_0 \times N \rightarrow M$ is a diffeomorphism in a neighborhood of $M_0 \times \{0\}$, so that f is a submersion in a neighborhood of $0 \in N$.

We wish to give a proof which works also in the PL context. We note that we can assume without loss of generality that the base N is a simplex. We now introduce a bit of terminology:

Definition 3. Let $p : M \rightarrow \Delta^n$ be a map of polyhedra, let $x \in N$, and let $K \subseteq p^{-1}\{x\}$ be a compact subpolyhedron. We will say that p *has a product structure near* K if there exists an open subset $V \subseteq N$ containing x and open subset $U \subseteq M$ containing K such that U is PL homeomorphic to a product $U_0 \times V$ where U_0 is a PL manifold (and p is given by the projection to the second factor).

We note that p is a submersion if and only if it has a product structure near every point. If p is proper and has a product structure near the inverse image $M_x = p^{-1}\{x\}$, then we can take $U_0 = M_x$ so we get an open embedding $M_0 \times V \hookrightarrow p^{-1}(V)$. Using the properness of p, we deduce that this map is a homeomorphism (possibly after shrinking V).

It p has a product structure near a subset $K \subseteq p^{-1}\{x\}$, then it has a product structure near a larger polyhedron containing K in its interior. In particular, if p is a submersion, then it has a product structure near every simplex of some sufficiently fine triangulation of $p^{-1}\{x\}$. It now suffices to show:

Proposition 4. *Let $p : M \to \Delta$ be a map of polyhedra (where Δ denotes a simplex), let $0 \in \Delta$ be a point, let $M_0 = p^{-1}(0)$, and let $A, B \subseteq M_0$ be compact subpolyhedra. If p has a product structure near A and B, then p has a product structure near $A \cup B$.*

The proof will be based on the following nontrivial result of piecewise linear topology:

Theorem 5 (Parametrized Isotopy Extension Theorem)**.** *Let M be a piecewise linear manifold, let K be a finite polyhedron, and let Δ be a simplex containing a point 0. Let $f : K \times \Delta \to M \times \Delta$ be a PL embedding compatible with the projection to Δ, which we think of as a family of embeddings $\{f_t : K \to M\}_{t \in \Delta}$. Assume that f is* locally *extendible to family of isotopies of M: that is, we can embed K as a closed subset of another polyhedron U and extend f to an open embedding $U \times \Delta \hookrightarrow M \times \Delta$. Then there exists a PL homeomorphism $h : M \times \Delta \to M \times \Delta$ (which we can think of as a family of PL homeomorphisms $\{h_t : M \to M\}_{t \in \Delta}$) such that $h_0 = \mathrm{id}_M$ and $h(f_t(k)) = (f_0(k), t)$.*

Proof of Proposition 4. Shrinking Δ if necessary, we may assume that there are open sets $U, V \subseteq M_0$ containing A and B, respectively, and open embeddings $f : U \times \Delta \hookrightarrow M$, $g : V \times \Delta \hookrightarrow M$ such that $f|U \times \{0\}$ and $g|V \times \{0\}$ are the inclusions $U, V \subseteq M_0 \subseteq M$. Let K be a compact polyhedron contained in $U \cap V$ which contains a neighborhood of $A \cap B$. Shrinking Δ if necessary, we may assume that $f(K \times \Delta)$ is contained in $g(V \times \Delta)$, so we that $g^{-1} \circ f$ gives a well-defined map $q : K \times \Delta \to V \times \Delta$ such that $q_0 : K \to V$ is the identity. Using Theorem 5, we can find a map $h : V \times \Delta \to V \times \Delta$ such that h_0 is the identity and $h \circ q$ is the canonical inclusion $K \times \Delta \to V \times \Delta$. Replacing g by $g \circ h^{-1}$, we can assume that f and g agree on $K \times \Delta$. Let $U_0 \subseteq U$ and $V_0 \subseteq V$ be smaller open subsets containing A and B such that $U_0 \cap V_0 \subseteq K$. Then $f|U_0 \times \Delta$ and $g|V_0 \times \Delta$ can be amalgamated to obtain a map $e : W \times \Delta \to M$, where $W = U_0 \cup V_0$. Shrinking W and Δ if necessary, we can arrange that e is an open embedding, which provides the desired product structure near $A \cup B$. □

Let us now return to the general case of Theorem 1. We will concentrate on the "only if" direction (since this is what is needed for the purposes described in the last lecture). The problem is local on N, so we may assume that N consists of a single simplex Δ. We therefore have a trivial fiber bundle $p : M \times \Delta \to \Delta$ of smooth manifolds, a Whitehead compatible triangulation of $M \times \Delta$ such that p is a piecewise linear map, and we wish to show that p is a PL fiber bundle.

Choose a proper smooth map $f : M \to \mathbb{R}_{>0}$. Modifying f slightly, we may assume that $1, 2, \ldots \in \mathbb{R}$ are regular values of f, so that the subsets $M_i = f^{-1}[0, i]$ are compact submanifolds M with boundary $B_i = f^{-1}\{i\}$. Choose disjoint collar neighborhoods $U_i \simeq B_i \times \mathbb{R} \subseteq M$ such that $U_i \cap M_i \simeq B_i \times \mathbb{R}_{\leq 0}$.

Fix a point $0 \in \Delta$, so that $p^{-1}\{0\} \simeq M$ inherits a PL structure. Choose any Whitehead compatible triangulation of B_i, so that $B_i \times \mathbb{R}$ inherits a PL structure. The inclusion $f : B_i \times \mathbb{R} \hookrightarrow M$ need not be a PL homeomorphism. However, we saw in Lecture 5 that f can be approximated arbitrarily well by a PL homeomorphism $f' : B_i \times \mathbb{R} \to U$. In particular, we can assume that $C = f'(B_i \times (-\infty, 0])$ is a PL manifold with boundary of whose interior contains $B_i \times (-\infty, -1]$ and which is contained in $B_i \times (-\infty, 1]$.

We now require the following consequence of a special case of Theorem 1, which we will prove in the next lecture:

Lemma 6. *Let B be a smooth manifold. Suppose we are given a Whitehead compatible triangulation of $B \times \mathbb{R} \times \Delta$, where Δ is a simplex, such that the projection $p : B \times \mathbb{R} \times \Delta \to \Delta$ is a piecewise linear. Then there exists an open subset $E \subseteq B \times \mathbb{R} \times \Delta$ containing $B \times [-1, 1] \times \Delta$ such that the projection $E \to \Delta$ is a PL fiber bundle.*

Applying the Lemma in the case $B = B_i$, we deduce the existence of an open subset $V \subseteq B_i \times \mathbb{R}$ containing $B_i \times [-1, 1]$ and a PL homeomorphism $E \simeq V \times \Delta$. Shrinking Δ is necessary, we may assume that this homeomorphism carries $B \times \{-1\} \times \Delta \subseteq E$ into the interior of $C \times \Delta \subseteq V \times \Delta$. Let X_i denote union of the image of $(C \cap V) \times \Delta$ under this map with $(M_i - (B_i \times (-1, 0])) \times \Delta$. We now have an increasing filtration

$$X_1 \subseteq X_2 \subseteq \ldots \subseteq M \times \Delta$$

by compact subpolyhedra, and each of the projections $X_i \to \Delta$ is a submersion whose fibers are PL manifolds with boundary. It follows from a variant of Proposition 2 (allowing for the case of manifolds with boundary) that each of the maps $X_i \to \Delta$ is a PL fiber bundle, so we have PL homeomorphisms $h_i : X_i \simeq P_i \times \Delta$ for some PL manifold with boundary P_i. Using the parametrized isotopy extension theorem, we can adjust h_i so that the induced maps $P_{i-1} \times \Delta \to P_i \times \Delta$ are induced by embeddings $P_{i-1} \hookrightarrow P_i$. Taking P to be the direct limit of the P_i, we obtain a PL homeomorphism $P_i \times \Delta \to M \times \Delta$, which proves that the projection map $p : M \times \Delta \to \Delta$ is a fiber bundle in the PL category.

Some Engulfing (Lecture 9)

February 22, 2009

Our goal in this lecture is to complete the proof that every Whitehead triangulation of a smooth fiber bundle yields a PL fiber bundle. Recall that we had reduced ourself to the case where the smooth fiber bundle in question was the projection $p : M \times \mathbb{R} \times \Delta \to \Delta$, where M is a compact smooth manifold and Δ is a simplex. We assume that we are given a Whitehead compatible triangulation of $M \times \mathbb{R} \times \Delta$ such that the map p is piecewise linear.

One strategy for analyzing the map p is to try to compare it with the projection $p' : M \times S^1 \times \Delta \to \Delta$. The map p' is proper, so any Whitehead compatible triangulation of $M \times S^1 \times \Delta$ making p' piecewise linear will automatically exhibit $M \times S^1 \times \Delta$ as a fiber bundle over Δ in the PL category. Unfortunately, the obvious translation map $T : (m, r, v) \mapsto (m, r - 2\pi, v)$ is probably not a piecewise linear map from $M \times \mathbb{R} \times \Delta$ to itself, so it is not clear that our Whitehead compatible triangulation of $M \times \mathbb{R} \times \Delta$ descends to give a Whitehead compatible triangulation of $M \times S^1 \times \Delta$. Our goal will be to find a good surrogate for the translation map T. Namely, we will prove the following:

Proposition 1. *There exists a PL homeomorphism $H : M \times \mathbb{R} \times \Delta \to M \times \mathbb{R} \times \Delta$ such that $p \circ H = p$ and $H(M \times (-\infty, 1] \times \Delta) \subseteq M \times (-\infty, -1) \times \Delta$.*

Assume Proposition 1 for the moment, and let D (for "fundamental domain") denote the closed set $(M \times (-\infty, 1] \times \Delta) - H(M \times (-\infty, 1) \times \Delta)$. Then the union $E = \bigcup_{n \in \mathbf{Z}} H^n D$ is an open subset of $M \times \mathbb{R} \times \Delta$ which is acted on by freely by the cyclic group $\{H^n\}_{n \in \mathbf{Z}} \simeq \mathbf{Z}$. Since H is a PL map, the quotient $E/\mathbf{Z}$ inherits a PL structure, and is equipped with a proper PL submersion $E/\mathbf{Z} \to \Delta$. We saw last time that this proper submersion must be a fiber bundle. It follows that E, being a cyclic cover of a fiber bundle, is also a fiber bundle. Since E contains $M \times [-1, 1] \times \Delta$, this will prove the lemma that we needed from the last lecture.

We now turn to the proof of Proposition 1. The idea is to construct H locally near each point of the simplex Δ, and then glue the resulting homeomorphisms together. To carry this out, we will need a more refined version of the statement of Proposition 1. Recall that a *piecewise linear isotopy* from $M \times \mathbb{R} \times \Delta$ to itself is a PL homeomorphism $h : M \times \mathbb{R} \times \Delta \times [0, 1] \to M \times \mathbb{R} \times \Delta \times [0, 1]$ which commutes with the projection to $[0, 1]$; we think of h as a family of PL homeomorphisms $h_t : M \times \mathbb{R} \times \Delta \to M \times \mathbb{R} \times \Delta$ parametrized by $t \in [0, 1]$. In what follows, we will assume that all of these homeomorphisms commute with the projection map $p : M \times \mathbb{R} \times \Delta$. We say that h is *supported* in a closed subset $K \subseteq M \times \mathbb{R} \times \Delta$ if each h_t is the identity on $M \times \mathbb{R} \times \Delta - K$.

Choose a large integer N (to be determined later). For each closed subset C and each integer $d > 0$, consider the following assertion:

$(P_{C,d})$ For every pair of integers $a \leq b \in \mathbf{Z}$ such that $-N \leq a - d \leq b + d \leq N$, there exists a PL isotopy $\{h_t : M \times \mathbb{R} \times \Delta \to M \times \mathbb{R} \times \Delta\}_{t \in [0,1]}$ (compatible with the projection p) having the following properties:

(1) The isotopy h is supported in a compact subset of $M \times (a - d, b + d) \times \Delta$.

(2) The map h_0 is the identity.

(3) The homeomorphism h_1 carries $M \times (-\infty, b] \times C$ into $M \times (-\infty, a) \times C$.

Remark 2. Note that there are only finitely many pairs of integers a, b satisfying the condition $-N \leq a - d \leq b + d \leq N$, so $P_{C,d}$ asserts the existence of only finitely many isotopies; this is the reason for introducing the parameter N.

We will prove that there exists an integer d such that $P_{\Delta,d}$ holds, where d does not depend on N. Then, if $N \geq d+1$, we can apply $P_{C,d}$ in the case $a = -1$, $b = 1$ to obtain an PL homeomorphism h_1 satisfying the requirements of Proposition 1.

The basic observation is the following:

Lemma 3. *If $P_{C,d}$ and $P_{C',d'}$ hold, then $P_{C \cup C', d+d'}$ holds.*

Proof. Assume that $-N \leq a - d - d' \leq b + d + d' \leq N$. Applying $P_{C,d}$, we can choose a PL isotopy h_t supported in $M \times (a-d, b+d+d') \times \Delta$ such that h_1 $M \times (-\infty, b+d'] \times C$ into $M \times (\infty, a) \times C$. Applying $P_{D,d'}$, we can choose a PL isotopy h'_t supported in $M \times (a-d-d', b+d') \times \Delta$ such that h'_1 carries $M \times (\infty, b] \times D$ into $M \times (\infty, a-d) \times D$. We claim that $h''_t = h_t \circ h'_t$ is an isotopy which verifies the conditions of $P_{C \cup D, d+d}$. ☐

Remark 4. In assertion $P_{C,d}$, we can assume that the isotopy h_t is supported in a compact subset $p^{-1}(U)$ for any fixed open neighborhood U of C: to achieve this, choose a PL function χ such that $\chi = 1$ on C and χ is supported in a compact subset of U, and replace $h_t(m, r, v)$ by $h_{\chi(v)t}(m, r, v)$.

It follows that if C is a union of closed subsets C_i with disjoint open neighborhoods U_i and $P_{C_i,d}$ holds for each i, then $P_{C,d}$ holds: we can define an isotopy $h_t(m, r, v)$ by the formula

$$h_t(m, r, v) = \begin{cases} h^i_t(m, r, v) & \text{if } v \in U_i \\ (m, r, v) & \text{otherwise,} \end{cases}$$

where each h^i_t is an isotopy verifying $P_{C_i,d}$ supported in $p^{-1}U_i$.

We will prove the following:

Lemma 5. *For point $v \in \Delta$, there is a closed neighborhood C of v such that $P_{C,1}$ holds.*

Assuming Lemma 5 for a moment, we can complete the proof. Note that the simplex Δ is homeomorphic to a cube $[0,1]^n$ for some integer n. Fix $k > 0$, and decompose this cube into smaller cubes

$$C_{i_1, i_2, \ldots, i_n} = \prod_{1 \leq j \leq n} [\frac{i_j}{k}, \frac{i_j + 1}{k}]$$

where $0 \leq i_1, \ldots, i_n < k$. For $k \gg 0$, Lemma 5 guarantees that for each cube $C = C_{i_1, \ldots, i_n}$, condition $P_{C,1}$ is satisfied. For any sequence of bits $b_1, \ldots, b_n \in \{0, 1\}$, let

$$C'_{b_1, \ldots, b_n} = \bigcup C_{i_1, \ldots, i_n}$$

where the union is taken over all sequences $(i_1, \ldots, i_n)$ such that i_j is congruent to b_j modulo 2 for each j. Applying Remark 4, we deduce that $P_{C',1}$ holds for each of the closed subsets $C' = C'_{b_1, \ldots, b_n}$. Applying Lemma 3, we deduce that $P_{\Delta, 2^n}$ holds, and the proof is complete.

It remains to prove Lemma 5. Since there are only finitely many pairs of integers a, b such that $-N \leq a - 1 \leq b + 1 \leq N$ (and since an finite intersection of closed neighborhoods of v is again a closed neighborhood of v), it will suffice to prove the existence of an isotopy h_t as in assertion $P_{C,1}$ for each pair (a, b) satisfying $-N \leq a - 1 \leq b + 1 \leq N$. We do this in a sequence of steps:

(1) Suppose first that Δ consists of a single point, and that $M \times \mathbb{R}$ is given the product PL structure (for some fixed Whitehead compatible triangulation on M). Then the existence of the desired isotopies is obvious: we can take $h_t(m, r) = (m, f_t(r))$, where $f_t : \mathbb{R} \to \mathbb{R}$ is a PL isotopy supported in a compact subset of $(a - 1, b + 1)$ which carries $(-\infty, b]$ into $(-\infty, a)$.

(2) Suppose again that Δ consists of a single point, but that the PL structure on $M \times \mathbb{R}$ is arbitrary. Choose a PD homeomorphism $f : K \to M$, and endow $K \times \mathbb{R}$ with the product PL structure. Our uniqueness result from Lecture 5 asserts that there exists a PL homeomorphism $g : K \times \mathbb{R} \to M \times \mathbb{R}$ which is arbitrarily close to the map $f \times \mathrm{id}_{\mathbb{R}}$. Step (1) shows the existence of a PL isotopy h_t of $K \times \mathbb{R}$ with the desired properties. We define a PL isotopy h'_t of $M \times \mathbb{R}$ by the formula $h'_t = g \circ h_t \circ g^{-1}$. It is easy to see that if g is close enough to $f \times \mathrm{id}_{\mathbb{R}}$, then h'_t will satisfy the requirements of $P_{\Delta,1}$.

(3) We now suppose that Δ is arbitrary. Since $M \times [a-1, b+1] \times \{v\}$ is a compact subset of the fiber $p^{-1}\{v\}$, it is contained in a finite polyhedron. Since p is a submersion, the results of the previous lecture show that $M \times [a-1, b+1] \times \{v\}$ has an open neighborhood which is PL homeomorphic to $U \times V$, where $U \subseteq M \times \mathbb{R}$ and $V \subseteq \Delta$ are open subsets containing $M \times [a-1, b+1]$ and v, respectively. Let h'_t be the isotopy constructed in (2). Since h'_t is supported in a compact subset K of $M \times (a-1, b+1)$, it restricts to an isotopy of U and therefore defines an isotopy h''_t of $U \times V$. Choose a compact neighborhood K' of v in V such that the map

$$K \times K' \to U \times V \to M \times \mathbb{R} \times \Delta \to \mathbb{R}$$

has image contained in a compact subset of $(a-1, b+1)$. Let $\chi : \Delta \to [0, 1]$ be a PL map such that $\chi = 0$ outside of K and $\chi = 1$ in a neighborhood of v, and define an isotopy k_t by the formula

$$k_t(m, r, v) = \begin{cases} h''_{\chi(v)t}(m, r, v) & \text{if } (m, r, v) \in U \times V \\ (m, r, v) & \text{otherwise.} \end{cases}$$

Then k_t is an isotopy of $M \times \mathbb{R} \times \Delta$ which is supported in a compact subset of $M \times (a-1, b+1) \times \Delta$, with $k_0 = \mathrm{id}$. We observe that k_1 carries $M \times (-\infty, b] \times \{v\}$ into $M \times (-\infty, a) \times \Delta$. It therefore does the same for $M \times (-\infty, b] \times C$ where C is any sufficiently small neighborhood of v, which completes the proof.

Smoothing PL Fiber Bundles (Lecture 10)

February 25, 2009

Recall our assertion:

Theorem 1. *Suppose given a commutative diagram*

$$\begin{array}{ccc} K & \xrightarrow{f} & M \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ L & \longrightarrow & N \end{array}$$

where K and L are polyhedra, M and N are smooth manifolds, and the horizontal maps are PD homeomorphisms. Assume that p is a submersion of smooth manifolds (so that q is a submersion of PL manifolds). Then p is a smooth fiber bundle if and only if q is a PL fiber bundle.

In the last two lectures, we proved the "only if" direction. However, almost exactly the same argument can be used to prove the converse. The only step that really changes is the step in which we were forced to actually construct an isotopy. Consequently, Theorem 1 is a consequence of the following:

Lemma 2. *Let M be a compact PL manifold, and suppose that $M \times \mathbb{R}$ is equipped with a compatible smooth structure. Then, for every pair of integers $a \leq b$, there exists a smooth isotopy h_t of $M \times \mathbb{R}$ supported on a compact subset of $M \times (a-1, b+1)$ such that h_1 $M \times (-\infty, b]$ into $M \times (-\infty, a)$.*

Choose PL homeomorphism of $(a-1, b+1)$ with $\mathbb{R}$ which carries a to 0 and b to 1. Then we are reduced to proving the following:

Lemma 3. *Let M be a compact PL manifold and suppose that $M \times \mathbb{R}$ is equipped with a compatible smooth structure. Then there exists a compactly supported smooth isotopy h_t of $M \times \mathbb{R}$ such that h_1 carries $M \times (-\infty, 1]$ into $M \times (-\infty, 0)$.*

To prove Lemma 3, let us consider the following condition on a pair of closed subpolyhedra $K \subseteq L \subseteq M \times \mathbb{R}$:

$(P_{K,L})$ For every open neighborhood U of K, there exists a compactly supported smooth isotopy h_t of $M \times \mathbb{R}$ such that $h_1(L) \subseteq U$.

Since isotopies can be concatenated, it is easy to see that conditions $P_{K,K'}$ and $P_{K',K''}$ imply $P_{K,K''}$. Moreover, Lemma 3 will follow if we can prove $P_{M\times(-\infty,-1],M\times(-\infty,1]}$. For this, we need to recall a bit of terminology from the theory of PL topology.

Definition 4. Let L be a polyhedron equipped with a triangulation. We say that a subpolyhedron $K \subseteq L$ is an *elementary collapse* of L if there exists a simplex σ of L with a face $\sigma_0 \subset \sigma$ having the following properties:

(i) The simplex σ is not contained as a face of any other simplex of L.

(*ii*) The simplex σ_0 is not contained as a face of any other simplex of L other than σ.

(*iii*) The polyhedron K is obtained from L by removing the interiors of σ and σ_0.

We say that K is a *collapse* of L if it can be obtained from L by a sequence of elementary collapses.

It turns out that the property that $K \subseteq L$ is a collapse does not depend strongly on the choice of a triangulation of L. More precisely, if K is a collapse of L with respect to one triangulation S of L, then K is also a collapse with respect to any sufficiently fine refinement of S. Consequently, we can define the notion of K being a collapse of L without mentioning a particular triangulation: it means that K is a collapse of L with respect to *some* triangulation of L.

The following assertions are not difficult to verify:

- The polyhedron $(-\infty, -1]$ is a collapse (in fact an elementary collapse) of $(-\infty, 1]$.
- If A is a collapse of B, then $M \times A$ is a collapse of $M \times B$, for any polyhedron M.

Combining these observations, we conclude that $M \times (-\infty, -1]$ is a collapse of $M \times (-\infty, 1]$. It follows that there exists a triangulation S of $M \times \mathbb{R}$ which contains $M \times (-\infty, 1]$ and $M \times (-\infty, -1]$ as subcomplexes such that each simplex of S is smoothly embedded in $M \times \mathbb{R}$, and such that $M \times (-\infty, -1]$ can be obtained from $M \times (-\infty, 1]$ by a finite sequence of elementary collapses. It will therefore suffice to prove the following:

Lemma 5. *Suppose that $K \subseteq L \subseteq M \times \mathbb{R}$, where K is obtained from L by an elementary collapse with respect to a simplex σ and a face σ_0 which are smoothly embedded in $M \times \mathbb{R}$. Then for every open set U containing K, there exists a smooth compactly supported isotopy h_t such that $h_1(L) \subseteq U$.*

The construction of this isotopy is now a local matter: we can choose it to be supported in a small tubular neighborhood of the smoothly embedded simplex σ (which is diffeomorphic to an open ball and therefore well-understood. We leave the details to the reader.

We have seen that the "only if" direction of Theorem 1 is a crucial step toward our understanding of the classification of PL structures on a given smooth manifold. Similarly, the "if" direction of Theorem 1 plays a vital role in understanding smooth structures on a given PL manifold. It is to this topic that we now turn.

The main result that we are heading toward is that the problem of smoothing a PL manifold is governed by an *h-principle*: that is, it can be reduced to a problem of homotopy theory. Roughly speaking, we would like to say that there is a space of smooth structures on M, which can be described as the space of sections of a fibration $E \to M$ such that the fiber E_x over a point $x \in M$ describes smooth structures on M near the point x.

To make this more precise, we would like to have a good understanding of a small neighborhoods of x in M, and how they depend on the choice of x. In the case where M is smooth, the theory of vector bundles provides such an understanding. Namely, there exists a vector bundle T_M on M (the tangent bundle) whose fiber at a point $x \in M$ is diffeomorphic to a small neighborhood of x in M. This diffeomorphism can be chosen canonically, for example, if a Riemannian metric on M has been specified. We would like to have a replacement for the theory of vector bundles in the piecewise linear setting. Milnor's theory of *microbundles* provides such a replacement.

Definition 6 (Milnor)**.** Let X be a topological space. An *topological microbundle* on X (of rank n) is a map $p : E \to X$ equipped with a section $s : X \to E$ satisfying the following condition:

(∗) For every point $x \in X$, there exists a neighborhood of $U \subseteq X$ containing x and an open subset of E homeomorphic to $U \times \mathbb{R}^n$, such that the section s can be identified with the zero section $U \simeq U \times \{0\} \hookrightarrow U \times \mathbb{R}^n$.

An *equivalence* of microbundles E and E' over X is a homeomorphism $h : U \simeq U'$ fitting into a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & U' \\ & \searrow \quad \swarrow & \\ & X, & \end{array}$$

where U is an open subset of E containing the image of the section $s : X \to E$, U' is an open subset of E' containing the image of $s' : X \to E'$, and the map $h \circ s = s'$.

Remark 7. There are similar definitions in the smooth and PL categories. For example, in the PL case we modify Definition 6 by requiring E and X to be polyhedra and all of the relevant maps to be piecewise linear. In the smooth case, we require E and X to be smooth manifolds and all of the homeomorphisms to be diffeomorphisms.

Example 8. Let M be a topological (PL, smooth) manifold. The *tangent microbundle* T_M is defined to be the product $M \times M$, mapping to M via the projection $\pi_1 : M \times M \to M$, with section $s : M \to M \times M$ given by the diagonal map.

Example 9. Let ζ be a (smooth) vector bundle over a (smooth) manifold M. Then the map $\zeta \to M$ is a (smooth) microbundle.

In the smooth case, the converse is true as well. Namely, suppose that $p : E \to M$ is a smooth microbundle. Replacing E by a small open neighborhood of $s(M)$, we can assume that p is a submersion of smooth manifolds, so that p has a relative tangent bundle $T_{E/M}$. The pullback $s^*T_{E/M}$ is then a smooth vector bundle over M, which can itself be regarded as a microbundle over M. In fact, this microbundle is equivalent to E: choosing a Riemannian metric on E allows us to define an "exponential spray" which identifies an open subset of $s^*T_{E/M}$ with an open subset of E containing $s(M)$.

This construction shows that the theory of microbundles is equivalent to the theory of vector bundles in the setting of smooth manifolds.

We will take up the theory of microbundles again in the next lecture.

Microbundles (Lecture 11)

February 27, 2009

In this lecture, we will continue our study of microbundles. Recall that a microbundle over X is a map $p : E \to X$ equipped with a section $s : X \to E$. We will sometimes abuse terminology and simply refer to $p : E \to X$ or just the space E as a microbundle.

Remark 1. Let $E \to X$ be a topological (PL, smooth) microbundle, and let $f : X' \to X$ be a continuous (PL, smooth) map. Then the pullback $X' \times_X E \to X'$ is a microbundle over X', which we will denote by f^*E.

Our goal for this lecture is to prove the following:

Theorem 2. *Let $f, f' : X \to X'$ be a pair of continuous maps between topological spaces, and let E be a microbundle over X'. If X is paracompact and the maps f and f' are homotopic, then the microbundles f^*E and ${f'}^*E$ are equivalent.*

Remark 3. We have stated Theorem 2 in the topological setting, but it has obvious analogues in the smooth and PL settings. These can be proven using the same arguments given below; we will stick to the topological case just to save words.

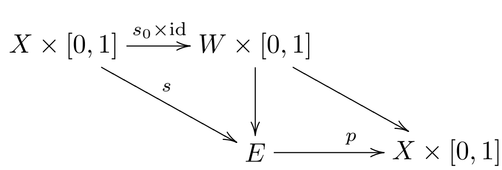
Corollary 4. *We say that a microbundle $E \to X$ is trivial if it is equivalent to a product $\mathbb{R}^n \times X$. If X is paracompact and contractible, then every microbundle over X is trivial.*

Proof. The identity map id_X is homotopic to a constant map $c : X \to X$ taking values at some point $x \in X$, so any microbundle E on X is equivalent to $c^*E = X \times E_x$. Since E is a microbundle, the fiber E_x has an open subset homeomorphic to $\mathbb{R}^n$ (containing $s(x)$). □

We now turn to the proof of Theorem 2. A homotopy between a pair of maps $f, f' : X \to X'$ is a map $h : X \times [0,1] \to X'$. To prove that $f^* \simeq {f'}^*E$, it will suffice to show that $h^*E \simeq \pi^*E_0$, where E_0 is a microbundle on X and $\pi : X \times [0,1] \to X$ is the projection map. We may therefore reformulate Theorem 2 as follows:

Proposition 5. *Let X be a paracompact space and let $E \to X \times [0,1]$ be a microbundle. Then there exists an equivalence of E with $E_0 \times [0,1]$, where E_0 denotes the fiber $E \times_{[0,1]} \{0\}$.*

In other words, there exists an open subset W of E_0 (containing the image of the section $s_0 : X \to E_0$) and an open embedding $W \times [0,1] \to E$ such that the diagram



is commutative.

We first treat the case where X is a point. In this case, E is a microbundle over the interval $[0,1]$ and we wish to prove that E is trivial. For each $x \in [0,1]$, there exists an open subset of E homeomorphic to a product $U \times V$, where V is a neighborhood of x in $[0,1]$ and $U \times V$ contains the image of the section s. Since $[0,1]$ is compact, we can cover $[0,1]$ by finitely many of the neighborhoods V. It follows that there exists an integer $N \gg 0$ and open embeddings

$$h(i) : U_i \times [\frac{i-1}{N}, \frac{i}{N}] \hookrightarrow E$$

for $1 \leq i \leq N$, where U_i is a space containing a base point $*$ and $h(i)$ carries $(*, t)$ to $s(t)$ for $t \in [\frac{i-1}{N}, \frac{i}{N}]$. We can think of each $h(i)$ as a family of open embeddings $h(i)_t : U_i \to E_t$, parametrized by $t \in [\frac{i-1}{N}, \frac{i}{N}]$. Using decreasing induction on $i < N$, we can assume (after shrinking U_i) that the map $h(i)_{\frac{i}{N}}(U_i) \subseteq h(i+1)_{\frac{i}{N}}(U_{i+1})$. We can then define a single map $f : U_0 \times [0,1] \to E$ by the following formula:

$$g(u,t) = h(i)_t h(i)^{-1}_{\frac{i-1}{N}} h(i-1)_{\frac{i-1}{N}} h(i-2)^{-1}_{\frac{i-2}{N}} \ldots h(1)_{\frac{1}{N}}(u)$$

where $\frac{i-1}{N} \leq t \leq \frac{i}{N}$. It is easy to see that g determines a trivialization of the microbundle E.

Now consider a general topological space X, and let $x \in X$ be a point. We can repeat the above argument to find an integer N and a finite sequence of open embeddings

$$h(i) : U_{i,x} \times [\frac{i-1}{N}, \frac{i}{N}] \times V_i \hookrightarrow E$$

where V_i is a sequence of open neighborhoods of x in X. Replacing each V_i by the intersection $V_x = \bigcap V_i$, we can assume that all of the open sets V_i are the same. After shrinking the open subsets $U_{i,x}$ as above, we can again define an open embedding $g_x : U_{0,x} \times V_x \times [0,1] \hookrightarrow E$ by setting

$$g_x(u,v,t) = h(i)_t h(i)^{-1}_{\frac{i-1}{N}} h(i-1)_{\frac{i-1}{N}} h(i-2)^{-1}_{\frac{i-2}{N}} \ldots h(1)_{\frac{1}{N}}(u,v)$$

where $\frac{i-1}{N} \leq t \leq \frac{i}{N}$. This open embedding determines a trivialization of the microbundle E on a neighborhood $[0,1] \times V_x$ of $[0,1] \times \{x\}$.

Since X is paracompact, we can choose a locally finite open covering $\{V_\alpha\}_{\alpha \in A}$ refining the covering $\{V_x\}_{x \in X}$ of X. For each $\alpha \in A$, choose a point $x \in X$ such that $V_\alpha \subseteq V_x$, let $W_\alpha = U_{0,x} \times V_x$ (which we identify with an open subset of E_0), let let $g_\alpha : W_\alpha \times [0,1] \to E$ be the restriction of g_x.

Each g_α determines an equivalence of E with the microbundle $\pi^* E_0$ over the open subset $W_\alpha \times [0,1] \subseteq X \times [0,1]$. We would like to "average" these equivalences to obtain a new equivalence G over all of $X \times [0,1]$. To this end, we choose choose a linear ordering on the set A and a partition of unity $\{\psi_\alpha : X \to [0,1]\}_{\alpha \in A}$ subordinate to the covering V_α. We attempt to define a map $G : E_0 \times [0,1] \to E$ as follows. Fix a point $e \in E_0$ lying over a point $x \in X$. Since the covering $\{V_\alpha\}$ is locally finite, x is contained V_α for only a finite number of indices $\alpha_1 < \alpha_2 < \ldots < \alpha_n$ of A. For each $t \in [0,1]$, choose an index i such that

$$\psi_{\alpha_1}(x) + \ldots + \psi_{\alpha_{i-1}}(x) \leq t \leq \psi_{\alpha_1}(x) + \ldots + \psi_{\alpha_i}(x)$$

and set

$$G(e,t) = g_{\alpha_i,t} g^{-1}_{\alpha_i, \psi_{\alpha_1}(x)+\ldots+\psi_{\alpha_{i-1}}(x)} g_{\alpha_{i-1}, \psi_{\alpha_1}(x)+\ldots+\psi_{\alpha_{i-1}}(x)} \cdots g_{\alpha_1, \psi_{\alpha_1}(x)}(e).$$

The map G is not everywhere defined, since the functions $g^{-1}_{\alpha,t}$ are defined only on open subsets of $E_t \times_X V_\alpha$ and the functions $g_{\alpha,t}$ are defined only on open subsets of $E_0 \times_X V_\alpha$. However, the composition is well-defined on the subset $s_0(X) \times [0,1] \subseteq E_0 \times [0,1]$, and therefore on an open neighborhood of this subset. Since $[0,1]$ is compact, we can choose this open neighborhood to be of the form $W \times [0,1]$, where W is an open subset of E_0 containing $s_0(X)$. Then $G : W \times [0,1] \to E$ is an open embedding which provides the desired equivalence $E \simeq \pi^* E_0$.

Classifying Spaces for Microbundles (Lecture 12)

March 1, 2009

In this lecture, we will discuss construct a classifying space for microbundles of rank n. For simplicity, we will restrict our attention to piecewise linear microbundles (since this will be the principal case of interest later). Up to this point, we have only defined the notion of a microbundle on a polyhedron K. In discussing microbundles, it is convenient to have a slightly more general definition.

Definition 1. Let $X_\bullet$ be a simplicial set. A *PL microbundle* (of rank n) on $X_\bullet$ consists of the following data:

(1) For every n-simplex $\sigma \in X_n$, a PL microbundle $E_\sigma \to \Delta^n$.

(2) For every nondecreasing map of linearly ordered sets $f : \{0, \ldots, m\} \to \{0, \ldots, n\}$ inducing a map $f^* : X_n \to X_m$ and every n-simplex σ in X_n, a PL isomorphism (not merely an equivalence) of microbundles $E_\sigma \times_{\Delta^n} \Delta^m \simeq E_{f^*\sigma}$.

(3) Given a pair of nondecreasing maps

$$\{0, \ldots, k\} \xrightarrow{g} \{0, \ldots, m\} \xrightarrow{f} \{0, \ldots, n\}$$

and an n-simplex $\sigma \in \Delta^n$, the associated diagram

$$\begin{array}{ccc} E_\sigma \times_{\Delta^n} \Delta^k & \longrightarrow & E_{f^*\sigma} \times_{\Delta^m} \Delta^k \\ & \searrow \quad \swarrow & \\ & E_{g^* f^* \sigma} & \end{array}$$

commutes.

There is a similar notion of an *equivalence* of microbundles on X: two microbundles on X are equivalent if they contain open submicrobundles with are isomorphic.

Remark 2. Let $f : X_\bullet \to Y_\bullet$ be a map of simplicial sets. If E is a PL microbundle on $Y_\bullet$, then we obtain a PL microbundle f^*E on $X_\bullet$, defined by the formula $(f^*E)_\sigma = E_{f(\sigma)}$.

Remark 3. Let $X_\bullet$ be a simplicial set with only finitely many nondegenerate simplices. Then the geometric realization $|X_\bullet|$ has the structure of a finite polyhedron. Unwinding the definition, we see that giving a PL microbundle E on $|X_\bullet|$ is equivalent to giving a PL microbundle on the simplicial set $X_\bullet$. Consequently, we can regard Definition 1 as a generalization of our earlier theory of PL microbundles (or at least a generalization of the theory of microbundles over finite polyhedra).

The main result of the previous lecture can be generalized to the present context: that is, every PL microbundle on $X_\bullet \times \Delta^1$ is equivalent to the pullback of a microbundle on $X_\bullet$.

Notation 4. Let $X_\bullet$ be a simplicial set. We let $M(X_\bullet)$ denote the set of equivalence classes of microbundles on $X_\bullet$.

The main result of this lecture is the following:

Theorem 5. *The functor $X_\bullet \to M(X_\bullet)$ is a representable functor on the homotopy category of simplicial sets. In other words, there exists a Kan complex $K_\bullet$ and a PL microbundle $\overline{E}$ on $K_\bullet$ with the following universal property: for every simplicial set $X_\bullet$, the construction*

$$(f : X_\bullet \to K_\bullet) \mapsto f^*\overline{E}$$

determines a bijection $\theta : [X_\bullet, K_\bullet] \to M(X_\bullet)$.

We first prove Theorem 5 by means of a specific construction.

Construction 6. *For each $n \geq 0$, let K_n denote the set of subpolyhedra $E \subseteq \Delta^n \times \mathbb{R}^\infty$ equipped with a map $s : \Delta^n \to E$ such that the pair $(E \to \Delta^n, s)$ is a PL microbundle over Δ^n. (recall that a subpolyhedron of $\Delta^n \times \mathbb{R}^\infty$ means a subpolyhedron of $\Delta^n \times V$, for some finite dimensional subspace $V \subseteq \mathbb{R}^\infty$).*

Our first step is to show that $K_\bullet$ is a Kan complex. In other words, we must show that every map $f : \Lambda^n_i \to K_\bullet$ can be extended to an n-simplex of $K_\bullet$. The map f classifies a PL microbundle $E \subseteq \Lambda^n_i \times \mathbb{R}^\infty$ over Λ^n_i (here we abuse notation by identifying a simplicial set with its geometric realization). We note that there is a PL retraction r from Δ^n onto Λ^n_i. Then r^*E is a PL microbundle over Δ^n equipped with an embedding $r^*E \hookrightarrow \Delta^n \times \mathbb{R}^\infty$.

By construction, the simplicial set $K_\bullet$ comes equipped with a tautological microbundle $\overline{E}$. This microbundle $\overline{E}$ gives a natural transformation of functors $\theta : [X_\bullet, K_\bullet] \to M(X_\bullet)$. Our next step is to show that θ is surjective for every simplicial set $X_\bullet$. In other words, we claim that every microbundle E on $X_\bullet$ is equivalent to $f^*\overline{E}$ for some map $X_\bullet \to K_\bullet$. We will prove something slightly stronger: every microbundle E on $X_\bullet$ is *isomorphic* to $f^*\overline{E}$ for some map $f : X_\bullet \to K_\bullet$. To prove this, we construct f one simplex at a time. At each stage, we are given a microbundle E_σ over the n-simplex Δ^n, and a PL embedding

$$i : E_\sigma \times_{\Delta^n} \partial\,\Delta^n \hookrightarrow \partial\,\Delta^n \times \mathbb{R}^\infty$$

(compatible with the projection to $\mathbb{R}^\infty$). To extend f over the simplex σ, we need to extend i to a PL embedding $E_\sigma \to \Delta^n \times \mathbb{R}^\infty$. The existence of this extension follows from general position argument.

We now prove the injectivity of θ. Suppose we are given two maps $f, f' : X_\bullet \to K_\bullet$ and equivalence of microbundles $f^*\overline{E} \simeq f'^*\overline{E}$. Then there exists a microbundle U on X and open embeddings $U \hookrightarrow f^*\overline{E}$ and $U \hookrightarrow f'^*\overline{E}$. We can then construct a microbundle E on $X_\bullet \times \Delta^1$ as a pushout

$$(f^*E \times [0, \frac{1}{2})) \coprod_{U \times [0, \frac{1}{2})} (U \times [0,1]) \coprod_{U \times (\frac{1}{2}, 1]} (f'^*E \times (\frac{1}{2}, 1])$$

We now construct a homotopy h from f to f' such that $h^*\overline{E}$ is isomorphic to the microbundle E on $X_\bullet \times \Delta^1$. The construction again proceeds one simplex σ at a time: at each stage, we are given a PL microbundle E_σ over $\Delta^n \times \Delta^1$ and an embedding

$$i : E_\sigma \times_{\Delta^n \times \Delta^1} \partial(\Delta^n \times \Delta^1) \hookrightarrow \partial(\Delta^n \times \Delta^1) \times \mathbb{R}^\infty,$$

and we wish to extend i to an embedding $E_\sigma \to (\Delta^n \times \Delta^1) \times \mathbb{R}^\infty$. The existence of the desired extension again follows from general position arguments. This completes the proof of Theorem 5.

The simplicial set $K_\bullet$ appearing in Theorem 5 is well-defined only up to homotopy equivalence. For some purposes it may be convenient to work with other models for the classifying space. It is therefore useful to have a criterion for determining whether or not a microbundle $\overline{E}$ on a simplicial set $K_\bullet$ satisfies the conclusions of Theorem 5.

Definition 7. We will say that that a microbundle E on a Kan complex $X_\bullet$ is *universal* if it satisfies the conclusions of Theorem 5.

Any microbundle E is classified by a map $f : X_\bullet \to K_\bullet$; we note that E is universal if and only if f is a homotopy equivalence. We can therefore adopt the following more general definition, which makes sense even when $X_\bullet$ is not a Kan complex: a microbundle E on $X_\bullet$ is *universal* if it is classified by a weak homotopy equivalence $f : X_\bullet \to K_\bullet$.

Remark 8. The simplicial set $K_\bullet$ is often denoted $BPL(n)$, for reasons which will become clear after the next lecture.

Remark 9. We can also consider classifying spaces for smooth or topological microbundles over simplicial sets. These admit classifying spaces $BTop(n)$ and $BSm(n)$. Since the theory of smooth microbundles is equivalent to the theory of vector bundles, we can take $BSm(n)$ to be a classifying space $BO(n)$ for the orthogonal group $O(n)$.

Remark 10. We can also consider what might be called "PD microbundles": that is, we can define a PD microbundle on a simplex Δ^n to consist of a smooth microbundle E over Δ^n, a PL microbundle E' over Δ^n, and a PD homeomorphism $E' \to E$ compatible with the projection to Δ^n, and a PD microbundle on $X_\bullet$ to be a compatible collection of PD microbundles over all simplices of $X_\bullet$. The above methods can be used to construct a classifying space for PD microbundles $BPD(n)$, equipped with forgetful maps

$$BO(n) \leftarrow BPD(n) \rightarrow BPL(n).$$

Our existence and uniqueness results for Whitehead compatible triangulations show that the left map is a homotopy equivalence (this is slightly easier than our results for manifolds, since we do not have guarantee that any maps are fiber bundles). This construction therefore yields a well-defined homotopy class of maps $BO(n) \to BPL(n)$.

Embeddings vs. Homeomorphisms (Lecture 13)

March 3, 2009

Our goal in this lecture is to carry out the main step in the proof of the Kister-Mazur theorem describing the relationship between microbundles and $\mathbb{R}^n$-bundles. Namely, we will prove the following:

Theorem 1. *Let* $\mathrm{Emb}(\mathbb{R}^n)$ *denote the simplicial set of open embeddings from* $\mathbb{R}^n$ *to itself (so a* k*-simplex of* $\mathrm{Emb}(\mathbb{R}^n)$ *is an open embedding* $j : \mathbb{R}^n \times \Delta^k \to \mathbb{R}^n \times \Delta^k$ *which commutes with the projection to* Δ^k*), and let* $\mathrm{Homeo}(\mathbb{R}^n) \subseteq \mathrm{Emb}(\mathbb{R}^n)$ *denote the simplicial subset of homeomorphisms from* $\mathbb{R}^n$ *to itself (so that a* k*-simplex of* $\mathrm{Homeo}(\mathbb{R}^n)$ *is a* k*-simplex of* $\mathrm{Emb}(\mathbb{R}^n)$ *for which the map* j *is a homeomorphism). Then the inclusion* $i : \mathrm{Homeo}(\mathbb{R}^n) \subseteq \mathrm{Emb}(\mathbb{R}^n)$ *is a homotopy equivalence of Kan complexes.*

Remark 2. We can also define topological spaces parametrizing homeomorphisms or open embeddings from $\mathbb{R}^n$ to itself: Theorem 1 is equivalent to the assertion that the inclusion between these topological spaces is a weak homotopy equivalence.

Remark 3. We can also define simplicial sets which parametrize PL embeddings and PL homeomorphisms from $\mathbb{R}^n$ to itself. Theorem 1 continues to hold in this case, using essentially the same proof that we will give below.

The main step in the proof of Theorem 1 is to establish that i is a surjection on π_0. In other words, every open embedding $f : \mathbb{R}^n \to \mathbb{R}^n$ is isotopic to a homeomorphism of $\mathbb{R}^n$ with itself. In fact, we will prove something more precise:

Proposition 4. *Let* f *be an open embedding from* $\mathbb{R}^n$ *to itself. Then there exists an isotopy* F_t *from* $f = F_0$ *to a homeomorphism* $f = F_1$*. Moreover, this isotopy is can be chosen to be constant on the unit ball* $B(1)$ *of* $\mathbb{R}^n$*.*

Notation 5. For every positive real number r, let $B(r) = \{x \in \mathbb{R}^n : |x| < r\}$ be the open ball of radius r around the origin (in giving the PL version of this proof, it is convenient to replace $B(r)$ by an open cube).

Here is the rough idea of the proof. The obstruction to an open embedding being a homeomorphism is that it might not be surjective. Our objective, therefore, is to use an isotopy to modify f so that its image becomes larger and larger. More precisely, we will construct a sequence of open embeddings

$$f^1, f^2, \ldots, : \mathbb{R}^n \to \mathbb{R}^n$$

and a sequence of isotopies $h^i : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ so that the following conditions are satisfied:

(1) The map $f^1 = f$.

(2) For each i, the map h^i is an isotopy from $f^i = h^i_0$ to $f^{i+1} = h^i_1$, which is constant on the open ball $B(i)$.

(3) For $i > 1$, we have $B(i) \subseteq f^i B(i)$.

Assuming that we can meet these requirements, we can define a homeomorphism $f' : \mathbb{R}^n \to \mathbb{R}^n$ by the formula $f'(x) = f^i(x)$ for any $i \geq |x|$. We get an isotopy from f to f' by concatenating the isotopies h^1, h^2, and so forth (this concatenation is well-defined since almost all of the isotopies h^i are constant on any given compact subset of $\mathbb{R}^n$).

To begin, we may assume without loss of generality that $f(0) = 0$ (otherwise, we can reduce to this case by conjugating by a relevant translation). Since f is an open embedding, the image $fB(1)$ contains an open ball $B(\epsilon)$ for some real number $\epsilon > 0$. Since f is continuous, there exists a positive real number $\delta < 1$ such that $f(B(\delta)) \subseteq B(\frac{\epsilon}{2})$.

To construct our isotopies h^i, we will need the following basic building blocks:

Notation 6. For every pair of real numbers $r < s$, we fix an isotopy $H(r,s)_t : \mathbb{R}^n \to \mathbb{R}^n$ from $\mathrm{id}_{\mathbb{R}^n}$ to $H(r,s)_1$ with the following properties:

(i) The isotopy $H(r,s)_t$ is trivial on $B(\frac{r}{2})$ and supported in a compact subset of $B(s+1)$.

(ii) The map $H(r,s)_1$ restricts to a homeomorphism $B(r)$ to $B(s)$.

We now proceed with the construction of the sequence $\{f^i\}$. Assume that f^i has already been constructed. We wish to construct an isotopy h^i from f^i to another map f^{i+1}, which is constant on $B(i)$. First, we define a homeomorhism c (for "contraction") from $\mathbb{R}^n$ to itself as follows:

$$c(x) = \begin{cases} x & \text{if } x \notin f^i(\mathbb{R}^n) \\ f^i(H(\delta, i)_1^{-1}(y)) & \text{if } x = f^i(y). \end{cases}$$

Since $f^i = f$ on $B(1)$ and f carries $B(\delta)$ into $B(\frac{\epsilon}{2})$, we deduce that $c(f^i(x)) \in B(\frac{\epsilon}{2})$ if $x \in B(i)$. Note that c is the identity outside a compact set, which we can take to be contained in $B(N_i)$ for some $N_i \gg i+1$.

We now define h^i_t by the formula

$$h^i_t = c^{-1} \circ H(\epsilon, N_i)_t \circ c \circ f^i.$$

It is clear that h^i_t is an isotopy from $f^i = h^i_0$ to another map $f^{i+1} = h^i_1$. Moreover, since $H(\epsilon, N_i)_t$ is the identity on $B(\frac{\epsilon}{2})$ and $c \circ f^i$ carries $B(i)$ into $B(\frac{\epsilon}{2})$, we deduce that h^i_t is constant on $B(i)$. It remains only to verify that $f^{i+1}B(i+1)$ contains $B(i+1)$. In fact, we claim that $f^{i+1}B(i+1)$ contains $B(N_i)$. Since c is supported in $B(N_i)$, it suffices to show that $(cf^{i+1})B(i+1) = (H(\epsilon, N_i)_1 \circ c \circ f^i)B(i)$ contains $B(N_i)$. For this, it suffices to show that $(c \circ f^i)B(i)$ contains $B(\epsilon) \subseteq fB(1) \subseteq f^iB(i+1)$. This is clear, since $H(\delta, i)_1$ induces a homeomorphism of $B(i+1)$ with itself. This completes the proof of Proposition 4.

Remark 7. In the above construction, each of the isotopies h^i is obtained by composing f^i with a 1-parameter family $c^{-1} \circ H(\epsilon, N_i)_t \circ c$ of homeomorphisms from $\mathbb{R}^n$ to itself. It follows that if the original map f is already a homeomorphism, then the isotopy F_t that we construct will be a path through the space of homeomorphisms.

Suppose now that we are given not a single open embedding $f : \mathbb{R}^n \to \mathbb{R}^n$, but a family of open embeddings $f : \mathbb{R}^n \times \Delta \to \mathbb{R}^n \times \Delta$ (compatible with the projection to Δ), where Δ is some parameter space. We might try to apply the above construction to each of the induced maps $\{f_v : \mathbb{R}^n \to \mathbb{R}^n\}_{v \in \Delta}$ to produce a family of isotopies $\{F_{v,t} : \mathbb{R}^n \to \mathbb{R}^n\}_{(v,t) \in \Delta \times [0,1]}$. We must be careful, since our construction depended on several choices. First of all, we needed to choose ϵ such that $f_vB(1)$ contains the open ball $B(\epsilon)$. We note that $f(B(1) \times \Delta)$ is an open neighborhood of $\{0\} \times \Delta$ in $\mathbb{R}^n \times \Delta$, which will contain some product neighborhood $B(\epsilon) \times \Delta$ provided that Δ is compact. We also needed to choose a constant δ such that $f_vB(\delta) \subseteq B(\frac{\epsilon}{2})$. Again, if Δ is compact, then a sufficiently small real number δ will work for all f_v's simultaneously. Finally, to constuct each h^i_v we needed to choose $N_i \gg i+1$, so that the relevant contraction c_v has compact support in $B(N_i)$. The support of c_v is contained in $f^i_vB(i+1)$. If Δ is compact, the image $f^i(B(i+1) \times \Delta)$ will be contained in a compact subset of $\mathbb{R}^n \times \Delta$, which is in turn contained in $B(N_i) \times \Delta$ for sufficiently large N_i. Consequently, we get the following more refined version of Proposition 4:

Proposition 8. *Let Δ be a compact topological space (for example, a simplex), and suppose we are given an open embedding $f : \mathbb{R}^n \times \Delta \to \mathbb{R}^n \times \Delta$ which is compatible with the projection to Δ. Then there exists an isotopy $F : \mathbb{R}^n \times \Delta \times [0,1] \to \mathbb{R}^n \times \Delta \times [0,1]$ with the following properties:*

(1) *The map F_0 coincides with f.*

(2) *The map F_1 is a homeomorphism.*

(3) *The isotopy F is constant along $B(1) \times \Delta$.*

(4) *If f_v is already a homeomorphism for some $v \in \Delta$, then the isotopy $F_v : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n \times [0,1]$ consists of homeomorphisms.*

We can now prove Theorem 1. The proof is based on the following criterion for detecting homotopy equivalences:

Proposition 9. *Let $i : K \subseteq K'$ be an inclusion of Kan complexes. Then i is a homotopy equivalence if and only if the following condition is satisfied:*

($*$) *For every n-simplex σ of K' whose boundary belongs to K, there exists a homotopy $h : \Delta^n \times \Delta^1 \to K'$ such that $h|\Delta^n \times \{0\} = \sigma$, $h|\Delta^n \times \{1\}$ factors through K, and $h|\partial \Delta^n \times \Delta^1$ factors through K.*

Roughly speaking, the simplex σ is a typical representative of a class in π_{n-1} of the homotopy fiber of the inclusion $K \to K'$, and condition ($*$) guarantees that any such class is trivial.

Theorem 1 follows immediately from Proposition 9 and Proposition 8.

In the next lecture, we will discuss the consequences of Theorem 1 for the classification of microbundles.

The Kister-Mazur Theorem (Lecture 14)

March 6, 2009

Our first goal in this lecture is to finish off the proof of the Kister-Mazur theorem, which guarantees that the theory of microbundles is equivalent to the theory of $\mathbb{R}^n$-bundles. We will work in the PL setting (where the result is due to Kuiper and Lashof). More precisely, we will prove the following:

Theorem 1. *Let X be a polyhedron and let $E \to X$ be a PL microbundle. Then there exists an open subset $U \subseteq E$ (containing the zero section) such that the projection $U \to X$ is a PL fiber bundle, with fiber $\mathbb{R}^n$.*

Remark 2. This theorem can be refined in various ways: for example, the fiber bundle U Is unique up to isomorphism. This can be proven using essentially the same arguments and is left as an exercise.

We first need the following result:

Lemma 3. *Let $E \to S^k$ be a PL fiber bundle with fiber $\mathbb{R}^n$ over the k-sphere. Suppose that E is trivial as a microbundle. Then E is trivial as a fiber bundle.*

Proof. We can decompose S^k into hemispheres H_+ and H_-. These are contractible, so we can choose trivializations $E \times_{S^k} H_+ \simeq \mathbb{R}^n \times H_+$ and $E \times_{S^k} H_- \simeq \mathbb{R}^n \times H^-$. These trivializations determine a family of homeomorphism $\{f_v : \mathbb{R}^n \to \mathbb{R}^n\}_{v \in S^{k-1}}$ by restricting to the equator S^{k-1} (in other words, a single PL homeomorphism $f : \mathbb{R}^n \times S^{k-1} \to \mathbb{R}^n \times S^{k-1}$ which commutes with the projectino to S^{k-1}). To prove that E is trivial, we must show that the family $\{f_v\}$ is isotopic to a constant family.

By assumption, E is trivial, so there exists an equivalence of microbundles $E \simeq \mathbb{R}^n \times S^k$. This gives an isomorphism of an open subset U of E with an open subset V of $\mathbb{R}^n \times S^k$. Shrinking U and V, we can assume that V has the form $B(\epsilon) \times S^k$, where $B(\epsilon)$ denotes the open box $(-\epsilon, \epsilon)^n$. Identifying $B(\epsilon)$ with $\mathbb{R}^n$, we get an open embedding $\mathbb{R}^n \times S^k \hookrightarrow E$. Over H_+, this gives us a family of open embeddings $\{g_v^+ : \mathbb{R}^n \to \mathbb{R}^n\}$. Over H_-, we get another family of embeddings $\{g_v^- : \mathbb{R}^n \to \mathbb{R}^n\}$. Along the equator, we have $g_v^+ = f_v \circ g_v^-$.

Since the families of embeddings g_v^- and g_v^+ are defined on the contractible sets H^+ and H^-, they are isotopic to constant families. Since every PL open embedding $\mathbb{R}^n \to \mathbb{R}^n$ is isotopic to a PL homeomorphism, we can take the constant values to be homeomorphisms g^+ and g^-. It follows that f_v is isotopic (through open embeddings) to $g^+ \circ (g^-)^{-1}$. Applying again the main result of last time, we conclude that f_v is isotopic through homeomorphisms to $g^+ \circ (g^-)^{-1}$, so that E is constant as desired. □

We now turn to the proof of Theorem 1. For every closed subpolyhedron $X_0 \in X$, let us say that an open subset $U_0 \subseteq E$ is *good near* X_0 if there exists an open neighborhood $V \subseteq X$ of X_0 such that U_0 is an $\mathbb{R}^n$-bundle over V. Fix a triangulation of X and write X as the union of an increasing sequence of compact subpolyhedra

$$\emptyset = X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$$

such that each X_i is obtained from X_{i-1} by adjoining a simplex whose boundary already belongs to X_{i-1}. We will prove that there exists a collection of open subsets $U_0, U_1, \dots \subseteq E$ with the following properties:

(a) The open set U_i is good near X_i.

(b) The open set U_{i+1} coincides with U_i over a neighborhood of X_i.

We will then obtain a proof of Theorem 1 by setting $U = \bigcup (U_i \times_X X_i)$.

We start the induction by setting $U_0 = \emptyset$. Assume that U_i has been defined, and let X_{i+1} be obtained from X_i by adjoining a single k-simplex σ. Let U_i be an $\mathbb{R}^n$ bundle over the neighborhood V of X_i. In particular, U_i determines an $\mathbb{R}^n$ bundle over $\partial\,\sigma$. This $\mathbb{R}^n$-bundle extends to a microbundle over the contractible space σ, and is therefore trivial as a microbundle. By Lemma 3, it is also trivial as an $\mathbb{R}^n$ bundle. It follows that there exists a compact neighborhood Z of $\partial\,\sigma$ contained in V, such that $U_i \times_X Z \to Z$ can be identified with the trivial bundle $Z \times \mathbb{R}^n$.

Let W be a contractible neighborhood of σ (for example, the star of σ), so that the microbundle E is trivial over E. As in the proof of Lemma 3, this means we can choose an open embedding $j : \mathbb{R}^n \times W \hookrightarrow E$. Choose $\epsilon > 0$ such that j carries $B(\epsilon) \times \partial\,\sigma$ into U_i. Shrinking ϵ and Z if necessary, we can assume that $j(B(\epsilon) \times Z) \subseteq U_i$. We can therefore think of j as providing a family of open embeddings $\{j_z : B(\epsilon) \to \mathbb{R}^n\}_{z \in Z}$. The main result of last time shows that there exists a family of isotopies $\{h_{z,t} : B(\epsilon) \to \mathbb{R}^n\}_{z \in Z}$ where $h_{z,0} = j_z$ and each $h_{z,1}$ is a PL homeomorphism.

Choose open subsets $V_0 \subseteq V$, $W_0 \subseteq W$ with the following properties:

(1) The union $V_0 \cup W_0$ contains X_{i+1}.

(2) The intersection $V_0 \cap W_0$ is contained in Z.

(3) The set W_0 is disjoint from X_i.

Choose a map $\chi : V_0 \cup W_0 \to [0, 1]$ such that $\chi = 1$ on a neighborhood of $(V_0 \cup W_0) - W_0$ and $\chi = 0$ on a neighborhood of $(V_0 \cup W_0) - V_0$. We now define U_{i+1} to be the open subset of E whose fiber over a point $x \in V_0 \cup W_0$ is defined as follows:

(a) If $\chi(x) = 1$, then the fiber of U_{i+1} over x is the fiber of U_i over x.

(b) If $\chi(x) = 0$, then the fiber of U_{i+1} over x is the image of j_x.

(c) If $x \in V_0 \cap W_0 \subseteq Z$, then the fiber of U_{i+1} over x is the image of $h_{x,\chi(x)}$.

It is not difficult to verify that this is a fiber bundle over $V_0 \cup W_0$ with the desired properties.

Remark 4. Using more elaborate reasoning of the same kind, we can show that the classifying space $BPL(n)$ for PL fiber bundles with fiber $\mathbb{R}^n$ is homotopy equivalent to the classifying space for PL microbundles constructed in Lecture 12. For this reason, the latter classifying space is typically denoted by $BPL(n)$. Analogous remarks apply in the smooth and topological setting. In the smooth case, microbundles are essentially the same as vector bundles, and the relevant classifying space is denoted by $BO(n)$.

Remark 5. Let E be a PL microbundle over a simplex Δ^n. Let us say that a *smoothing* of E is a smoothing of an open subset $U \subseteq E$ containing the zero section, so that the projection $U \to \Delta^n$ is submersive. We regard two smoothings as identical if they agree on a neighborhood of the zero section of E. Let $X_\bullet$ be the simplicial set whose n-simplices are pairs (σ, S), where σ is an n-simplex of $BPL(n)$ and S is a smoothing of the associated microbundle $E \to \Delta^n$. There is an evident forgetful map $f : X_\bullet \to BPL(n)$.

We also have a canonical vector bundle ζ over the simplicial set $X_\bullet$: it assigns to each simplex (σ, S) the vector bundle $\zeta_\sigma \to \Delta^n$ obtained by taking the vertical tangent space to E along the zero section (the tangent space is defined using the smoothing S). This vector bundle is classified by a map $X_\bullet \to BO(n)$. We will see in the next lecture that ζ is universal: that is, the classifying map χ is a homotopy equivalence. We can therefore identify $X_\bullet$ itself with a classifying space $BO(n)$ for vector bundles of rank n, and f with a map $BO(n) \to BPL(n)$. Informally, we think of this as coming from a group homomorphism $O(n) \to PL(n)$. (In fact, we do have an evident morphism from $O(n)$ to $PL(n)$ as discrete groups: every orthogonal transformation of $\mathbb{R}^n$ is in particular a piecewise linear homeomorphism.) The fiber of f is often denoted $PL(n)/O(n)$; it can be thought of as the space of all smoothings of the PL manifold $\mathbb{R}^n$.

The main result of the last lecture has another consequence:

Proposition 6. *Let $f : D^n \to \mathbb{R}^n$ be a tame embedding (in other words, an embedding such that $f(S^{n-1})$ admits a bicollar in $\mathbb{R}^n$). Then there is a homeomorphism of $\mathbb{R}^n$ to itself that carries $f(D^n)$ to the standard disk D^n.*

Proof. Let us identify D^n with the closure of the open box $B(1)$. Choosing an "outer collar" of $f(S^{n-1})$, we obtain an open embedding $f_0 : B(1 + \epsilon) \to \mathbb{R}^n$. The main result of the last lecture shows that f_0 is isotopic to a homeomorphism f_1, via an isotopy fixed on $B(1 + \frac{\epsilon}{2})$. Then f_1^{-1} carries $f_0(D^n) = f_1(D^n)$ to the standard disk. □

Smoothings and Microbundles (Lecture 15)

March 11, 2009

We now return to the problem of smoothing piecewise linear manifolds. Recall the diagram

$$\mathrm{Man}^m_{PL} \xrightarrow{\theta} \mathrm{Man}^m_{PD} \xrightarrow{\theta'} \mathrm{Man}^m_{\mathrm{sm}}$$

of Lecture 6. We have shown that θ' is a trivial Kan fibration, so that we can also regard Man^m_{PD} as a classifying space for smooth manifolds. Then we can regard θ as assigning to each smooth manifold an underlying PL manifold. The fiber of θ over a vertex of Man^m_{PL} corresponding to a PL manifold $M \subseteq \mathbb{R}^\infty$ can be viewed as a "space" of smooth structures on M. The following guarantees that these "spaces" of smooth structures are well-behaved:

Lemma 1. *The map θ is a Kan fibration.*

In fact, we will factor θ in two steps. Let $\mathrm{Man}^m_{PD'}$ denote the simplicial set whose k-simplices are fiber bundles of PL manifolds $E \to \Delta^k$ where $E \subseteq \Delta^k \times \mathbb{R}^\infty$, together with a Whitehead compatible smooth structure on E such that the map $E \to \Delta^k$ is a submersion (and therefore a fiber bundle) in the smooth category. This differs only slightly from our definition of Man^m_{PD}, in that we do not require an additional smooth embedding of E into $\Delta^k \times \mathbb{R}^\infty$. By general position arguments, this difference is immaterial: the map $\mathrm{Man}^m_{PD} \to \mathrm{Man}^m_{PD'}$ is a trivial Kan fibration. Consequently, it suffices to prove the following analogue of Lemma 1:

Lemma 2. *The map $\mathrm{Man}^m_{PD'} \to \mathrm{Man}^m_{PL}$ is a Kan fibration.*

Proof. We must show that we can solve lifting problems of the form

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \mathrm{Man}^m_{PD'} \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & \mathrm{Man}^m_{PL}. \end{array}$$

In more concrete terms: we are given a bundle of PL manifolds $K \subseteq \Delta^n \times \mathbb{R}^\infty$, and a PD homeomorphism of the subbundle $K_0 = K \times_{\Delta^n} \Lambda^n_i$ with a smooth fiber bundle $M_0 \to \Lambda^n_i$. We need to construct the following:

(1) A fiber bundle $M \to \Delta^n$ of smooth manifolds extending the given bundle $M_0 \to \Lambda^n_i$.

(2) A PD homeomorphism $K \to M$ which commutes with the projection to Δ^n.

To satisfy (1), we observe that Λ^n_i is trivial, so we can write M_0 as a product $\Lambda^n_i \times N$ for some smooth manifold N. We then define $M = \Delta^n \times N$. To construct (2), we observe that Δ^n is PL homeomorphic to $\Lambda^n_i \times \Delta^1$. We can lift this to a PL homeomorphism $K \simeq K_0 \times \Delta^1$. We now have a unique map $K \to \Delta^n \times N$ which commutes with the projection to Δ^n, and such that the map $K \to N$ is given by the composition

$$K \simeq K_0 \times \Delta^1 \to K_0 \to M_0 \simeq N \times \Lambda^n_i \to N.$$

It is easy to see that this map is a PD homeomorphism. □

Notation 3. Given a PL manifold M (which we implicitly assume to be given as a polyhedron in $\mathbb{R}^\infty$, so that it defines a vertex of Man^m_{PL}), we let $\mathrm{Smooth}(M)$ denote the fiber of the Kan fibration $\mathrm{Man}^m_{PD'} \to \mathrm{Man}^m_{PL}$ over M. The vertices of $\mathrm{Smooth}(M)$ are smooth structures on M which are Whitehead compatible with the given PL structure on M.

The theory of microbundles allows us to set up a local version of the same story. Namely, let $BPL(m)$ denote the classifying space (= simplicial set) for PL microbundles of rank m constructed in Lecture 12: an n-simplex of $BPL(m)$ is a microbundle $E \to \Delta^n$, where E is given as a subpolyhedron of $\Delta^n \times \mathbb{R}^\infty$. (The Kister-Mazur theorem, in its PL incarnation, allows us to identify this space with the classifying space of a simplicial group $PL(m)$).

Definition 4. Let E be a PL microbundle over a simplex Δ^n. Let us say that a *smoothing* of E is a smoothing of an open subset $U \subseteq E$ containing the zero section, so that the projection $U \to \Delta^n$ is submersive. We regard two smoothings as identical if they agree on a neighborhood of the zero section of E. Let $X_\bullet$ be the simplicial set whose n-simplices are pairs (σ, S), where σ is an n-simplex of $BPL(m)$ and S is a smoothing of the associated microbundle $E \to \Delta^n$. There is an evident forgetful map $f : X_\bullet \to BPL(m)$.

We can regard the map f as a "local version" of the Kan fibration $\theta : \mathrm{Man}^m_{PD} \to \mathrm{Man}^m_{PL}$. A slight modification of the proof of Lemma 2 shows that f is also a Kan fibration.

Lemma 5. *The vector bundle ζ over $X_\bullet$ constructed above is universal: that is, it exhibits $X_\bullet$ as a classifying space for vector bundles of rank m.*

Proof. By an argument which should be familiar from previous lectures, it will suffice to prove the following: given a map $\chi_0 : \partial\,\Delta^n \to X_\bullet$ and a vector bundle ζ' over Δ^n with an isomorphism $\alpha_0 : \zeta' |\, \partial\,\Delta^n \simeq \chi_0^* \zeta$, we can extend χ_0 to a map $\chi : \Delta^n \to X_\bullet$ and α to an isomorphism $\zeta' \simeq \chi^* \zeta$.

Since Δ^n is contractible, we can assume that ζ' is a trivial bundle of rank m. The map χ_0 classifies a PL microbundle $E_0 \to \partial\,\Delta^n$ (together with an embedding $E_0 \hookrightarrow \partial\,\Delta^n \times \mathbb{R}^\infty$), and a smoothing S of a neighborhood U_0 of the zero section of E_0. The map α_0 gives a trivialization of vertical tangent space to U_0 along the zero section. As we have seen, this is equivalent to trivializing U_0 as a smooth microbundle. We may therefore assume, after shrinking U, that $U_0 \simeq \partial\,\Delta^n \times \mathbb{R}^m$ as a smooth fiber bundle over $\partial\,\Delta^n$.

We wish to show that we can extend E_0 to a PL microbundle $E \to \Delta^n$ (which we can then embed in $\Delta^n \times \mathbb{R}^\infty$ using general position arguments) and U_0 to an open subset $U \subseteq E$ containing the zero section, equipped with a PD homeomorphism $U \to \mathbb{R}^m \times \Delta^n$. To construct this, choose a finite polyhedral neighborhood V of $\partial\,\Delta^n$ in Δ^n for which there exists a retraction $r : V \to \Delta^n$. Let V_0 denote the interior of V, and let r_0 be the restriction of r to V_0, and let $\partial\,V = V - V_0$. Let E denote the pushout

$$(r_0^* E_0) \coprod_{r_0^* U_0} (\Delta^n \times \mathbb{R}^\infty)$$

Over V, this set is equipped with a natural polyhedral structure by identifying it with an open subset of $r^* E_0$. In particular, we get a PL structure on $E \times_{\Delta^n} \partial\,V \simeq (\partial\,V) \times \mathbb{R}^m$ which is Whitehead compatible with the smooth structure on $\mathbb{R}^m$. We now simply extend this to a triangulation of the smooth fiber bundle

$$E \times_{\Delta^n} (\Delta^n - V_0) \simeq \mathbb{R}^m \times (\Delta^n - V_0) \to \Delta^n - V_0$$

to obtain the desired PL microbundle E. □

Since $X_\bullet$ is classifying space for vector bundles, we will denote it by $BO(m)$: it is homotopy equivalent to any other model for the classifying space $BO(m)$ (for example, one constructed using the singular complex of the topological group $O(m)$). By construction, we have a Kan fibration $\theta_0 : BO(m) \to BPL(m)$. Informally, we think of this as coming from a group homomorphism $O(n) \to PL(n)$. (In fact, we do have an evident morphism from $O(n)$ to $PL(n)$ as discrete groups: every orthogonal transformation of $\mathbb{R}^n$ is in particular a piecewise linear homeomorphism.) The fiber of f is often denoted $PL(n)/O(n)$; it can be thought of as the space of all smoothings of the PL manifold $\mathbb{R}^n$.

Let us adopt the following convention: if M is a polyhedron and $Y_\bullet$ is a simplicial set, then a *map* from M into $Y_\bullet$ means a map of simplicial sets from the PL singular complex $\mathrm{Sing}_\bullet^{PL} M$ into $Y_\bullet$. The collection of all such maps can itself be organized into a simplicial set which we will denote by $Y_\bullet^M$.

If M is a PL manifold of dimension m, then there is a natural map $\chi : M \to BPL(m)$: namely, it assigns to each n-simplex $\sigma : \Delta^n \to M$ the product $M \times \Delta^n$, regarded as a PL microbundle over Δ^n with the section supplied by σ. Any smoothing of M determines a smoothing of this PL microbundle: in other words, it allows us to produce a lifting

$$\begin{array}{ccc} & & BO(m) \\ & \nearrow & \downarrow \\ M & \longrightarrow & BPL(m). \end{array}$$

Our goal in the next few lectures is to prove the converse. More precisely, we will show the following:

Theorem 6. *Let M be a PL manifold. The above construction determines a homotopy equivalence from the simplicial set* $\mathrm{Smooth}(M)$ *of smooth structures on M to the simplicial set*

$$BO(m)^M \times_{BPL(m)^M} \{\chi\}$$

of liftings of χ. In particular, M admits a smoothing if and only if there exists a commutative diagram

$$\begin{array}{ccc} & & BO(m) \\ & \overset{L}{\nearrow} & \downarrow \\ M & \xrightarrow{\chi} & BPL(m). \end{array}$$

The virtue of this result is that it reduces the classification of smooth structures on M to a problem of homotopy theory. The existence of the arrow L can in principle be attacked by methods of obstruction theory. Namely, consider the fiber of the Kan fibration $BO(m)- > BPL(m)$, which we will suggestively denote by $PL(m)/O(m)$ (it can be thought of as the space of all smooth structures on the trivial PL microbundle $\mathbb{R}^m \to *$). Obstruction theory tells us that L will exist provided that a sequence of cohomology classes $\mathrm{H}^k(M; \pi_{k-1} PL(m)/O(m))$ vanish Similarly, the uniqueness of L can be studied by computing cohomology groups of the form $\mathrm{H}^k(M; \pi_k PL(m)/O(m))$. In particular, if the homotopy groups of $PL(m)/O(m)$ vanish, then M admits an essentially unique smooth structure. This is what happens for $m \leq 3$, as we will see later.

Flexibility (Lecture 16)

March 11, 2009

Recall that our goal is to prove the following result:

Theorem 1. *Let M be a PL manifold. The above construction determines a homotopy equivalence from the simplicial set* $\mathrm{Smooth}(M)$ *of smooth structures on M to the simplicial set*

$$BO(m)^M \times_{BPL(m)^M} \{\chi\}$$

of liftings of χ. In particular, M admits a smoothing if and only if there exists a commutative diagram

$$\begin{array}{ccc} & & BO(m) \\ & \overset{L}{\nearrow} & \downarrow \\ M & \xrightarrow{\chi} & BPL(m). \end{array}$$

To prove Theorem 1, it will be convenient to formulate a more local version. For every open subset $U \subseteq M$, let $\mathrm{Smooth}(U)$ denote the simplicial set of smooth structures on U. The assignment $U \mapsto \mathrm{Smooth}(U)$ defines a sheaf of simplicial sets on M. We can extend the definition of this sheaf to closed subpolyhedra $K \subseteq M$ by the formula $\mathrm{Smooth}(K) = \varinjlim_{K \subseteq U} \mathrm{Smooth}(U)$. We now have the following generalization of Theorem 1:

Theorem 2. *Let M be a PL manifold and $K \subseteq M$ a closed subpolyhedron. Then the above construction determines a homotopy equivalence from the simplicial set* $\mathrm{Smooth}(K)$ *of smooth structures on M to the simplicial set*

$$BO(m)^K \times_{BPL(m)^K} \{\chi|K\}$$

of liftings of $\chi|K$.

We observe that Theorem 2 is trivial in the case where K is a point: in this case, the map $\mathrm{Smooth}(K) \to BO(m) \times_{BPL(m)} *$ is an isomorphism of simplicial sets.

In the statement of Theorem 2, the right hand side has a description in terms of sections of fibrations, and is thus under good homotopy-theoretic control. To prove Theorem 2, we will need a similar understanding of the left hand side. This is furnished by the following fact, which is the main objective of this lecture:

Proposition 3 (Flexibility)**.** *Let $K \subseteq K'$ be compact subpolyhedra of M. Then the restriction map* $\mathrm{Smooth}(K') \to \mathrm{Smooth}(K)$ *is a Kan fibration.*

Note that $\mathrm{Smooth}(K') = \varinjlim_{K' \subseteq V} \mathrm{Smooth}(V)$. Since a direct limit of Kan fibrations is a Kan fibration, it will suffice to prove that each of the maps $\mathrm{Smooth}(V) \to \mathrm{Smooth}(K)$ is a Kan fibration. Replacing M by V, we are reduced to proving the following:

Proposition 4. *Let K be a compact subpolyhedron of M. Then the restriction map* $\mathrm{Smooth}(M) \to \mathrm{Smooth}(K)$ *is a Kan fibration.*

We must show that every lifting problem of the form

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \mathrm{Smooth}(M) \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & \mathrm{Smooth}(K) \end{array}$$

has a solution. The top map determines a PD homeomorphism $\Lambda^n_i \times M \to N$, where N is a smooth fiber bundle over Λ^n_i. Since the horn Λ^n_i is contractible, we can write $N = \Lambda^n_i \times N_0$, where N_0 is a smooth manifold. The bottom map determines an open subset U of $M \times \Delta^n$ containing $K \times \Delta^n$ and a PD homeomorphism $U \to W$, where W is a smooth fiber bundle over Δ^n whose restriction to Λ^n_i can be identified with an open subset of $\Lambda^n_i \times N_0$. Since Δ^n is trivial, we can write $W = W_0 \times \Delta^n$, where W_0 is a smooth manifold. Unwinding everything, we have the following data:

(1) A PD family $\{f_v : M \to N_0\}_{v \in \Lambda^n_i}$ of PD homeomorphisms.

(2) A PD homeomorphism $g : U \simeq \Delta^n \times W_0$, compatible with the projection to Δ^n.

(3) A smooth family of open embeddings $\{h_v : W_0 \to N_0\}_{v \in \Lambda^n_i}$ such that the following diagrams commute:

$$\begin{array}{ccc} U \times_{\Delta^n} \{v\} & \longrightarrow & M \\ \downarrow{\scriptstyle g_v} & & \downarrow{\scriptstyle f_v} \\ W_0 & \xrightarrow{h_v} & N_0. \end{array}$$

Let $B \subseteq N_0$ be a compact set containing the image of $K \times \Delta^n$ in its interior. Enlarging B, we may suppose that B is a smooth submanifold with boundary of N with codimention zero. Fix a point $0 \in \Lambda^n_i$. Using the parametrized isotopy extension theorem (in the smooth category), we can find a smooth family of diffeomorphisms $\{h'_v : M \to M\}_{v \in \Lambda^n_i}$ such that $(h'_v h_0)|B = h_v|B$. Replacing h_v by ${h'_v}^{-1} h_v$ and f_v by ${h'}^{-1}_v f_v$, we can assume that h_v is constant on the interior B. Replacing W_0 by the interior of B and shrinking U, we may assume that h_v is actually constant. We may therefore identify W_0 with an open subset of N_0.

To prove the existence of the desired extension, it will suffice to show that we can extend f_v to a PD family of PD homeomorphisms $\{f'_v : M \to N_0\}_{v \in \Delta^n}$, such that the families $\{f'_v\}$ and g agree in a neighborhood of K. Enlarging K, it will suffice to guarantee that we can arrange these maps to agree on K itself. Choose a PL homeomorphism $\Delta^n \simeq C \times [0,1]$, where $C = \Lambda^n_i$, and view $\{g_v\}_{v \in \Delta^n}$ as a two-parameter family $\{g_{c,t}\}_{c \in C, t \in [0,1]}$.

Note that f_v and g determine a polyhedral structure S on

$$(N_0 \times C) \coprod_{g(P) \times_{[0,1]} \{0\}} g(P)$$

where P is any closed subpolyhedron of U. Choose P to contain $K \times \Delta^n$. Our existence results for triangulations show that we can find a Whitehead compatible triangulation of $N_0 \times C \times [0,1]$ which is compatible with the projection to $C \times [0,1]$ and agrees with S near $N_0 \times C \times \{0\}$ and near $g(K \times C \times [0,1])$. Since the projection $\pi : N_0 \times C \times [0,1] \to C \times [0,1]$ is a fiber bundle in the smooth category, it is also a fiber bundle in the PL category, and can therefore be identified with $\pi^{-1}(C \times \{0\}) \times [0,1] \simeq M \times C \times [0,1]$. Using the parametrized isotopy extension theorem (in the PL category), we can adjust this identification so that it agrees with g on $K \times C \times [0,1]$. This provides the desired extension $\{f'_{c,t}\}_{c \in C, t \in [0,1]}$ of $\{f_c\}_{c \in C}$ and completes the proof.

Classification of Smooth Structures (Lecture 17)

March 16, 2009

Recall that our goal is to prove the following result:

Theorem 1. *Let M be a PL manifold and $K \subseteq M$ a closed subpolyhedron. Then the above construction determines a homotopy equivalence from the simplicial set* $\mathrm{Smooth}(K)$ *of smooth structures on M to the simplicial set*

$$BO(m)^K \times_{BPL(m)^K} \{\chi | K\}$$

of liftings of $\chi | K$.

Lemma 2. *Theorem 1 is true when K consists of a single simplex.*

Proof. Choose a point $v \in K$. Restriction to v determines a commutative diagram

$$\begin{array}{ccc} \mathrm{Smooth}(K) & \longrightarrow & \mathrm{Smooth}(\{v\}) \\ \downarrow & & \downarrow \\ BO(m)^K \times_{BPL(m)^K} * & \longrightarrow & BO(m) \times_{BPL(m)} *. \end{array}$$

The right vertical map is an isomorphism of simplicial sets, and the bottom horizontal map is a homotopy equivalence because the inclusion $\{v\} \hookrightarrow K$ is a homotopy equivalence. Consequently, it will suffice to show that the restriction map $r : \mathrm{Smooth}(K) \to \mathrm{Smooth}(\{v\})$ is a trivial Kan fibration. In other words, we must show that every lifting problem of the form

$$\begin{array}{ccc} \partial \Delta^n & \xrightarrow{f} & \mathrm{Smooth}(K) \\ \downarrow & \overset{F}{\nearrow} & \downarrow \\ \Delta^n & \xrightarrow{f} & \mathrm{Smooth}(\{v\}) \end{array}$$

has a solution. The map f determines a smooth structure on $U \times \partial \Delta^n$ (fibered over $\partial \Delta^n$), where U is some neighrbood of K in M. Similarly, g determines a smooth structure on $V \times \Delta^n$, where V is a neighborhood of v in M; without loss of generality we may assume that $V \subseteq U$. Since r is a Kan fibration, we are free to replace g by any map which is homotopic (relative to the boundary $\partial \Delta^n$); we may therefore assume that the smooth structure is a product of the smooth structure determined by $g| \partial \Delta^n$ over a collar $C = \partial \Delta^n \times [0, 1)$ of $\partial \Delta^n$ in Δ^n. This smooth structure therefore extends over U, so we obtain a smooth structure S on $W = (U \times C) \coprod_{V \times C} (V \times \Delta^n)$.

Choose a PL isotopy h_t of M supported in U from the identity id_M to a map h_1 which carries Δ^n into V. Let $\chi : \Delta^n \to [0, 1]$ be the map which is equal to 1 on $\Delta^n - C$ and equal to the projection $C \to [0, 1)$ on C. The map $(x, z) \mapsto (h_{\chi(z)}(x), z)$ determines a PL map $H : M \times \Delta^n \to M \times \Delta^n$. Let $W' = H^{-1}(W)$. Our smooth structure on W determines a smooth structure on $H^{-1}(W)$, which contains $K \times \Delta^n$ and therefore determines a map $F : \Delta^n \to \mathrm{Smooth}(K)$. It is easy to see that this map has the desired properties. □

Now fix a triangulation S of the PL manifold M. We prove the following:

Lemma 3. *Let $K \subseteq M$ be a finite union of simplices of the triangulation S. Then Theorem 1 is true for K.*

Proof. We use induction on the number of simplices of S which belong to K. If K is empty, there is nothing to prove. Otherwise, choose a simplex σ belonging to K having maximal dimension, so we can write K as a pushout

$$\begin{array}{ccc} \partial\,\sigma & \longrightarrow & \sigma \\ \downarrow & & \downarrow \\ K_0 & \longrightarrow & K. \end{array}$$

The theorem holds for $\partial\,\sigma$ and K_0 by the inductive hypothesis, and it holds for σ by Lemma 2. We have diagrams

$$\begin{array}{ccc} \mathrm{Smooth}(K) & \longrightarrow & \mathrm{Smooth}(K_0) \\ \downarrow & & \downarrow \\ \mathrm{Smooth}(\sigma) & \longrightarrow & \mathrm{Smooth}(\partial\,\sigma) \end{array} \qquad \begin{array}{ccc} BO(m)^K \times_{BPL(m)^K} * & \longrightarrow & BO(m)^{K_0} \times_{BPL(m)^{K_0}} * \\ \downarrow & & \downarrow \\ BO(m)^\sigma \times_{BPL(m)^\sigma} * & \longrightarrow & BO(m)^{\partial\,\sigma} \times_{BPL(m)^{\partial\,\sigma}} *. \end{array}$$

The square on the right is a homotopy pullback square since the diagram above is a homotopy pushout square of polyhedra. The square on the left is a homotopy pullback square since it is a pullback square in which each of the morphisms is a Kan fibration (by the main result of last time). We therefore have a map of homotopy pullback squares which induces a homotopy equivalence everywhere except perhaps in the upper left hand corner. It follows that it induces a homotopy equivalence in the upper left hand corner as well: that is, the map $\mathrm{Smooth}(K) \to BO(m)^K \times_{BPL(m)^K} *$ is a homotopy equivalence as desired. □

We can now prove Proposition 1 in general. Let K be an arbitrary closed subpolyhedron of M (for example, M itself). We can choose a filtration of K

$$K_0 \subseteq K_1 \subseteq \dots$$

with $K = \bigcup_i K_i$, where each K_i is a finite subpolyhedron. We have a homotopy equivalence of towers $\{\mathrm{Smooth}(K_i)\} \to \{BO(m)^{K_i} \times_{BPL(m)^{K_i}} *\}$. All of the transition maps in these towers are Kan fibrations (for the left tower, this follows from the main result of last time; for the right tower, it follows from the observation that each map of PL singular complexes $\mathrm{Sing}^{PL}_\bullet X_i \to \mathrm{Sing}^{PL}_\bullet X_{i+1}$ is a monomorphism of simplicial sets). It follows that the homotopy inverse limits of these towers can be identified with the ordinary inverse limits, so we get a homotopy equivalence

$$\mathrm{Smooth}(K) \simeq \varprojlim \mathrm{Smooth}(K_i) \simeq \varprojlim BO(m)^{K_i} \times_{BPL(m)^{K_i}} * \simeq BO(m)^K \times_{BPL(m)^K} *.$$

This completes the proof of Theorem 1.

We can informally summarize Theorem 1 by saying that smooth structures on a PL manifold M can be identified with liftings of the canonical map $\chi : M \to BPL(m)$ to a map $\widetilde{\chi} : M \to BO(m)$. More precisely, we get a bijection of the set of homotopy classes of such liftings with the set $\pi_0 \mathrm{Smooth}(M)$. It is therefore of interest to describe the latter set more explicitly. In other words, we ask the following question: given two smooth structures s_0 and s_1 (compatible with the given PL structure) on M, when do they belong to the same connected component of $\mathrm{Smooth}(M)$? This is true if and only if s_0 and s_1 can be joined by an edge in $\mathrm{Smooth}(M)$. In other words, if and only if there exists a PD homeomorphism $M \times [0,1] \to N$ (compatible with the projection to $[0,1]$), where $p : N \to [0,1]$ is a fiber bundle of smooth manifolds. In this case, we can identify N with the trivial fiber bundle $N_0 \times [0,1]$, where $N_0 = p^{-1}\{0\}$ is the smooth manifold determined by the smoothing s_0. We can summarize the situation as follows:

Claim 4. *Let M be a PL manifold equipped with a Whitehead compatible smooth structure s_0. Then another smooth structure s_1 is equivalent to s_0 (in other words, it belongs to the same connected component of* $\mathrm{Smooth}(M)$*) if and only if there exists a PD isotopy $h : M \times [0,1] \to M$, where $h_0 = \mathrm{id}_M$ and s_1 is the smooth structure obtained by pulling back s_0 along the homeomorphism h_1.*

Variant 5. *Suppose that M is a PL manifold, K a closed subset, and the smooth structures s_0 and s_1 coincide in a neighborhood of K. Then s_0 and s_1 belong to the same connected component of the fiber* $\mathrm{Smooth}(M) \times_{\mathrm{Smooth}(K)} *$ *if and only if there exists a PD isotopy h_t as above, which is constant in a neighborhood of K. This can be proven by essentially the same argument, together with the smooth version of the isotopy extension theorem.*

Product Structure Theorems (Lecture 18)

March 16, 2009

Our goal in this lecture is to study the relative connectivity properties of the quotient spaces $PL(m)/O(m)$. Our basic observation is the following:

Remark 1. Let $K \subseteq \mathbb{R}^m$ be a closed subpolyhedron. Then the mapping space $(PL(m)/O(m))^K$ can be identified with the simplicial set $\text{Smooth}(K)$ of germs of smooth structures on $\mathbb{R}^m$ near K. This follows from the main result of the last lecture, together with the observation that the standard PL structure on $\mathbb{R}^m$ determines a *constant* map $\chi : \mathbb{R}^m \to BPL(m)$.

Proposition 2. *Fix an integer $m \geq 0$. The following conditions are equivalent:*

(1) *All homotopy fibers of the map $PL(m)/O(m) \to PL(m+1)/O(m+1)$ are $(m-1)$-connected.*

(2) *All homotopy fibers of the map $BO(m) \to BO(m+1) \times_{BPL(m+1)} BPL(m)$ are $(m-1)$-connected.*

(3) *The following weak product structure theorem holds:*

   (∗) *Let M be a PL manifold of dimension m, let $K \subseteq M$ be a closed subpolyhedron, and suppose we are given a smooth structure on $M \times \mathbb{R}$ which is the product of a smooth structure on M with the standard smooth structure on $\mathbb{R}$ in a neighborhood of $K \times \mathbb{R}$. Then, after modifying the smooth structure by a suitable PD isotopy which is trivial in a neighborhood of $K \times \mathbb{R}$, we can arrange that the smooth structure on $M \times \mathbb{R}$ is the product of a smooth structure on M with the standard smooth structure on $\mathbb{R}$.*

Proof. We have a natural transformation of homotopy fiber sequences

$$\begin{array}{ccccc} PL(m)/O(m) & \longrightarrow & BO(m) & \longrightarrow & BPL(m) \\ \downarrow \phi & & \downarrow \psi & & \downarrow \\ PL(m+1)/O(m+1) & \xrightarrow{\theta} & BO(m+1) \times_{BPL(m+1)} BPL(m) & \longrightarrow & BPL(m). \end{array}$$

It follows that every homotopy fiber of ϕ is also a homotopy fiber of ψ, so the implication $(2) \Rightarrow (1)$ is clear. To prove the converse, it suffices to show that every homotopy fiber of ψ is equivalent to a homotopy fiber of ϕ. This will follow if the map θ is surjective on π_0. This surjectivity follows from the fiber sequence, since $BPL(m)$ is connected.

We now prove that $(2) \Rightarrow (3)$. In the situation of (3), the smooth structure on $M \times \mathbb{R}$ is classified by a map $M \times \mathbb{R} \to BO(m+1) \times_{PL(m+1)} PL(m)$. Finding a PD isotopy to a smooth structure on $M \times \mathbb{R}$ which is a product with $\mathbb{R}$ is equivalent to solving the lifting problem

$$\begin{array}{ccc} & & BO(m) \\ & \nearrow & \downarrow \\ M \times \mathbb{R} & \longrightarrow & BO(m) \times_{BPL(m)} BPL(m). \end{array}$$

If we wish to do achieve this via an isotopy fixed near K, then we must solve instead a relative lifting problem of the form

$$\begin{array}{ccc} K \times \mathbb{R} & \longrightarrow & BO(m) \\ \downarrow & \nearrow & \downarrow j \\ M \times \mathbb{R} & \longrightarrow & BO(m+1) \times_{BPL(m+1)} BPL(m). \end{array}$$

This is a purely homotopy theoretic problem; we may therefore replace the inclusion $K \times \mathbb{R} \subseteq M \times \mathbb{R}$ by $K \subseteq M$. Since M is a PL m-manifold, it can be obtained from K by successive cell attachments where the cells have dimension $\leq m$. Working cell-by-cell, we are reduced to solving lifting problems of the form

$$\begin{array}{ccc} \partial D^k & \longrightarrow & BO(m) \\ \downarrow & \nearrow & \downarrow \psi \\ D^k & \longrightarrow & BO(m+1) \times_{BPL(m+1)} BPL(m) \end{array}$$

where D^k indicates a disk of dimension $\leq k$. The obstruction to solving such a problem is equivalent to the vanishing of a class in π_{k-1} of a homotopy fiber F of ψ. This class automatically vanishes by virtue of our assumption that F is $(m-1)$-connected.

We now prove that (3) $\Rightarrow$ (1). We must show that every lifting problem of the form

$$\begin{array}{ccc} \partial D^k & \longrightarrow & PL(m)/O(m) \\ \downarrow & \nearrow & \downarrow \psi \\ D^k & \longrightarrow & PL(m+1)/O(m+1) \end{array}$$

has a solution, provided that $k \leq m$. In this case, we can choose a PL embedding of ∂D^k into $\mathbb{R}^m$ and obtain an equivalent lifting problem

$$\begin{array}{ccc} \partial D^k \times \mathbb{R} & \longrightarrow & PL(m)/O(m) \\ \downarrow & \nearrow & \downarrow \psi \\ \mathbb{R}^{m+1} & \longrightarrow & PL(m+1)/O(m+1). \end{array}$$

The diagram determines a smoothing of $\mathbb{R}^{m+1}$ which is a product smoothing in a neighborhood of $\partial D^k \times \mathbb{R}$, and a solution to the indicated lifting problem is equivalent to giving a PD isotopy (fixed near $\partial D^k \times \mathbb{R}$) to a product smoothing. $\square$

Remark 3. If the equivalent conditions of Proposition 2 are satisfied, then the map $PL(m)/O(m) \to PL(m+1)/O(m+1)$ is surjective on π_0 for $m \geq 0$. Since $PL(0)/O(0) = *$ is connected, we it follows by induction that $PL(m)/O(m)$ is connected for each m. In other words, Euclidean space $\mathbb{R}^m$ admits a unique smooth structure compatible with its standard PL structure, up to PD isotopy.

The connectivity estimate given in Proposition 2 is not the best possible. We now describe how to do a little better. We need a variation on the main result of the last lecture, which applies to manifolds with boundary.

Variant 4. *Let M be a PL $(m+1)$-manifold with boundary ∂M. We can define the notion of a smoothing of M as before. Smoothings of M can be organized into a simplicial set* $\mathrm{Smooth}(M)$. *Every smoothing of M determines a smoothing of the boundary of M; this is given by a Kan fibration* $\mathrm{Smooth}(M) \to \mathrm{Smooth}(\partial M)$. *Given a smooth structure on the boundary of M, we denote the fiber of this map by* $\mathrm{Smooth}(M; \partial)$. *Given* *such a smoothing of ∂M, we get a map $\partial M \to BO(m)$. Then, up to homotopy, smoothings of M compatible with this smooth structure on ∂M are given by solutions to the lifting problem*

$$\begin{array}{ccc} \partial M & \longrightarrow & BO(m+1) \\ \big\downarrow & \nearrow & \big\downarrow \\ M & \longrightarrow & BPL(m+1). \end{array}$$

Smoothings of M itself (without boundary data) can be identified with solutions to the lifting problem of pairs

$$\begin{array}{ccc} & & (BO(m+1), BO(m)) \\ & \nearrow & \big\downarrow \\ (M, \partial M) & \longrightarrow & (BPL(m+1), BPL(m)). \end{array}$$

Notation 5. Fix an integer $m \geq 0$. We let Δ_m^{PL} denote the homotopy fiber product

$$BPL(m) \times^h_{BPL(m+1)} BPL(m) = BPL(m) \times_{BPL(m+1)^{\{0\}}} BPL(m+1)^{[0,1]} \times_{BPL(m+1)^{\{1\}}} BPL(m).$$

Similarly, define Δ_m^O to be the homotopy fiber product

$$BO(m) \times^h_{BO(m+1)} BO(m) = BO(m) \times_{BO(m+1)^{\{0\}}} BO(m+1)^{[0,1]} \times_{BO(m+1)^{\{1\}}} BO(m).$$

We have a Kan fibration $\Delta^O \to \Delta^{PL}$. For every PL m-manifold M, the tangent microbundle to $M \times [0,1]$ and its boundary determines a map $M \to \Delta_m^{PL}$. According to Variation 4, we can identify smoothings of $M \times [0,1]$ with solutions to the lifting problem

$$\begin{array}{ccc} & & \Delta_m^O \\ & \nearrow & \big\downarrow \\ M & \longrightarrow & \Delta_m^{PL}. \end{array}$$

The proof of Proposition 2 adapts without essential change to show the following:

Proposition 6. *Fix an integer $m \geq 0$. The following conditions are equivalent:*

(1) *Let F denote the homotopy fiber of the map $\Delta_m^O \to \Delta_m^{PL}$. Then all $PL(m)/O(m) \to F$ are $(m-1)$-connected.*

(2) *All homotopy fibers of the map $BO(m) \to \Delta_m^O \times_{\Delta_m^{PL}} BPL(m)$ are $(m-1)$-connected.*

(3) *The following strong product structure theorem holds:*

 ($*$) *Let M be a PL manifold of dimension m, let $K \subseteq M$ be a closed subpolyhedron, and suppose we are given a smooth structure on $M \times [0,1]$ which is the product of a smooth structure on M with the standard smooth structure on $[0,1]$ in a neighborhood of $K \times \mathbb{R}$. Then, after modifying the smooth structure by a suitable PD isotopy which is trivial in a neighborhood of $K \times [0,1]$, we can arrange that the smooth structure on $M \times [0,1]$ is the product of a smooth structure on M with the standard smooth structure on $[0,1]$*

Remark 7. Let F be as in Proposition 6. Then the homotopy fibers of the map $PL(m)/O(m) \to F$ can be identified with *path spaces* in the space in homotopy fibers of the map $\psi : PL(m)/O(m) \to PL(m+1)/O(m+1)$. Consequently, if we grant that the homotopy fibers of ψ are nonempty (which follows from

Proposition 2 if $m \geq 0$), then Proposition 6 asserts that the homotopy fibers of ψ are m-connected. This is a slightly better connectivity estimate than we get from Proposition 2 itself, which is why the geometric assertion of part (3) of Proposition 6 is called the *strong* product structure theorem to contrast it with the corresponding *weak* product structure theorem of Proposition 2. However, the terminology is slightly misleading: Proposition 6 does not quite formally imply Proposition 2, since it does not guarantee that the homotopy fibers of ψ are nonempty. This missing strength is equivalent to the assertion of Remark 3: we need to know that every smooth structure on $\mathbb{R}^m$ is PD isotopic to the product with $\mathbb{R}$ of a smooth structure on $\mathbb{R}^{m-1}$, and thus (using induction on m) PD isotopic to the standard smooth structure on $\mathbb{R}^m$.

Product Structure Theorem: First Steps (Lecture 19)

March 18, 2009

In the last lecture, we saw that the connectivity properties of the map $PL(m)/O(m) \to PL(m+1)/O(m+1)$ could be phrased geometrically as follows:

Theorem 1 (Product Structure Theorems)**.** *Let M be a PL manifold of dimension m, let $K \subseteq M$ be a closed subpolyhedron, and suppose we are given a smooth structure on $M \times \mathbb{R}$ which is the product of a smooth structure on M with the standard smooth structure on $\mathbb{R}$ in a neighborhood of $K \times \mathbb{R}$. Then, after modifying the smooth structure by a suitable PD isotopy which is trivial in a neighborhood of $K \times \mathbb{R}$, we can arrange that the smooth structure on $M \times \mathbb{R}$ is the product of a smooth structure on M with the standard smooth structure on $\mathbb{R}$. The same result holds if we replace $\mathbb{R}$ by $[0, 1]$.*

Our goal in the next few lectures is to sketch a proof of this result. The argument is essentially the same whether we use $\mathbb{R}$ or $[0, 1]$; we will therefore switch from one case to the other as convenient. To simplify the exposition, we will assume that $K = \emptyset$. The case where K is nonempty can be treated by more careful versions of the same arguments.

To begin, let us assume that we are given a smooth structure on the product $M \times [0, 1]$. Let $X = M \times [0, 1]$, and let $\pi : X \to [0, 1]$ denote the projection. The easiest case of Theorem 1 is the following:

Lemma 2. *Theorem 1 is true if π is a smooth submersion.*

Proof. If π is a smooth submersion, then it exhibits X as a smooth fiber bundle over $[0, 1]$. Let $M_0 = M$, equipped with the smooth structure given by the identification $M_0 \simeq \pi^{-1}\{0\}$. We have a diffeomorphism $f : X \simeq M_0 \times [0, 1]$.. In other words, X is diffeomorphic to a product with $[0, 1]$. This is not quite the full strength of Theorem 1: we must show that this diffeomorphism can be chosen to be PD isotopic to the identity map on X. Let us think of f as a PD family $\{f_t : M \to M_0\}_{t \in [0,1]}$ of PD homeomorphisms from M to M_0, where f_0 is the identity. Define a PD isotopy $\{h_t : X \to M_0 \times [0, 1]\}_{t \in [0,1]}$ by the formula

$$h_t(m, s) = \begin{cases} (f_{s-t}(m), s) & \text{if } t \leq s \\ (f_0(m), s) & \text{if } t \geq s. \end{cases}$$

Then h_0 is the diffeomorphism f, which gives the original smooth structure on X. The map h_1 is the identity map $X \simeq M \times [0, 1] \simeq M_0 \times [0, 1]$, which gives a product smooth structure on X. □

If π is a smooth map, then we can test whether or not π is a submersion by checking whether the derivative of π does not vanish at any point. Of course, the condition that π is smooth is very strong: in our situation, we only know that π is piecewise linear with respect to some Whitehead compatible triangulation of X. In other words, we know that π is piecewise differentiable on X: that is, there is a smooth triangulation of X such that π is differentiable on each simplex. In this case, it is still possible to salvage something of the theory of derivatives:

Definition 3. Let X be a smooth manifold, and let $f : X \to \mathbb{R}$ be a piecewise differentiable map. (In the case of interest, X is a smoothing of $M \times \mathbb{R}$ for some PL manifold M, and f is the projection onto the second factor.) Let $x \in X$ be a point and let v be a tangent vector to X and x. We define $D_v(f)$ to be the minimum value of the derivatives $D_v(f|\sigma)$, where σ ranges over all simplices containing x of some triangulation of X for which f is smooth on each simplex.

The map $(v, x) \mapsto D_v(f)$ is not generally continuous if f is not a smooth function. However, it is lower semicontinuous. In other words, for every real number ϵ, the subset of the tangent bundle T_X consisting of pairs (x, v) for which $D_v(f) > \epsilon$ is an open set. We will say that a tangent vector v to X is *regular for* f if $D_v(f) > 0$. Lower semicontinuity guarantees that the set of regular tangent vectors is open in T_X.

Definition 4. Let X be a smooth manifold and $f : X \to \mathbb{R}$ a piecewise differentiable function. We will say that f is *regular* if, for every point $x \in X$, there exists a tangent vector $v \in T_{X,x}$ such that (x, v) is regular (in other words, such that $D_v(f) > 0$).

Example 5. If f is smooth, then f is regular if and only if it is a smooth submersion.

Lemma 6. *Let X be a smooth manifold and $f : X \to \mathbb{R}$ a regular piecewise differentiable function. Then there exists a smooth tangent field $v : X \to T_X$ such that, for every $x \in X$, the tangent vector $v(x)$ is regular for f.*

Proof. Since f is regular, we can find for each x a tangent vector w_x at x such that $D_{v_x}(f) > 0$. Let $v_x : X \to T_X$ be a smooth tangent field such that $v_x(x) = w_x$. Since the collection of regular tangent vectors is open, there exists an open neighborhood U_x of x such that $v_x(y)$ is f-regular for $y \in U_x$. Since X is paracompact, the open covering $\{U_x\}_{x \in X}$ has a locally finite refinement. Choose a smooth partition of unity ψ_i subordinate to this refinement, so that each ψ_i is supported in U_{x_i}. Then the smooth vector field $v = \sum_i \psi_i v_{x_i}$ has the desired property. □

In the situation of Lemma 6, we will say that the vector field f is *transverse* to f.

Lemma 7. *Let $f : X \to \mathbb{R}$ be a piecewise differentiable function, and let $v : X \to T_X$ be a smooth vector field which is transverse to f. Then for any continuous function $\epsilon : X \to \mathbb{R}_{>0}$, there exists a smooth map $g : X \to \mathbb{R}$ such that*

$$D_{v(x)}(g) > D_{v(x)}(f) - \epsilon(x)$$
$$g(x) - f(x) < \epsilon(x).$$

(Choosing ϵ sufficiently small will guarantee that v is also transverse to g.)

Proof. Choose a partition of unity ψ_i on X subordinate to a locally finite cover of X by compact sets K_i, each of which is contained in a coordinate chart U_i. Suppose we are given smooth maps $g_i : U_i \to \mathbb{R}$, and define g by the formula

$$g = \sum \psi_i g_i.$$

Then $g(x) - f(x) < \epsilon(x)$ will be satisfied provided that $g_i(x) - f(x) < \epsilon(x)$ holds for $x \in U_i$. The other condition is a bit more subtle: we have

$$\begin{aligned} D_{v(x)} g &= \sum_i (D_{v(x)} \psi_i) g_i + \sum_i \psi_i D_{v(x)}(g_i) \\ &= \sum_i (D_{v(x)} \psi_i)(g_i - f) + D_{v_x}(\sum_i \psi_i) f + \sum_i \psi_i D_{v(x)}(g_i) \\ &\geq \sum_i \psi_i D_{v(x)}(g_i) - \sum_i C_i (g_i - f) \end{aligned}$$

where $C_i > 0$ is an upper bound for the compactly supported function $D_{v(x)} \psi_i$. If the inequalities

$$D_{v(x)}(g_i) > D_{v(x)}(f) - \frac{\epsilon(x)}{2}$$

$$\sum_{x \in K_j \cap K_i} C_j (g_j(x) - f(x)) < \frac{\epsilon(x)}{2}$$

hold for $x \in K_i$, then g will satisfy the desired inequality. Since only finitely many intersections $K_j \cap K_i$ are nonempty, the latter inequality can be achieved by ensuring that each g_i is a close approximation to f on K_i.

In other words, we may reduce to the case where $X = \mathbb{R}^n$, and the inequalities

$$D_{v(x)}(g) > D_{v(x)}(f) - \epsilon(x)$$

$$g(x) - f(x) < \epsilon(x).$$

only need to be satisfied when x lies in some compact subset $K \subseteq \mathbb{R}^n$. Let $k : \mathbb{R}^n \to \mathbb{R}_{>0}$ be a smooth function with total integral 1, which is supported in a small ball of radius δ. Define $g(x) = \int_y f(y)k(x-y)$. Then g is a smooth function. It is not difficult to see that the conditions

$$D_{v(x)}(g) > D_{v(x)}(f) - \epsilon(x)$$

$$g(x) - f(x) < \epsilon(x).$$

will be satisfied on any compact subset K, provided that δ is chosen sufficiently small. □

We now come to the main goal of this lecture:

Proposition 8. *Theorem 1 is true in the case where the projection $\pi : M \times \mathbb{R} \to \mathbb{R}$ is a regular (but not necessarily smooth with respect the smoothing of $M \times \mathbb{R}$).*

Proof. We will show that, after adjusting the smooth structure on $M \times \mathbb{R}$ by a PD isotopy, we can arrange that π is a smooth submersion; the desired result will then follow from Lemma 2. First, choose a smooth Riemannian metric on $X = M \times \mathbb{R}$. Let $\epsilon : X \to \mathbb{R}_{>0}$ be a smooth function such that each of the closed balls $B_{\epsilon(x)}(x) \subseteq X$ of radius $\epsilon(x)$ around x is compact. Let $v : X \to T_X$ be a smooth tangent field which is transverse to π. Rescaling v, we can assume that each $v(x)$ has unit length.

Choose a smooth function $\delta : X \to \mathbb{R}_{>0}$ such that

$$D_{v(x)}(f) > \delta(x)$$

for $x \in X$. Let $\delta' : X \to \mathbb{R}_{>0}$ be another smooth function such that if $d(x,y) \leq \epsilon$, then $\delta'(x) \leq \delta(y)$. Using the previous Lemma, we can choose a smooth map $g : X \to \mathbb{R}$ with the following properties:

$$D_{v(x)}(g) > \frac{\delta(x)}{2}$$

$$\pi(x) - g(x) < \epsilon(x)\frac{\delta'(x)}{2}.$$

In particular, $\lambda(x) = D_{v(x)}(g)$ is a smooth function of x satisfying $\pi(x) - g(x) < \epsilon(x)\lambda(y)$ whenever $d(x,y) < \epsilon(x)$.

Since v is a unit vector field and each of the $\epsilon(x)$-balls around x is compact, the flow along the vector field v gives a well-defined map

$$F : \{(x,t) \in X \times \mathbb{R} : |t| < \epsilon(x)\} \to X.$$

Moreover, for fixed x, $F(x,t)$ stays in a ball of radius ϵ around x. It follows that the t-derivative of $g(F(x,t))$ coincides with $\lambda(F(x,t)) > \frac{f(x)-g(x)}{\epsilon(x)}$. Consequently, for $s \in [0,1]$, we can find a unique $t = t(x,s)$ such that $g(F(x,t)) - g(x) = s(\pi(x) - g(x))$. We now define a map $h_s : X \to X$ by the formula

$$h_s(x) = F(x, t(x,s)).$$

The family $\{h_s : X \to X\}_{s \in [0,1]}$ is then a PD isotopy from X to itself, where h_0 is the identity and $g \circ h_1 = f$, so that f is smooth with respect to the smooth structure on X determined by h_1. □

Product Structure Theorem: Isolating Singularities (Lecture 20)

March 30, 2009

In this lecture, we will continue our efforts to prove the product structure theorem. As in the last lecture, we will be content to treat the special case where the set K is empty, and the product is with $\mathbb{R}$ rather than with $[0,1]$. In the last lecture, we reduced this to proving the following assertion:

Proposition 1. *Let M be a PL manifold, and suppose we are given a compatible smooth structure on $X = M \times \mathbb{R}$. Let $\pi : X \to \mathbb{R}$ denote the projection onto the second factor (so that π is a PD map). Then, after altering the smooth structure on X by a PD isotopy, we can arrange that the map π is regular.*

To prove this, it is useful to have a criterion for testing whether or not a map is regular. Fix a smooth triangulation of X for which π is PL (and therefore smooth) on each simplex. Let $x \in X$, and let σ denote the simplex containing x in its interior. The tangent space $T_{X,x}$ to X at x contains the tangent space $T_{\sigma,x}$ as a linear subspace. Let $v \in T_{\sigma,x}$. Note that every simplex τ containing x contains σ, so the derivatives $D_v(\pi|\tau)$ all agree with $D_v(\pi|\sigma)$. It follows that $D_v(\pi) = D_v(\pi|\sigma)$. It follows that π is regular at x unless the derivative of $\pi|\sigma$ is identically zero. We have proven:

Lemma 2. *If $x \in X$ is a point where π is not regular and σ is as above, then σ lies in a fiber of π.*

Corollary 3. *Fix a triangulation of the polyhedron $X \simeq M \times \mathbb{R}$, and suppose that the restriction of π to the set of vertices of this triangulation is* injective. *Then π is regular away from the set of vertices of the triangulation. In particular, π is regular away from an isolated set of points.*

We can always arrange to be in the situation of Corollary 3. To see this, choose any triangulation of $M \times \mathbb{R}$ which is sufficiently fine that the star of each vertex has a neighborhood with a PL product chart $\mathbb{R}^m \times \mathbb{R}$. For each vertex v, let $L(v)$ denote the link of v and $\mathrm{St}(v)$ its star. We define a PL isotopy h_t of $M \times \mathbb{R}$, supported in the star $\mathrm{St}(v)$, which we view as a closed subset of $\mathbb{R}^m \times \mathbb{R} \simeq \mathbb{R}^{m+1}$. Fix $v' \in \mathbb{R}^{m+1}$. For each $t \in [0,1]$, there is a unique map $h_t : \mathrm{St}(v) \to \mathbb{R}^m \times \mathbb{R} \subseteq M \times \mathbb{R}$ which is linear on each simplex, the identity on $L(v)$, and carries v to $(1-t)v + tv'$. If v' is chosen sufficiently close to v, then this defines a PL isotopy of M, where h_1 moves v to v'. We can assume that $\pi(v')$ is distinct from $\pi(w)$, for any other vertex w of the triangulation. Applying this construction repeatedly and concatenating the resulting isotopies (note that only finitely many isotopies have support near any fixed point of $M \times \mathbb{R}$, so the concatenation is well-defined), we can arrange that π is injective when restricted to vertices, as desired.

We may now assume that π is regular away from the set of vertices with respect to some smooth triangulation of X. We would like to adjust the smooth structure on X by a PD isotopy to arrange that π is everywhere regular. Since the set of vertices of X is isolated, it will suffice to construct these isotopies one vertex at a time. More precisely, we will prove the following:

Proposition 4. *Let v be a vertex with respect to some smooth triangulation of X, and let K denote the star of v. Assume that π is injective on vertices of X, so that π is regular on the interior of K except perhaps at v. Then it is possible to alter the smooth structure on X by means of a PD isotopy supported on the interior of K, so that π is regular on the whole interior of K.*

Applying this proposition repeatedly and concatenating the resulting isotopies, we will obtain a proof of Proposition 1. We are therefore reduced to proving Proposition 4. Moreover, we may assume without loss of generality that our triangulation of X is sufficiently fine that the star of each vertex is contained in a PL product chart $\mathbb{R}^m \times \mathbb{R}$ and also a smooth chart. We will identify K with its image in $\mathbb{R}^{m+1}$. Without loss of generality, we may assume that $v \mapsto 0$, so that K can be identified with the cone on the link $L(v) = \partial K$, which is an m-sphere equipped with a PL embedding into $\mathbb{R}^{m+1} - \{0\}$. The map $\pi | K : K \to \mathbb{R}$ is given by projection onto the $(m+1)$st coordinate. As above, we may assume that π is injective on vertices. In particular, $\pi(w) \neq 0$ whenever w is a vertex of $L(v)$.

The smooth structure on X is given by a PD embedding $f : K \to \mathbb{R}^{m+1}$. We wish to modify f by a PD isotopy which is the identity near ∂K, so that the map $\pi \circ f^{-1} : f(K) \to \mathbb{R}$ is regular on the interior of K.

We can therefore rephrase our problem as follows:

Problem 5. *Let $K \subseteq \mathbb{R}^{m+1}$ be a polyhedron which is the cone (with cone point 0) on its boundary ∂K, let $\pi : K \to \mathbb{R}$ be projection onto the last factor, and assume that π is injective on the vertices of K. Let $f : K \to \mathbb{R}^{m+1}$ be a PD embedding, and assume $f(0) = 0$. Then, after adjusting f by a PD isotopy which is fixed near ∂K, we can arrange that $\pi \circ f^{-1}$ is regular on the interior of $f(K)$.*

Remark 6. In the course of solving Problem 5, we are free to replace K by its image rK for $r \in (0,1)$: any PD isotopy of $f|rK$ can then be extended to a PD isotopy of f by declaring it to be the identity on $K - rK$.

Our first step is to "linearize" the map f. Since f is differentiable on each simplex of K, we can define a map $f' : K \to \mathbb{R}^{m+1}$ which is linear on each simplex by taking the derivatives of f at the origin. There is a PD homotopy from f' to f, given by the formula

$$f_t(x) = \begin{cases} t^{-1} f(tx) & \text{if} t \neq 0 \\ f'(x) & \text{if } t = 0. \end{cases}$$

This homotopy is generally not trivial on the boundary ∂K. To fix this, choose a smooth map $\chi : K \to [0,1]$ which is supported in a small neighborhood U of the origin, such that χ is identically equal to 1 in an open set $V \subseteq U$ containing 1, and define

$$g_t(x) = \chi(\frac{x}{N}) f_t(x) + (1 - \chi(\frac{x}{N})) f(x).$$

By choosing N sufficiently large, we can arrange that each g_t is arbitrarily close to f in the C^1-sense, and therefore a PD embedding. Then g_t is a PD isotopy from f to a map g_1, where $g_1|V$ is linear on each simplex. Using Remark 6, we obtain the following:

Claim 7. *It suffices to solve Problem 5 in the special case where f is linear on each simplex.*

For $x \in K$. Choose a function $\chi : K \to \mathbb{R}_{>0}$ which is smooth on each simplex, nondecreasing on each ray from the origin, and satisfies the following conditions:

(1) The map χ is constant in a neighborhood of 0.

(2) The map χ is equal to 1 near ∂K.

(3) The map χ is given by $\chi(x) = \frac{s\epsilon}{|f(x)|}$ for $x \in s\,\partial K$ if $s \in [\frac{1}{4}, \frac{1}{2}]$, for some $\epsilon > 0$.

We define a PD isotopy f_t by the formula

$$f_t(x) = (1-t) f(x) + t\chi(x) f(x).$$

Then f_1 carries $s\,\partial K$ to the sphere of radius $s\epsilon$ for $s \in [\frac{1}{4}, \frac{1}{2}]$. Replacing f by f_1, applying an appropriate dilation to the target space $\mathbb{R}^{m+1}$, and invoking Remark 6, we are reduced to the following situation:

Claim 8. *It suffices to solve Problem 5 in the special case where $f(K)$ is the unit ball $B(1)$, and $f(tx) = tf(x)$ for $t \in [\frac{1}{2}, 1]$, $x \in \partial K$.*

The advantage of our present situation is that the image of ∂K now inherits a smooth structure from the map f. We will exploit this in the next lecture.

Product Structure Theorem: Inductive Step (Lecture 21)

March 30, 2009

Recall that we are in the process of proving the product structure theorem for smooth structures on PL manifolds, which (by virtue of smoothing theory) is equivalent to the following connectivity estimate:

Theorem 1. *Let $m \geq 0$. Then all homotopy fibers of the map $PL(m)/O(m) \to PL(m+1)/O(m+1)$ are m-connected.*

We have reduced the proof to the following statement:

Proposition 2. *Let $K \subseteq \mathbb{R}^m \times \mathbb{R}$ be a polyhedron which is the closed star of the origin 0 with respect to some PL triangulation of $\mathbb{R}^{m+1}$ (so that K is the cone on ∂K, with the origin as the cone point), let $\pi : K \to \mathbb{R}$ denote the projection onto the last factor. Let $f : K \to \mathbb{R}^{m+1}$ be a PD embedding satisfying the following conditions:*

(1) *The image of f is the unit ball $B(1) \subseteq \mathbb{R}^{m+1}$.*

(2) *For $\frac{1}{2} \leq t \leq 1$ and $x \in \partial K$, we have $f(tx) = tf(x)$.*

(3) *The projection π is injective when restricted to the vertices of K (with respect to some PL triangulation), so that $\pi \circ f^{-1}$ is regular on the interior of the unit ball except possibly at the origin.*

Then, after modifying f by a PD isotopy which is trivial on ∂K, we can arrange that $\pi \circ f^{-1}$ is regular on the interior of the unit ball.

Let $S^m \subseteq B(1)$ denote the unit sphere. Condition (1) implies that f restricts to a PD homeomorphism $f_0 : \partial K \to S^m$. Since π is injective on vertices, the composition $\pi \circ f_0^{-1} : S^m \to \mathbb{R}$ is regular except possibly at the images of the vertices of ∂K. In particular, it is regular in a neighborhood of $\partial K \cap \pi^{-1}\{0\}$. Using the arguments of Lecture 19, we deduce that there is a PD isotopy $\{g_t : \partial K \to S^m\}_{t \in [0,1]}$ such that $g_0 = f_0$ and $\pi \circ g_1^{-1} : S^m \to \mathbb{R}$ has 0 as a regular value. This map decomposes the sphere S^m into two smooth submanifolds

$$D_- = \{g_1 \pi^{-1} \mathbb{R}_{\leq 0}\} \qquad D_+ = \{g_1 \pi^{-1} \mathbb{R}_{\geq 0}\}.$$

The map g_1 provides PD homeomorphisms of D_- and D_+ with PL m-disks.

We will now use the product smoothing theorem for $(m-1)$-manifolds (which we may assume as an inductive hypothesis) to verify the following:

Lemma 3. *Let $X = [0,1]^m$ be a PL m-disk. Then, up to PD isotopy and X has a unique smooth structure (in other words, there are no exotic smooth structures on PL m-disks).*

Proof. Smoothing theory tells us that smooth structures on X are classified by the following homotopy-theoretic data:

(a) Solutions to the lifting problem

$$\begin{array}{ccc} & & BO(m) \\ & \nearrow & \downarrow \\ X & \longrightarrow & BPL(m). \end{array}$$

(b) Solutions to the induced lifting problem

$$\begin{array}{ccc} & & BO(m-1) \\ & \nearrow & \downarrow \\ \partial X & & BO(m) \times_{BPL(m)} BPL(m-1). \end{array}$$

Using Theorem 1 in dimensions $< m$, we deduce that $PL(m)/O(m)$ is connected. Since X is contractible, problem (a) has a unique solution up to homotopy. Solutions to problem (b) can be described as sections of a fibration $\phi : \widetilde{\partial X} \to \partial X$ whose fibers are homotopy fibers of the map $PL(m-1)/O(m-1) \to PL(m)/O(m)$. Invoking Theorem 1 again (in dimension $m-1$), we deduce that these fibers are $(m-1)$-connected. Since ∂X has dimension $(m-1)$, the fibration ϕ has a unique section up to homotopy. □

Returning to our problem, we deduce that the smooth submanifolds $D_-.D_+ \subseteq S^m$ are diffeomorphic to smooth disks. We now need the following:

Lemma 4. *Let $B(1)$ denote the open unit ball in $\mathbb{R}^m$, and suppose we are given a smooth orientation-preserving embedding $i : B(1) \to S^m$. Then i is isotopic to the standard embedding.*

We can identify S^m with the one-point compactification of $\mathbb{R}^m$. Without loss of generality, we may assume that the image of i does not contain the point at infinity (since i is not surjective, we can always reduce to this situation by applying a rotation of the sphere S^m). Then Lemma 4 is an immediate consequence of the following:

Lemma 5. *Let $B(1)$ denote the open unit ball in $\mathbb{R}^m$, and let $i : B(1) \to \mathbb{R}^m$ be a smooth orientation-preserving embedding. Then i is isotopic to the standard embedding.*

Proof. Applying a translation of $\mathbb{R}^m$, we can arrange that $i(0) = 0$. Acting by a linear map, we can arrange that the derivative of i is equal to zero near the origin (since i is orientation preserving, this linear map can be chosen to lie in the identity component of $GL(n, \mathbb{R})$). Define a smooth homotopy $\{i_t : B(1) \to \mathbb{R}^m\}$ from $i_0 = i$ to the standard inclusion by the formula $i_t(x) = (1-t)i(x) + tx$. This map is generally not an isotopy. However, it is an isotopy near 0, and therefore on a ball $B(\epsilon)$ for ϵ sufficiently small. Let $j : B(1) \to \mathbb{R}^m$ be the map given by $j(x) = \frac{i(\epsilon x)}{\epsilon}$. Then i_t determines an isotopy from j to the standard embedding. Moreover, i is isotopic to j, since we have a smooth family of maps

$$\{j_t(x) = \frac{i(tx)}{t}\}_{t \in [\epsilon, 1]}.$$

□

Remark 6. We can carry out a version of the proof of Lemma 5 with parameters, given an appropriate generalization of the condition that i be orientation-preserving (we need to be able to arrange that the derivative of i is the identity near the origin). This argument can be used to prove the following fact: any smooth microbundle contains an (essentially unique) smooth disk bundle. This is the key difference between the smooth and PL categories: a PL microbundle always contains a PL $\mathbb{R}^n$-bundle, but this generally cannot be refined to a PL disk bundle.

We now return to the proof of Proposition 2. Lemma 4 implies that we can adjust the PD isotopy $\{g_t\}$ by a smooth isotopy of S^m to arrange that D_- (and therefore D_+) can be identified with the standard disks in S^m. It follows that $D_- \cap D_+$ is the standard equator $S^{m-1} \subseteq S^m$, given by the zero locus of the projection $\pi : S^m \hookrightarrow \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ onto the last factor. In other words, we can assume that $\pi \circ g_1^{-1}$ coincides with π on S^{m-1}. Using the uniqueness of smooth collars, we may further adjust our isotopy so that $\pi \circ g_1^{-1}$ coincides with π on a neighborhood $\pi^{-1}(-\epsilon, \epsilon)$ of S^{m-1}.

We now define a PD isotopy $\{f_t : K \to B(1)\}_{t \in [0,1]}$ by the formula

$$f_t(sx) = \begin{cases} f(sx) & \text{if } s \leq \frac{1}{2} \\ sg_{tt'}(x) & \text{if } s = \frac{1}{2} + \frac{t'}{6}, 0 \leq t' \leq 1 \\ sg_1(x) & \text{if } \frac{4}{6} \leq s \leq \frac{5}{6} \\ sg_{tt'}(x) & \text{if } s = 1 - \frac{t'}{6}, 0 \leq t' \leq 1. \end{cases}$$

We can then replace $f = f_0$ by f_1 in the statement of Proposition 2. Replacing K by $\frac{5}{6}K$ and multiplying f by $\frac{6}{5}$, we are reduced to proving the following analogue of Proposition 2:

Proposition 7. *Let $K \subseteq \mathbb{R}^m \times \mathbb{R}$ be a polyhedron which is the closed star of the origin 0 with respect to some PL triangulation of $\mathbb{R}^{m+1}$ (so that K is the cone on ∂K, with the origin as the cone point), let $\pi : K \to \mathbb{R}$ denote the projection onto the last factor. Let $f : K \to \mathbb{R}^{m+1}$ be a PD embedding satisfying the following conditions:*

(1) *The image of f is the unit ball $B(1) \subseteq \mathbb{R}^{m+1}$.*

(2) *For $\frac{4}{5} \leq t \leq 1$ and $x \in \partial K$, we have $f(tx) = tf(x)$.*

(3) *The projection π is injective when restricted to the vertices of K (with respect to some PL triangulation), so that $\pi \circ f^{-1}$ is regular on the interior of the unit ball except possibly at the origin.*

(4) *The maps π and $\pi \circ f^{-1}$ coincide on $S^m \cap \pi^{-1}(-\epsilon, \epsilon) \subseteq B(1)$ for some $\epsilon > 0$.*

We will prove Proposition 7 in the next lecture, thereby completing the proof of the product structure theorem.

Product Structure Theorem: End of the Proof (Lecture 22)

March 31, 2009

We continue our proof of the product structure theorem for smooth structures on PL manifolds. Recall that we are reduced to proving the following:

Proposition 1. *Let $K \subseteq \mathbb{R}^m \times \mathbb{R}$ be a polyhedron which is the closed star of the origin 0 with respect to some PL triangulation of $\mathbb{R}^{m+1}$ (so that K is the cone on ∂K, with the origin as the cone point), let $\pi : K \rightarrow \mathbb{R}$ denote the projection onto the last factor. Let $f : K \rightarrow \mathbb{R}^{m+1}$ be a PD embedding satisfying the following conditions:*

(1) *The image of f is the unit ball $B(1) \subseteq \mathbb{R}^{m+1}$ and $f(0) = 0$.*

(2) *For $.8 \leq t \leq 1$ and $x \in \partial K$, we have $f(tx) = tf(x)$.*

(3) *The projection π is injective when restricted to the vertices of K (with respect to some PL triangulation), so that $\pi \circ f^{-1}$ is regular on the interior of the unit ball except possibly at the origin.*

(4) *The map $\pi \circ f$ coincides with π on $\pi^{-1}(-\epsilon, \epsilon) \cap S^m$ for ϵ sufficiently small.*

(5) *The map f is PL in a neighborhood of the origin.*

Then, after modifying f by a PD isotopy which is trivial on ∂K, we can arrange that $\pi \circ f^{-1}$ is regular on the interior of the unit ball.

Replacing f by its restriction to tK for t close to 1, we can assume that $\pi \circ f$ is regular on $B(1) - \{0\}$. Let $C_0 = \partial K \cap \pi^{-1}[-\epsilon, \epsilon]$ and let $C = [.8, 1] \times C_0 \subseteq K$. Conditions (4) and (2) guarantee that $\pi | C = (\pi \circ f) | C$. Let $D \subseteq K$ be a PL neighborhood of the origin on which f is PL. Choose a triangulation S of K with the following properties:

(1) The subpolyhedra C and D of K are unions of simplices.

(2) The map π is injective on the vertices of K.

Let Lf denote the linearized version of f with respect to the triangulation S (that is, the unique map which is linear on each simplex of S and which agrees with f on vertices). Choose a PL function $\chi : K \rightarrow [0, 1]$ such that $\chi = 1$ on $.8K$ and $\chi = 0$ on $[.9, 1] \times \partial K$, and define a homotopy $\{f_t : K \rightarrow \mathbb{R}^{m+1}\}$ by the formula

$$f_t(x) = t\chi(x) Lf(x) + (1 - t\chi(x)) f(x).$$

We have seen that if S is a sufficiently fine triangulation, then f_t is a PD isotopy from f to f_1, where f_1 is a map which is PL on $.8K$ and agrees with f on $[.9, 1] \times \partial K$. Since f is already PL on D, we have $f = f_1$ on D, so that $\pi \circ f_1^{-1}$ is regular on $f_1(D - \{0\})$. Similar reasoning shows that $\pi \circ f_1 = \pi \circ f = \pi$ on $C \subseteq K$. Choosing S sufficiently fine, we can arrange that f_1 is an arbitrarily close approximation to f (in the C^1-sense). In particular, we can arrange that:

(a) The map $\pi \circ f_1^{-1}$ is regular on $B_1 - f_1(D)$ (and therefore on $B(1) - \{0\}$).

(b) For every point $x \in f_1(.8K)$, we have $tx \in f_1(.8K)$ for $0 \leq t \leq 1$ (since $f_1(.8K)$ closely approximates $f(.8K)$, which is the ball $B(.8)$).

(c) For $x \notin C$, we have $|(\pi \circ f)(x)| \geq \frac{\epsilon}{2}$.

We define another map $f_2 : K \to \mathbb{R}^{m+1}$ so that for $x \in \partial K$, we have

$$f_2(tx) = \begin{cases} f_1(tx) & \text{if } .8 \leq t \leq 1 \\ \frac{t}{.8} f(.8x) & \text{if } 0 \leq t \leq .8. \end{cases}$$

Using the assumption that $\pi \circ f_1^{-1}$ is regular on $B(1) - \{0\}$, it is easy to check that $\pi \circ f_2^{-1}$ is regular on $B(1) - \{0\}$ (if $v \in \mathbb{R}^{m+1}$ is a regular vector for $\pi \circ f_1^{-1}$ at a point $x \in f_1(.8K)$, then v is regular for $\pi \circ f_2^{-1}$ at tx for $t \in (0, 1]$). In order to proceed, we need to know the following:

Claim 2. *There exists a PD isotopy from f_1 to f_2, fixed near ∂K.*

In fact, there exists a PL isotopy from f_1 to f_2 which is supported on $.8K$. This is an obvious consequence of the following result:

Theorem 3 (The Alexander Trick)**.** *Let $\phi, \phi' : D^n \to D^n$ be two PL homeomorphisms from the PL n-disk to itself. If ϕ and ϕ' agree on the boundary ∂D^n, then ϕ is PL isotopic to the identity.*

Composing with an inverse to ϕ', we are reduced to proving that if ϕ is the identity on ∂D^n, then ϕ is PL isotopic to the identity. We will give a proof in the topological category: the PL version of Theorem 3 can be established using a construction of the same flavor. Let us identify D^n with the unit ball $B(1) \subseteq \mathbb{R}^n$. We define an isotopy $\{\phi_t : B(1) \to B(1)\}$ by the formula

$$\phi_t(sx) = \begin{cases} sx & \text{if } t \leq s \\ t\phi(\frac{sx}{t}) & \text{if } t > s. \end{cases}$$

where $x \in \partial B(1)$. It is easy to see that ϕ_t is an isotopy from $\phi_0 = \text{id}$ to $\phi_1 = \phi$.

Remark 4. The Alexander trick does not work in the smooth category; the map described above exhibits essential nondifferentiable behavior when $t = 0$.

We now return to the proof of Proposition 1. Note that f_2 has the following properties:

- If $x \in C_0 \subseteq \partial K$, then $\pi f_2(x) = \pi(x)$.
- If $x \in \partial K - C_0$, then $|(\pi \circ f_2)(tx)| \geq \frac{t\epsilon}{2}$.

We are free to replace f by f_2. Since $\pi \circ f_2^{-1}$ is regular away from the origin, we are free to replace K by any smaller neighborhood of the identity. In particular, we can replace K by the star of the origin with respect to some triangulation of $.8K$ with respect to which $f_2|.8K$ is PL. We are thereby reduced to proving the following version of Proposition 1

Proposition 5. *Let $K \subseteq \mathbb{R}^m \times \mathbb{R}$ be a polyhedron which is the closed star of the origin 0 with respect to some PL triangulation of $\mathbb{R}^{m+1}$ (so that K is the cone on ∂K, with the origin as the cone point), let $\pi : K \to \mathbb{R}$ denote the projection onto the last factor. Let $f : K \to \mathbb{R}^{m+1}$ be a PL embedding satisfying the following conditions:*

(1) *The image of f is the unit ball $B(1) \subseteq \mathbb{R}^{m+1}$ and $f(0) = 0$.*

(2) *The projection π is injective when restricted to the vertices of K.*

(3) *There exists a subpolyhedron $C_0 \subseteq \partial K$ and a constant ϵ such that $|\pi(tx)|, |\pi \circ f(tx)| \geq t\epsilon$ for $x \notin C_0$.*

(4) *The maps* $\pi \circ f$ *and* π *agree on* C_0 *(and therefore on the cone* $\overline{C} = \{tx : x \in C_0, t \in [0,1]\}$*).*

Then, after modifying f *by a PD isotopy which is trivial on* ∂K*, we can arrange that* $\pi \circ f^{-1}$ *is regular on the interior of* $f(K)$*.*

We will construct a PD isotopy $\{f_t\}$ of f with the following properties:

(i) For every simplex σ of our triangulation of K, the $\{f_t|\sigma\}$ is a smooth isotopy from σ to $f(\sigma)$.

(ii) The isotopy $\{f_t\}$ is fixed on ∂K.

(iii) We have $\pi \circ f_1 = \pi$ in a neighborhood of the origin.

Since π is injective on the vertices of K, the map $\pi \circ f_1^{-1}$ will automatically be regular on the interior of K except possibly at the origin; condition (iii) will guarantee regularity at the origin as well. It therefore suffices to construct $\{f_t\}$. Since π is injective on the vertices of K, the set V of vertices of ∂K can be partitioned into two subsets $V_+ = \{v \in V : \pi(v) > 0\}$ and $V_- = \{v \in V : \pi(v) < 0\}$. Refining our triangulation of ∂K if necessary, we may assume that every simplex τ of ∂K which contains vertices from both V_+ and V_- belongs to C_0. For each simplex τ of ∂K, let $\widehat{\tau}$ denote the cone of this simplex (with cone point the origin). We construct the isotopies $\{f_t|\widehat{\tau}\}$ one simplex at a time. If τ is a simplex of C_0, then we let $\{f_t|\widehat{\tau}\}$ be the trivial isotopy (this satisfies (iii) since f satisfies (4)). Otherwise, we may assume without loss of generality that each vertex v of τ belongs to V_+. Let $v_1, \ldots, v_k$ be the vertices of τ. There exist positive constants $\{a_i\}_{1 \leq i \leq k}$ such that $\pi(v_i) = a_i(\pi \circ f)v_i$. We define a homotopy $\{g_t : \widehat{\sigma} \rightarrow \mathbb{R}_{\geq 0} f(\widehat{\sigma})\}$ by the formula

$$g_t(\lambda_1 v_1 + \ldots + \lambda_k v_k) = \sum \lambda_k f(v_i)(ta_i + (1-t)).$$

Then g_t is a homotopy from $f|\widehat{\tau} = g_0$ to a map g_1 satisfying $\pi \circ g_1 = \pi$. Note that g_t carries a neighborhood of the origin in $\widehat{\tau}$ into $f(\widehat{\tau})$. Using a relative version of the smooth isotopy extension theorem, we can find an isotopy $\{f_t|\widehat{\tau} \rightarrow f(\widehat{\tau})\}$ which is supported in a compact subset of $\widehat{\tau} - \tau$, agrees with g_t near the origin, and agrees with the isotopies we have already constructed on the cone of $\partial \tau$.

Comparison of Smooth and PL Structures (Lecture 23)

April 3, 2009

In this lecture, we will attempt to prove that the theories of smooth and PL manifolds are equivalent. In view of the smoothing theory we have already developed, this is equivalent to the assertion that the spaces $PL(n)/O(n)$ are contractible for $n \geq 0$. We will attempt to prove this using induction on n. Of course, this attempt is doomed to failure, since there are PL manifolds which cannot be smoothed and PL manifolds which admit inequivalent smooth structures (such as Milnor's exotic spheres).

Let us assume that the space $PL(n-1)/O(n-1)$ is contractible, and attempt to prove that $PL(n)/O(n)$ is contractible. Consider the map

$$\phi : PL(n-1)/O(n-1) \to PL(n)/O(n).$$

The product smoothing theorem implies that all the homotopy fibers of ϕ are $(n-1)$-connected. In particular, they are connected, so that $PL(n)/O(n)$ is connected. Hence ϕ really only has one homotopy fiber up to equivalence, which can be identified with the loop space $\Omega PL(n)/O(n)$. Since this loop space is $(n-1)$-connected, we have proven the following:

Lemma 1. *If $PL(n-1)/O(n-1)$ is contractible, then $PL(n)/O(n)$ is n-connected.*

Consequently, $PL(n)/O(n)$ is contractible if and only if the loop space $\Omega^{n+1} PL(n)/O(n)$ is contractible. Let us try to understand this loop space.

First, consider the loop space $\Omega^n PL(n)/O(n)$. Let D^n be an n-dimensional disk in the PL setting, and equip the boundary ∂D^n with its standard smooth structure. Smoothing theory implies that the space of smoothings of D^n (compatible with our given smoothing on the boundary) can be identified the space of solutions to the lifting problem

$$\begin{array}{ccc} \partial D^n & \longrightarrow & BO(n) \\ \downarrow & \nearrow & \downarrow \\ D^n & \longrightarrow & BPL(n). \end{array}$$

Since the horizontal maps are constant (the disk D^n has trivial tangent microbundle in both the smooth and PL settings), this space of solutions can be identified with $\Omega^n PL(n)/O(n)$.

When we loop the space $\Omega^n PL(n)/O(n)$ one more time, we encounter not classifying spaces of smooth structures but classifying spaces for their automorphisms. More precisely, let $\mathrm{Diff}(D^n, \partial)$ denote the space of diffeomorphisms of the standard smooth disk D^n which are the identity near the boundary ∂D^n, and let $\mathrm{Homeo}_{PL}(D^n, \partial)$ be defined similarly. Then the spaces $B\,\mathrm{Diff}(D^n, \partial)$ and $B\,\mathrm{Homeo}_{PL}(D^n, \partial)$ can be identified with connected components of the classifying for smooth and PL manifolds which are bounded by the sphere S^{n-1}. As we have seen, there is a map (well-defined up to homotopy) $B\,\mathrm{Diff}(D^n, \partial) \to B\,\mathrm{Homeo}_{PL}(D^n, \partial)$. Denote the homotopy fiber of this map by $\mathrm{Homeo}_{PL}(D^n, \partial)/\mathrm{Diff}(D^n, \partial)$, so that $\mathrm{Homeo}_{PL}(D^n, \partial)/\mathrm{Diff}(D^n, \partial) \simeq \Omega^n PL(n)/O(n)$. We have a fibration sequence

$$\mathrm{Diff}(D^n, \partial) \to \mathrm{Homeo}_{PL}(D^n, \partial) \to \mathrm{Homeo}_{PL}(D^n, \partial)/\mathrm{Diff}(D^n, \partial).$$

Lemma 2. *The space $\mathrm{Homeo}_{PL}(D^n, \partial)$ is contractible.*

Lemma 2 is just an articulation of the Alexander trick, which we described in the last lecture: every PL homeomorphism of D^n which is the identity on the boundary is canonically isotopic to the identity.

It follows from Lemma 2 that we can identify $\mathrm{Diff}(D^n, \partial)$ with the loop space of $\mathrm{Homeo}_{PL}(D^n, \partial)/\mathrm{Diff}(D^n, \partial)$, and therefore with $\Omega^{n+1} PL(n)/O(n)$. We have proven:

Proposition 3. *Assume that $PL(n-1)/O(n-1)$ is contractible. Then $PL(n)/O(n)$ is contractible if and only if $\mathrm{Diff}(D^n, \partial)$ is contractible: in other words, if and only if the Alexander trick works in the smooth category.*

We can massage the criterion of Proposition 3 further. Let S^n denote the n-sphere, and choose a point $x \in S^n$. We can identify D^n with the submanifold obtained from S^n by removing the interior of a small disk around x. We have seen that, in the smooth category, this small disk is determined up to contractible ambiguity (this is not true in the PL category). Here is another way to articulate this idea: given a point $x \in S^n$, we can define a new smooth manifold M by forming the *real blow-up* of S^n at x. Namely, we let $M = (S^n - \{x\}) \coprod (T_{S^n,x} - \{0\})/\mathbb{R}_{>}$ be the space obtained from S^n by replacing the point x by the collection of all directed rays in the tangent space $T_{S^n,x}$. Then M has the structure of a smooth manifold, which depends functorially on the pair (S^n, x). This smooth manifold is simply a smooth n-disk D^n. Moreover, this construction determines an isomorphism of $\mathrm{Diff}(D^n, \partial)$ with the group $\mathrm{Diff}(S^n, \{x\})$ of diffeomorphisms of S^n which coincide with the identity near $\{x\}$. Thus:

Proposition 4. *Assume that $PL(n-1)/O(n-1)$ is contractible. Then $PL(n)/O(n)$ is contractible if and only if $\mathrm{Diff}(S^n, \{x\})$ is contractible.*

Let $\mathrm{Diff}_x(S^n)$ denote the group of diffeomorphisms ϕ of S^n which satisfy $\phi(x) = x$. We have a homotopy fiber sequence

$$\mathrm{Diff}(S^n, \{x\}) \to \mathrm{Diff}_x(S^n) \to G$$

where G denotes the monoid of equivalences from the smooth microbundle of S^n at x. Since a smooth microbundle is canonically determined by its tangent space along the zero section, this gives us a fiber sequence

$$\mathrm{Diff}(S^n, \{x\}) \to \mathrm{Diff}_x(S^n) \xrightarrow{\theta} GL_n(\mathbb{R})$$

where θ is given by differentiation at x. It follows that $\mathrm{Diff}(S^n, \{x\})$ is contractible if and only if θ is a homotopy equivalence.

Note that the group $O(n)$ acts on S^n by diffeomorphisms fixing the point x. We have a commutative diagram

$$\begin{array}{ccccc} & & \mathrm{Diff}_x(S^n) & & \\ & \overset{\theta'}{\nearrow} & & \overset{\theta}{\searrow} & \\ O(n) & & \xrightarrow{\theta''} & & GL_n(\mathbb{R}). \end{array}$$

Since θ'' is a homotopy equivalence, we deduce that θ is a homotopy equivalence if and only if θ' is a homotopy equivalence. In other words:

Proposition 5. *Assume that $PL(n-1)/O(n-1)$ is contractible. Then $PL(n)/O(n)$ is contractible if and only if the inclusion $O(n) \to \mathrm{Diff}_x(S^n)$ is a homotopy equivalence.*

The group $\mathrm{Diff}(S^n)$ acts on S^n. This gives rise to homotopy fiber sequences

$$\begin{array}{ccccc} O(n) & \longrightarrow & O(n+1) & \longrightarrow & S^n \\ \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \psi} & & \downarrow \\ \mathrm{Diff}_x(S^n) & \longrightarrow & \mathrm{Diff}(S^n) & \longrightarrow & S^n. \end{array}$$

It follows that θ' is a homotopy equivalence if and only if ψ is a homotopy equivalence. This proves the following:

Theorem 6. *Assume that $PL(n-1)/O(n-1)$ is contractible. Then $PL(n)/O(n)$ is contractible if and only if the inclusion $O(n+1) \to \mathrm{Diff}(S^n)$ is a homotopy equivalence.*

Example 7. The conditions of Theorem 6 are satisfied when $n = 1$: the space $PL(0)/O(0)$ is obviously contractible, while $O(2) \simeq \mathrm{Diff}(S^1)$ by the arguments given on the first day of class. This proves that the theory of smooth and PL manifolds are the same in dimension 1.

Example 8. To apply Theorem 6 when $n = 2$, we must show that $O(3) \simeq \mathrm{Diff}(S^2)$. This is a theorem of Smale, which we will prove in the next lecture.

Example 9. Theorem 6 also applies when $n = 3$. For this, we need to show that $O(4) \simeq \mathrm{Diff}(S^3)$. This assertion is known as the *Smale conjecture*. It was proven by Hatcher, but we will not present the details in class.

Example 10. It is unknown (at least by me) whether Theorem 6 applies when $n = 4$. This is equivalent to the question of whether the inclusion $O(5) \to \mathrm{Diff}(S^4)$ is a homotopy equivalence. Even the simplest consequence of this assertion is a difficult open question: is every orientation-preserving diffeomorphism of S^4 isotopic to the identity?

Remark 11. Even if the second hypothesis of Theorem 6 fails, the contractibility of $PL(n-1)/O(n-1)$ still has powerful consequences for the theory of n-manifolds. Namely, it implies that $PL(n)/O(n)$ is n-connected (Lemma 1). The smooth structures on a PL n-manifold M are classified by sections of a fibration over M with fiber $PL(n)/O(n)$. Since M is n-dimensional and these fibers are n-connected, we deduce that the space of sections is nonempty and connected: in other words, M admits a smooth structure which is unique up to PD isotopy. We can proceed further to argue that $PL(n+1)/O(n+1)$ must again be n-connected, so that every PL $(n+1)$-manifold M admits a compatible smooth structure (though we will not know that this smooth structure is unique).

For example, our present state of knowledge is enough to guarantee that every PL 2-manifold can be smoothed in an essentially unique way, and that every PL 3-manifold admits a smoothing. After we prove Smale's theorem in the next lecture, we will know that PL 3-manifolds admit essentially unique smoothings, and that PL 4-manifolds can be smoothed. Assuming the Smale conjecture, we can go further to say that PL 4-manifolds admit essentially unique smoothings, and that PL 5-manifolds can be smoothed.

These results are not optimal: as it turns out, PL manifolds of dimension ≤ 7 can be smoothed, and these smoothings are essentially unique in dimensions ≤ 6.

Diffeomorphisms of the 2-Sphere (Lecture 24)

April 6, 2009

The goal of this lecture is to compute the homotopy type of the diffeomorphism group of the 2-sphere S^2. The idea is to endow the 2-sphere with some additional structure (a conformal structure). We will show that this structure is essentially unique, and it will follow that the diffeomorphism group $\mathrm{Diff}(S^2)$ is homotopy equivalent to the group of automorphisms which respect this additional structure. The latter group is finite dimensional and easy to describe.

Definition 1. Let M be a smooth manifold. A *(Riemannian) metric* on M consists of a positive definite inner product on each tangent space $T_{M,x}$ which varies smoothly with the chosen point $x \in M$. We will denote the collection of Riemannian metrics on M by $\mathrm{Met}(M)$.

Given a metric g on M and a smooth function $\lambda : M \to \mathbb{R}_{>0}$, the product λg is another metric on M. We will say that two metrics g and g' are *conformally equivalent* if $g = \lambda g'$ for some smooth function $\lambda : M \to \mathbb{R}_{>0}$. The relation of conformal equivalence is an equivalence relation on $\mathrm{Met}(M)$; we will denote the set of equivalence classes by $\mathrm{Conf}(M)$.

There is a natural topology on $\mathrm{Met}(M)$ (we can identify $\mathrm{Met}(M)$ with an open subset of the Frechet space of all smooth sections of the bundle $\mathrm{Sym}^2 T_M^\vee$); we endow $\mathrm{Conf}(M)$ with the quotient topology.

Remark 2. The exact topologies that we place on $\mathrm{Met}(M)$ and $\mathrm{Conf}(M)$ are not really important in what follows: for our purposes it will be enough to work with the singular simplicial sets of $\mathrm{Met}(M)$ and $\mathrm{Conf}(M)$.

Lemma 3. *Let M be a smooth manifold. Then the spaces $\mathrm{Met}(M)$ and $\mathrm{Conf}(M)$ are contractible.*

Proof. The contractibility of $\mathrm{Met}(M)$ follows from the fact that it is a convex subset of a topological vector space. More concretely, choose a metric g_0 on M (such a metric can be constructed by choosing standard metrics on Euclidean charts and averaging them using a partition of unity). Then any other metric g on M can be joined to g_0 by a canonical path of metrics: we simply choose a straight line $g_t = (1-t)g_0 + tg$.

Let G denote the collection of smooth maps from M to $\mathbb{R}_{>0}$. We regard G as a group with respect to pointwise multiplication. The group G is contractible: again, it is a convex subset of the Frechet space of all smooth real-valued functions on M, so every function $f \in G$ is connected to the constant function 1 by a straight-line homotopy $f_t(x) = (1-t)f(x) + t$. The group G acts freely on $\mathrm{Met}(M)$ with quotient $\mathrm{Conf}(M)$. We therefore have a fibration sequence

$$G \to \mathrm{Met}(M) \to \mathrm{Conf}(M)$$

Since G and $\mathrm{Met}(M)$ are contractible (and the map $\mathrm{Met}(M) \to \mathrm{Conf}(M)$ is surjective), we conclude that $\mathrm{Conf}(M)$ is also contractible. □

A conformal structure on an n-manifold M can be thought of as a reduction of the structure group of the tangent bundle of M from $GL_n(\mathbb{R})$ to $\mathbb{R}_{\geq 0} \times O(n)$. If M is an oriented 2-manifold endowed with a conformal structure, then its tangent bundle has structure group reduced to $\mathbb{R}_{\geq 0} \times SO(2)$. If we choose an identification $\mathbb{R}^2 = \mathbf{C}$ (endowing the latter with its standard notion of length), then we can identify $\mathbb{R}_{\geq 0} \times SO(2)$ with the group $\mathbf{C}^*$ of nonzero complex numbers, acting on $\mathbf{C}$ by conjugation. In other words, an orientation of M together with a conformal structure on M give us a reduction of the structure group of M from $GL_2(\mathbb{R})$ to $GL_1(\mathbf{C})$: that is, they give an almost complex structure on M.

Theorem 4 (Existence of Isothermal Coordinates)**.** *Let M be a 2-manifold equipped with an almost complex structure. Then M is a complex manifold: in other words, near each point $x \in M$ we can choose an open neighborhood U and an open embedding $U \hookrightarrow \mathbf{C}$ of almost complex structures.*

Remark 5. In the situation of Theorem 4, suppose that we think of the almost complex structure on M as being given by an orientation together with a conformal structure, where the latter is given by some metric g on M. The assertion of Theorem 4 is equivalent to the assertion that we can choose local coordinate systems on M in which g is *conformally flat*: that is, it has the form λg_0 where g_0 denotes the standard metric on $\mathbb{R}^2 \simeq \mathbf{C}$.

Remark 6. Theorem 4 is a consequence of the Newlander-Nirenberg theorem, which asserts that an almost complex structure on a manifold M is a complex structure if and only if a certain obstruction (called the Nijenhuis tensor) vanishes. When M has dimension 2, the vanishing of this tensor is automatic. However, Theorem 4 is much more elementary. Nevertheless, we will not give a proof here.

Now suppose that M is the 2-sphere S^2, which we regard as an oriented smooth manifold. Every choice of conformal structure $\eta \in \mathrm{Conf}(M)$ endows M with the structure of a complex manifold: that is, a Riemann surface.

Proposition 7. *Up to isomorphism, the 2-sphere S^2 admits a unique complex structure. That is, if X is a Riemann surface which is diffeomorphic to S^2, then X is biholomorphic to the Riemann sphere $\mathbf{C}\mathrm{P}^1$.*

Proof. Let $\mathcal{O}_X$ denote the sheaf of holomorphic functions on X. Since X is compact, it has a well-defined holomorphic Euler characteristic

$$\chi(\mathcal{O}_X) = \dim \mathrm{H}^0(X, \mathcal{O}_X) - \dim \mathrm{H}^1(X, \mathcal{O}_X).$$

This Euler characteristic can be computed using the Riemann-Roch theorem: it is given by $1 - g = \frac{\chi(X)}{2} = 1$, since X has genus 0. The space $\mathrm{H}^0(X, \mathcal{O}_X)$ consists of globally defined holomorphic functions on X. By the maximum principle (and the fact that X is compact), every such function must be constant, so that $\mathrm{H}^0(X, \mathcal{O}_X) \simeq \mathbf{C}$. It follows from the Euler characteristic estimate that $\mathrm{H}^1(X, \mathcal{O}_X)$ vanishes.

Now choose a point $x \in X$, and consider the sheaf $\mathcal{O}_X(x)$ of functions on X which are holomorphic except possibly at the point x, and have a pole of order at most 1 at x. We have an exact sequence of sheaves

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(x) \to x_* \mathbf{C} \to 0$$

Since the cohomology group $\mathrm{H}^1(X, \mathcal{O}_X)$ vanishes, we get a short exact sequence

$$0 \to \mathrm{H}^0(X, \mathcal{O}_X) \to \mathrm{H}^0(X, \mathcal{O}_X(x)) \to \mathrm{H}^0(X, x_* \mathbf{C}) \simeq \mathrm{H}^0(\{x\}, \mathbf{C}) \simeq \mathbf{C} \to 0.$$

This proves that $\mathrm{H}^0(X, \mathcal{O}_X(x))$ is 2-dimensional. In particular, there exists a nonconstant meromorphic function f on X having at most a simple pole at x. Since f cannot be holomorphic (otherwise it would be constant), it must have a pole of exact order 1 at x.

We can regard f as a holomorphic map $X \to \mathbf{C}\mathrm{P}^1$ with $f(x) = \infty$. Since f has unique simple pole at x, this map has degree 1 and is therefore an isomorphism of X with $\mathbf{C}\mathrm{P}^1$. □

Proposition 7 implies that the group $\mathrm{Diff}(S^2)$ acts transitively on the collection $\mathrm{Conf}(S^2)$ of conformal structures on S^2. We have a fiber sequence

$$\mathrm{Diff}_{\mathrm{Conf}}(S^2) \to \mathrm{Diff}(S^2) \to \mathrm{Conf}(S^2),$$

where $\mathrm{Diff}_{\mathrm{Conf}}(S^2)$ denotes the subgroup of $\mathrm{Diff}(S^2)$ consisting of diffeomorphisms which preserve the standard conformal structure on $S^2 = \mathbf{C}\mathrm{P}^1$. Since $\mathrm{Conf}(S^2)$ is contractible, we conclude that the inclusion $\mathrm{Diff}_{\mathrm{Conf}}(S^2) \subseteq \mathrm{Diff}(S^2)$ is a homotopy equivalence.

The group $\mathrm{Diff}_{\mathrm{Conf}}(S^2)$ can be written as a union $\mathrm{Diff}^+_{\mathrm{Conf}}(S^2) \cup \mathrm{Diff}^-_{\mathrm{Conf}}(S^2)$, where $\mathrm{Diff}^+_{\mathrm{Conf}}(S^2)$ denotes the subgroup of orientation preserving conformal diffeomorphisms of S^2 (that is, holomorphic automorphisms of $\mathbf{C}\mathrm{P}^1$), while $\mathrm{Diff}^-_{\mathrm{Conf}}(S^2)$ consists of orientation reversing conformal diffeomorphisms (antiholomorphic automorphisms).

Theorem 8. *The inclusion $O(3) \hookrightarrow \mathrm{Diff}(S^2)$ is a homotopy equivalence.*

Proof. It will suffice to show that the inclusion $O(3) \hookrightarrow \mathrm{Diff}_{\mathrm{Conf}}(S^2)$ is a homotopy equivalence. For this, we will show that $SO(3) \hookrightarrow \mathrm{Diff}^+_{\mathrm{Conf}}(S^2)$ is a homotopy equivalence. Both groups act transitively on the sphere S^2, giving rise to a map of fiber sequences

$$\begin{array}{ccccc} \mathrm{SO}(2) & \longrightarrow & \mathrm{SO}(3) & \longrightarrow & S^2 \\ \downarrow{\scriptstyle\theta} & & \downarrow & & \downarrow \\ G & \longrightarrow & \mathrm{Diff}^+_{\mathrm{Conf}}(S^2) & \longrightarrow & S^2, \end{array}$$

where G denotes the group of holomorphic automorphisms of $\mathbf{C}\mathrm{P}^1$ that preserve the point ∞. We will prove that θ is a homotopy equivalence.

Elements of G can be identified with biholomorphic maps $f : \mathbf{C}\mathrm{P}^1 \to \mathbf{C}\mathrm{P}^1$ carrying ∞ to itself. Such a map can be viewed as a meromophic function on $\mathbf{C}\mathrm{P}^1$ having a pole of order at most 1 at ∞. The collection of all such meromorphic functions forms a vector space which, by the proof of Proposition 7, has dimension 2. We can write down these meromorphic functions explicitly: they are precisely the maps of the form $z \mapsto az + b$, where $a, b \in \mathbf{C}$. Such a map determines an automorphism of $\mathbf{C}\mathrm{P}^1$ if and only if $a \neq 0$. Consequently, we can identify G with the product $\mathbf{C}^* \times \mathbf{C} = \{(a, b) \in \mathbf{C}^2 : a \neq 0\}$. The map θ has image $S^1 = \{(a, b) \in \mathbf{C}^2 : |a| = 1, b = 0\}$. It is now clear that θ is a homotopy equivalence. □

Remark 9. The automorphism group $\mathrm{Diff}^+_{\mathrm{Conf}}(S^2)$ can be identified with $PGL_2(\mathbf{C})$, which contains $SO(3)$ as a maximal compact subgroup.

We can use the same methods to compute the diffeomorphism group of a surface of genus 1. Such a surface looks like a torus $T = \mathbb{R}^2 / \mathbf{Z}^2$. This description of T as a quotient makes it evident that two different groups act on T:

(*i*) The group T acts on itself by translations.

(*ii*) The group $\mathrm{GL}_2(\mathbf{Z})$ acts on T.

These group actions are in fact compatible with one another, and give a rise to a map $G \to \mathrm{Diff}(T)$, where G denotes the semidirect product of T with $\mathrm{GL}_2(\mathbf{Z})$.

Proposition 10. *The map $G \to \mathrm{Diff}(T)$ is a homotopy equivalence.*

The proof proceeds in several steps.

(*a*) The groups G and $\mathrm{Diff}(T)$ both act transitively on T. It will therefore suffice to show that we have a homotopy equivalence $G_0 \to \mathrm{Diff}_0(T)$, where G_0 and $\mathrm{Diff}_0(T)$ denote the subgroups of G and $\mathrm{Diff}(T)$ consisting of maps which fix the origin $0 \in T$. In other words, we must show that the inclusion $\phi : \mathrm{GL}_2(\mathbf{Z}) \to \mathrm{Diff}_0(T)$ is a homotopy equivalence.

(*b*) The map ϕ has an obvious splitting, since $\mathrm{Diff}_0(T)$ maps to $\mathrm{GL}_2(\mathbf{Z})$ via its action on the homology group $\mathrm{H}_1(T; \mathbf{Z})$. It will therefore suffice to show that $\mathrm{Diff}_1(T)$ is contractible, where $\mathrm{Diff}_1(T)$ denotes the group of diffeomorphisms of T which fix the origin 0 and act trivially on the homology of T.

(*c*) The group $\mathrm{Diff}_1(T)$ does not act transitively on $\mathrm{Conf}(T)$. However, it does act freely: suppose that we fix a point of $\mathrm{Conf}(T)$, which endows T with a complex structure. The fixed point $0 \in T$ endows T with the structure of an *elliptic curve*. In particular, it acquires a canonical group structure. If we let $\mathfrak{t}$ denote the (complex) Lie algebra of T at the origin, then we get an exponential map $\mathfrak{t} \to T$ which exhibits T as a quotient $\mathfrak{t}/\Lambda$. Any element f of $\mathrm{Diff}_1(T)$ which preserves the conformal structure must act by a group automorphism of T (since it is complex analytic and fixed the origin), and is therefore determined by its derivative $df : \mathfrak{t} \to \mathfrak{t}$. Since f is required to act trivially on $\mathrm{H}_1(T; \mathbf{Z}) \simeq \Lambda$, we deduce that $df = \mathrm{id}$ so that $f = \mathrm{id}$.

(*d*) We now have a fiber sequence

$$\mathrm{Diff}_1(T) \to \mathrm{Conf}(T) \to M,$$

where $M = \mathrm{Conf}(T)/\mathrm{Diff}_1(T)$ can be thought of as a moduli space for genus 1 Riemann surfaces Σ equipped with a marked point and an oriented trivialization $\mathrm{H}_1(\Sigma, \mathbf{Z}) \simeq \mathbf{Z}^2$. Again, any such Σ must be an elliptic curve and therefore has the form V/Λ, where V is the tangent space to Σ at the origin (a 1-dimensional complex vector space) and $\Lambda \subseteq V$ is a lattice. Our trivialization $\mathbf{Z}^2 \simeq \mathrm{H}_1(\Sigma, \mathbf{Z})$ gives an oriented basis (u, v) for Λ, so a point of M can be identified with an isomorphism class of triples (V, u, v). Any such triple is uniquely isomorphic to $(\mathbf{C}, 1, \tau)$ (namely, the choice of an element u trivializes the vector space V), where τ is an element of the upper half plane $\{x + iy : y > 0\} \subseteq \mathbf{C}$. It follows that M is contractible. Since $\mathrm{Conf}(T)$ is also contractible, we deduce that $\mathrm{Diff}_1(T)$ is contractible, as desired.

We will give a different proof of Proposition 10 shortly.

Existence of Prime Decompositions (Lecture 25)

April 8, 2009

In this lecture, we begin our study of 3-manifolds. Our ultimate goal is to say something about the classification of 3-manifolds. To this end, we begin by considering an arbitrary compact 3-manifold M: what might it look like?

We observe that M can be written as a disjoint union of finitely many connected 3-manifold. Consequently, there is no harm in assuming (as we will from now on) that all of our 3-manifolds M are connected, so that $\pi_0 M \simeq *$. Consider now the fundamental group $\pi_1 M$. Let $\widetilde{M}$ denote the universal cover of M. If $\pi_1 M$ is finite, then $\widetilde{M}$ is a compact, simply connected 3-manifold. In this case, the structure of M is understood:

Theorem 1 (Perelman; Poincare Conjecture)**.** *Let $\widetilde{M}$ be a simply connected compact 3-manifold. Then $\widetilde{M} \simeq S^3$.*

The manifold M itself can be recovered as a quotient $S^3/\pi_1 M$, for some free action of the finite group $\pi_1 M$ on the 3-sphere S^3. There are a number of possibilities for what such an action can look like (for example, Lens spaces can be obtained via this construction); we will return to this point in a later lecture. For present purposes, we will regard these examples as "understood" and move on the case where the fundamental group $\pi_1 M$ is infinite.

If $\pi_1 M$ is infinite, the universal cover $\widetilde{M}$ is noncompact. It follows that $\mathrm{H}_3(\widetilde{M}; \mathbf{Z}) \simeq \mathrm{H}^0_c(\widetilde{M}; \mathbf{Z}) \simeq 0$, by Poincare duality. Since $\widetilde{M}$ is a simply connected space of dimension 3, we have two possibilties:

(i) The second homology group $\mathrm{H}_2(\widetilde{M}; \mathbf{Z})$ does not vanish. By the Hurewicz theorem, this group is isomorphic to $\pi_2 \widetilde{M} \simeq \pi_2 M$, so that there are nontrivial maps $S^2 \to M$.

(ii) The universal cover $\widetilde{M}$ is contractible, so that $M = \widetilde{M}/\pi_1(M)$ is homotopy equivalent to the classifying space $B\pi_1 M$.

Our goal in the next few lectures is to show that the study of 3-manifolds in general can be reduced to the case (ii). As a first step, we consider the prototypical example of 3-manifolds M which do *not* satisfy (ii). Let M_0 and M_1 be a pair of 3-manifolds containing points x and y. Let M'_0 and M'_1 denote the 3-manifolds with boundary S^2 obtained by removing small balls around x and y (or by performing real blow-ups at x and y). We denote the amalgam $M'_0 \coprod_{S^2} M'_1$ by $M_0 \# M_1$; this 3-manifold is called the *connect sum of* M_0 *and* M_1.

Warning 2. The connect sum $M_0 \# M_1$ depends not only on M_0 and M_1, but on a choice of identification of the boundaries $\partial M'_0 \simeq S^2 \simeq \partial M'_1$. This choice of identification only matters up to isotopy (if we are interested only in the diffeomorphism class of the connect sum $M_0 \# M_1$), but the space $\mathrm{Diff}(S^2) \simeq O(3)$ has two different connected components, as we saw in the last lecture. Note however that if M_0 and M_1 are oriented, then there is a unique isotopy class of identifications such that $M_0 \# M_1$ admits an orientation compatible with those of M_0 and M_1. For simplicity, we will restrict our attention to the oriented case.

The operation $\#$ is commutative and associative up to diffeomorphism. Moreover, it has a unit given by the 3-sphere S^3: we have $S^3 \# M \simeq M$ for any 3-manifold M.

Definition 3. Let M be a 3-manifold which is not a 3-sphere. We say that M is *prime*, for any decomposition $M \simeq M_0 \# M_1$, either M_0 or M_1 is diffeomorphic to S^3.

Our goal in this lecture is to prove the following:

Theorem 4. *Let M be a 3-manifold. Then M admits a decomposition*

$$M \simeq M_1 \# M_2 \# \cdots \# M_n$$

where each M_i is prime. (Here we allow the degenerate possibility that $n = 0$, in which case the expression on the right side means the 3-manifold S^3.)

In the next lecture, we will prove a theorem of Milnor which asserts that the prime factors M_i of M are unique up to diffeomorphism. For the moment, we will be content to prove the existence of a prime factorization asserted by Theorem 4.

Notation 5. Let M be a compact 3-manifold. The fundamental group $\pi_1 M$ is a finitely generated group. We let $n(M)$ denote the minimal number of generators for $\pi_1 M$. Note that $n(M) = 0$ if and only if $\pi_1 M \simeq *$, which (by virtue of the Poincare conjecture) is equivalent to the assertion that M is a 3-sphere.

We will prove Theorem 4 using induction on $n(M)$. If $n(M) = 0$, then $M \simeq S^3$ and there is nothing to prove. Similarly, if M is prime then we are done. Otherwise, we can write $M \simeq M' \# M''$ where M' and M'' are not diffeomorphic to S^3, so that $n(M'), n(M'') > 0$. If M' and M'' admit prime factorizations, then these together give a prime factorization of M. The existence of these prime factorizations follows immediately from the inductive hypothesis and the following:

Lemma 6. *For any pair of compact 3-manifolds M' and M'', we have $n(M' \# M'') = n(M') + n(M'')$.*

Remark 7. The proof of Theorem 4 sketched above depends on the Poincare conjecture. However, Theorem 4 was known long before the Poincare conjecture. To give a proof independent of the Poincare conjecture, special considerations are needed to show the existence of prime factorizations when $n(M) = 0$. We will not pursue the point further here.

To prove Lemma 6 we observe that since S^2 is simply connected, van Kampen's theorem implies that $\pi_1(M' \# M'')$ is the free product $\pi_1 M' \star \pi_1 M''$ of the groups M' and M''. The inequality $n(M' \# M'') \leq n(M') + n(M'')$ is obvious, since any if $\{g_i\}$ is a collection of generators for $\pi_1 M'$ and $\{h_j\}$ is a collection of generators for $\pi_1 M''$, then $\{g_i, h_j\}$ is a collection of generators for $\pi_1 M' \star \pi_1 M''$. The reverse inequality follows from the following:

Theorem 8 (Grushko)**.** *Let F be a finitely generated free group, and let $\phi : F \to G \star H$ be a surjection of groups. Then F can be decomposed as a free product $F_0 \star F_1$ so that ϕ is a free product of maps $\phi_0 : F_0 \to G$, $\phi_1 : F_1 \to H$.*

Remark 9. In the situation of Theorem 8, the groups F_0 and F_1 are automatically free (since they are subgroups of the free group F) and finitely generated (since the rank of F is the sum of the ranks of F_0 and F_1). Since ϕ is surjective, ϕ_0 and ϕ_1 are also surjective, so that $n(G) + n(H) \leq n(F_0) + n(F_1) = n(F)$, where $n(X)$ denotes the minimal number of generators for a group X.

We will describe a geometric proof of Grushko's theorem, due to Stallings. Let BG and BH denote classifying spaces for G and H, and consider the space $X = BG \vee BH$ obtained by gluing BG and BH together along a point which we will denote by $*$. By van Kampen's theorem we have $\pi_1 X = G \star H$. In fact, X is a classifying space $B(G \star H)$, though we will not need to know this.

Choose a system of generators $\{v_1, \ldots, v_k\}$ for the group F. We regard ϕ as a map $F \to \pi_1(X, *)$, so that each $\phi(v_i)$ is represented by a loop L_i from $*$ to itself in X. Without loss of generality, we can assume that L_i is a composition of finitely many loops $L_i = L_{i,0} \circ \ldots \circ L_{i,n_i}$ where each L_{i,n_i} belongs entirely to BG or to BH. Let Y denote the bouquet of circles $\vee_i S^1$, so that the maps $\{L_i\}_{1 \leq i \leq k}$ determine a continuous map $f : Y \to X$. Using the above formulas for L_i, we conclude that Y can be written as a union of subgraphs

$$Y_G \coprod_{Y_0} Y_H$$

where $f(Y_G) \subseteq BG$, $f(Y_H) \subseteq BH$, Y_0 is a finite number of points, and $f(Y_0) = \{*\}$.

To prove Theorem 8, we will construct the following:

(1) A homotopy equivalence $Y \hookrightarrow K$.

(2) A decomposition $K \simeq K_G \coprod_{K_0} K_H$ extending the decomposition $Y \simeq Y_G \coprod_{Y_0} Y_H$, where the topological space K_0 is a graph without loops (in other words, a union of finitely many trees) and therefore homotopy equivalent to finitely many points.

(3) A map $f' : K \to X$ extending f, which carries K_G into BG, K_H into BH, and K_0 to $*$.

so that the following condition is satisfied:

(4) The space K_0 is a tree.

Then $F \simeq \pi_1 Y \simeq \pi_1 K$ by (1), and the map $\phi : F \to G \star H$ can be identified with $f'_* : \pi_1 K \to \pi_1 X$ by (3). Using (4) and van Kampen's theorem, we deduce that $F = \pi_1 K \simeq \pi_1 K_G \star \pi_1 K_H$, and we will have the desired decomposition of F.

If Y_0 is connected, we can take $K = Y$ and there is nothing to prove. In the general case, we proceed in several steps. We first show that it is possible to construct the data described in (1), (2), and (3) so that the following weaker version of condition (4) holds:

(4′) There exist two different connected components of Y_0 which belong to the same component of K_0.

If we can satisfy this condition, we then replace Y by K and repeat the same argument. The cardinality of the sets $\pi_0 K_0$ will form a decreasing chain as we proceed, and must eventually stabilize to the case where K_0 is connected (and therefore a tree, by virtue of (2).

Let $C_1, \ldots, C_m$ denote the set of path components of Y_0, and choose a point $y_i \in C_i$ for $1 \leq i \leq m$. Since Y is path connected, we can choose a path γ in Y from y_1 to y_2. Note that $f(\gamma)$ is a loop in X based at the point $*$, and therefore represents an element of $\pi_1 X$. Since ϕ is surjective, we can compose the original path γ with a loop based at y_1, and thereby arrange that $f(\gamma)$ is nullhomotopic. We have a homotopy

$$\gamma \simeq \gamma_1 \circ \ldots \circ \gamma_p$$

where each of the paths γ_i is supported entirely in Y_G or in Y_H and has endpoints in $\{y_1, \ldots, y_m\}$. We may assume that if any path γ_a begins and ends at the same point y_j, then $f(\gamma_a)$ is not nullhomotopic: otherwise, we can replace γ by the path

$$\gamma_1 \circ \ldots \circ \gamma_{a-1} \circ \gamma_{a+1} \circ \ldots \circ \gamma_p.$$

Concatenating the paths γ_a if necessary (and possibly swapping G with H), we can assume that γ_a is a path in Y_G when a is odd and a path in Y_H when a is even. We observe that each $f(\gamma_a)$ is a closed loop in X, so we have

$$1 = [f(\gamma_1)] \ldots [f(\gamma_p)] \in \pi_1 X \simeq G \star H.$$

Using the structure of the free product $G \star H$, we deduce that some factor $[f(\gamma_a)]$ must vanish. Without loss of generality, we may assume that γ_a is a path in Y_G from y_i to y_j; since $f(\gamma_a)$ is nullhomotopic we have $i \neq j$. Note that since the map $G \to G \star H$ is injective, the map $f(\gamma_a)$ is already nullhomotopic as a map in BG.

Let K_0 be the space obtained from Y_0 by adjoining a path τ from y_j to y_i, let $K_H = K_0 \coprod_{Y_0} Y_H$. We now let K_G be the space obtained from $K_0 \coprod_{Y_0} Y_G$ by attaching a 2-cell bounding the loop $\tau \circ \gamma_a$. Since $f(\gamma_a)$ is nullhomotopic in BG, we can extend f to a map $f'_G : K_G \to BG$ which takes the constant value $*$ on the path τ. Then f'_G and f determine a map $f' : K \to X$, which is easily seen to satisfy (1), (2), (3), and (4′).

Uniqueness of Prime Decompositions (Lecture 26)

April 13, 2009

In the last lecture, we introduced the notion of a *prime* 3-manifold, and showed that every 3-manifold can be obtained as a connected sum of prime factors. In this lecture, we will prove a theorem of Milnor which asserts that this decomposition is unique. We will assume for convenience that all of our 3-manifolds are connected and oriented.

Note that a prime 3-manifold need not have $\pi_2 M \simeq *$. For example, if $M = S^2 \times S^1$, then $\pi_2 M$ does not vanish, but M is prime (since $\pi_1 M$ cannot be factored nontrivially as a free product). However, this is essentially the only counterexample.

Definition 1. Let M be a 3-manifold which is not a 3-sphere. We will say that M Is *irreducible* if every embedded 2-sphere $S^2 \hookrightarrow M$ bounds a disk on one side or the other.

Remark 2. We say that an embedding $S^2 \hookrightarrow M$ is *separating* if $M - S^2$ is disconnected. Note that S^2 is separating if and only if its fundamental class $[S^2] \in \mathrm{H}_2(M; \mathbf{Z}/2\mathbf{Z})$ vanishes.

By definition, $M \neq S^3$ is prime if and only if every *separating* 2*-sphere* of M bounds a 3-disk. Consequently, every irreducible 3-manifold is prime. The product $S^2 \times S^1$ is an example of a prime 3-manifold which is not irreducible, but this example is unique (provided we stick to oriented 3-manifolds):

Proposition 3. *Let M be a compact, connected, oriented 3-manifold. Suppose that M contains a nonseparating 2-sphere S. Then M can be written as a connect sum $M_1 \# (S^2 \times S^1)$. In particular, if M is prime, then $M \simeq S^2 \times S^1$.*

Proof. Since S is nonseparating, there exists a loop L in M which intersects the 2-sphere S transversely in exactly one point. Let M_2' denote the union of a tubular neighborhood of L and a tubular neighborhood of S. Then the boundary of M_2' is equivalent to a connect sum of 2-spheres, so that $\partial M_2' \simeq S^2$. Let M_2 be the 3-manifold obtained by capping off this boundary 2-sphere with a disk. Then M_2 has the structure of an S^2-bundle over the loop L. Since M is orientable, this 2-sphere bundle must be trivial, so that $M_2 \simeq S^2 \times S^1$ and cutting along $\partial M_2'$ gives the desired connect sum decomposition of M. □

We now turn to the uniqueness of prime factorizations. Suppose that M is a compact connected 3-manifold and we have two prime decompositions

$$M_1 \# M_2 \# \cdots \# M_n \simeq M \simeq M_1' \# \cdots \# M_m'.$$

We will show that $m = n$ and that the diffeomorphism types of the prime factors agree up to a permutation. Our first step is to give a criterion which allows us to intrinsically detect if $S^2 \times S^1$ appears as a factor on one side. Note that if $M \simeq M' \# (S^2 \times S^1)$, then M contains a nonseparating 2-sphere. Conversely:

Proposition 4. *Let M be a compact, connected, oriented 3-manifold with a prime decomposition $M \simeq M_1 \# \ldots \# M_n$, and suppose that each M_i is irreducible. Then M contains no nonseparating 2-sphere.*

Proof. It will suffice to show that if M and N are 3-manifolds containing no nonseparating 2-spheres, then $M\#N$ likewise contains no nonseparating 2-sphere. Assume for a contradiction $M\#N$ contains a nonseparating 2-sphere S, and let T denote the separating 2-sphere given by the connect sum decomposition of $M\#N$. Without loss of generality we may assume that S and T meet transversely. Let k be the number of connected components of $S\cap T$, and assume that S has been chosen to minimize k. If $k=0$, then without loss of generality we have $S\subseteq M$. Since $[S]$ is nontrivial in $\mathrm{H}_2(M\#N)$, it is nontrivial in $\mathrm{H}_2(M-D^3)\simeq \mathrm{H}_2(M)$ so that S is a separating 2-sphere of M, contrary to our assumption.

We may therefore assume that $k>0$. Regard the intersection $S\cap T$ as a union of finitely many circles in $T\simeq S^2$. Choose an "innermost" circle $C\subseteq S\cap T$, so that C bounds a disk D in T whose interior does not intersect S. This circle also cuts S into 2-disks E_+ and E_-. Let $S_+ = D\cup E_+$ and $S_- = D\cup E_+$. Then $[S]=[S_+]+[S_-]\neq 0$, so that either S_+ or S_- is also a nonseparating 2-sphere in $M\#N$. Without loss of generality S_+ is nonseparating. Moving S_+ by a small isotopy, we can arrange that it intersects T in fewer than k components, contradicting the minimality of k. □

Returning to our decomposition

$$M_1\#M_2\#\cdots\#M_n\simeq M\simeq M'_1\#\cdots\#M'_m,$$

we deduce that if some $M_i\simeq S^2\times S^1$, then also some $M'_j\simeq S^2\times S^1$. Reordering the decompositions, we may assume $i=j=1$. We would like to assert that the complementary summands $M_2\#\ldots\#M_n$ and $M'_2\#\ldots\#M'_m$ are diffeomorphic. These complementary summands can be obtained by cutting M along nonseparating 2-spheres in the factors $S^2\times S^1$, and then capping of the resulting boundary spheres by disks. To prove that the resulting manifold is unique up to diffeomorphism, it suffices to prove the following:

Proposition 5. *Let M be a compact, connected, oriented 3-manifold containing a pair of nonseparating 2-spheres S and T. Then there is an (orientation preserving) diffeomorphism of M with itself that carries S to T.*

Proof. Moving S by an isotopy, we can assume that S and T meet transversely. We work by induction on the number k of connected components of $S\cap T$. If $k>0$, then we can write $[S]=[S_+]+[S_-]$ as before, so that either S_+ or S_- is a nonseparating k-sphere in M; without loss of generality, S_+ is nonseparating. Moving S_+ by a small isotopy, we can arrange that it is disjoint from S and intersects T in fewer than k components. Applying the inductive hypothesis, we obtain diffeomorphisms of M carrying S to S_+ and S_+ to T; the composition of these diffeomorphisms then does the job.

If $k=0$, then S and T are disjoint. Since $M-S$ is connected, $M-(S\cup T)$ has at most 2 components. Assume first that $M-(S\cup T)=N\coprod N'$, and let $\overline{N}$ and $\overline{N'}$ be the 3-manifolds obtained by capping off the boundary 2-spheres of N and N'. Since the orientation-preserving diffeomorphism group $\mathrm{Diff}^+(\overline{N})$ acts transitively on pairs of distinct points of $\overline{N}$, we can find a diffeomorphism which restricts to a diffeomorphism of N which exchanges the two boundary components. Similarly, we can find such a diffeomorphism of N. Modifying them by an isotopy if necessary (using the connectedness of $\mathrm{Diff}^+(S^2)$), we can assume that they glue to give a diffeomorphism of M which exchanges S and T.

The proof when $M-(S\cup T)$ is similar: we let $\overline{M}$ denote the 3-manifold obtained by capping off the boundary 3-spheres in $M-(S\cup T)$, and use the fact that $\mathrm{Diff}^+(\overline{M})$ acts transitively on quadruples of distinct points in $\overline{M}$. □

By repeatedly applying the above result, we are reduced to proving the uniqueness of prime decompositions

$$M_1\#M_2\#\cdots\#M_n\simeq M\simeq M'_1\#\cdots\#M'_m$$

in which each factor (on either side) is irreducible. Without loss of generality $n,m>1$ (otherwise $M=S^3$ or is irreducible, and there is nothing to prove). Let T be the separating 2-sphere of M corresponding to the decomposition $M'_1\#(M'_2\#\cdots\#M'_m)$. Similarly, we can choose nonintersecting 2-spheres $S_1,\ldots,S_{n-1}$ giving rise to the first decomposition. Without loss of generality, T meets $\bigcup S_i$ transversely in k circles. We assume that the system of spheres $\{S_i\}$ has been chosen to minimize k. If $k=0$, then T is contained in some M_i.

Since M_i is irreducible, T bounds a 3-disk in M_i. Let $\{M_{j_1}, \ldots, M_{j_k}\}$ denote the collection of those M_j which are attached to M_i via spheres contained in this 3-disk. Reindexing, we can assume that $j_1 = 1, \ldots, j_k = k$. Then T separates M into pieces $M_1 \# \ldots \# M_k$ and $M_{k+1} \# \ldots \# M_n$. It follows either that $k = 1$, $M_1 \simeq M'_1$, and $M_2 \# \ldots \# M_n \simeq M'_2 \# \ldots \# M'_m$, or that $k = n - 1$, $M_n \simeq M'_1$, and $M_1 \# \ldots \# M_{n-1} \simeq M'_2 \# \ldots \# M'_m$. In either case, we can conclude by induction that the prime factors agree up to reindexing.

If $k > 0$, then as before we can choose an innermost circle C in the intersection $(\bigcup S_i) \cap T$, so that C bounds a disk D in T which does not intersect any S_i; this disk is then contained in $M_j - B^3 \subseteq M$. Let $S = S_i$ be the the boundary sphere of $M_j - B^3$ containing C, so that C cuts S into two disks E_- and E_+. Let $S_+ = E_+ \cup D$. Then S_+ is a 2-sphere in M_j; since M_j is irreducible we conclude that S_+ bounds a 3-disk X. Replacing E_+ by E_- if necessary, we can assume that $B^3 \subseteq X$. By a small isotopy, we can arrange that S_+ does not intersect T along C. Replacing S_i by S_+, we obtain a system of spheres which intersects T in fewer than k circles, and can conclude by the inductive hypothesis.

Irreducibility and π_2 (Lecture 27)

April 15, 2009

In the last lecture, we introduced the notion of an irreducible 3-manifold: a 3-manifold M is said to be irreducible if every embedded 2-sphere in M bounds a disk (on exactly one side). Our stated motivation was that embedded 2-spheres were good candidates to represent nontrivial classes in $\pi_2 M$. Our first goal in this lecture is to show that this is indeed the case.

Proposition 1. *Let M be a 3-manifold, and let $S \hookrightarrow M$ be an embedded 2-sphere. The following conditions are equivalent:*

(1) *The sphere S bounds a disk in M.*

(2) *The sphere S represents a trivial class in $\pi_2 M$.*

Remark 2. The statement of Proposition 1 is a little sloppy: the homotopy group $\pi_2 M$ is really only well-defined after we have chosen a base point on M. If M is connected, then the groups $\pi_2(X, x)$ and $\pi_2(X, y)$ can be related by choosing a path from x to y, but the identification depends on this choice of path via the action of $\pi_1 M$ on $\pi_2 M$. This means that the class of S in $\pi_2 M$ is only well-defined up to the action of $\pi_1 M$; however, the condition that this class vanishes is invariant under the action of $\pi_1 M$ (the vanishing is equivalent to the requirement that $S \hookrightarrow M$ is homotopic to a constant map, ignoring the base points).

Proof. (In what follows, we do not assume that M is compact.) It is clear that if S bounds a disk, then S is nullhomotopic. Conversely, suppose that S is nullhomotopic. Suppose first that M is simply connected. Since $[S] = 0 \in \mathrm{H}_2(M; \mathbf{Z}/2\mathbf{Z})$, the 2-sphere S is separating (though the converse can fail in the noncompact setting); we can therefore write $M = M_0 \coprod_{S^2} M_1$ where M_0 and M_1 are 3-manifolds with 2-sphere boundary. We have an exact sequence

$$\mathrm{H}_2(S) \xrightarrow{j} \mathrm{H}_2(M_0) \oplus \mathrm{H}_2(M_1) \to \mathrm{H}_2(M)$$

(all homology computed with $\mathbf{Z}/2\mathbf{Z}$ coefficients). Since $[S]$ vanishes in $\mathrm{H}_2(M)$, we deduce that the class $([S], 0)$ lies in the image of j: in other words, either $([S], 0)$ or $(0, [S])$ vanishes. Assume the former, and let $\widehat{M_0}$ be the 3-manifold obtained from M_0 by capping off the boundary sphere. We have an exact sequence

$$\mathrm{H}_3(\widehat{M_0}) \to \mathrm{H}_2(S^2) \xrightarrow{i} \mathrm{H}_2(D^3) \oplus \mathrm{H}_2(M_0).$$

Since the map i is not injective, we deduce that $\mathrm{H}_3(\widehat{M_0})$ is nonzero. By Poincare duality (the simple connectivity of $\widehat{M_0}$ guarantees orientability), we deduce that $\mathrm{H}^0_c(\widehat{M_0})$ does not vanish, so that $\widehat{M_0}$ is a compact, simply connected 3-manifold. By the Poincare conjecture, $\widehat{M_0}$ is a 3-sphere, so that M_0 is a disk bounded by S.

Suppose now that M is not simply connected; we still have $M = M_0 \coprod_S M_1$ as above. Let $\widetilde{M}$ be a universal cover of M, and $\pi : \widetilde{M} \to M$ the projection map. Since S is simply connected, we can lift S to a 2-sphere $\widetilde{S}$ in $\widetilde{M}$. Since $\pi_2 M \simeq \pi_2 \widetilde{M}$, the sphere $\widetilde{S}$ is nullhomotopic and therefore bounds a disk. This disk might contain other preimages of S: however, by adjusting our choice of $\widetilde{S}$ we can arrange that $\widetilde{S}$ contains a disk D which intersects the inverse image of $\pi^{-1} S$ only in $\widetilde{S}$. It follows that $\pi(D) \subseteq M_0$ or $\pi_D \subseteq M_1$;

without loss of generality we may assume the former. The map π induces a local homeomorphism $D \to M_0$. Since D is compact, this local homeomorphism is proper, and is therefore a finite-sheeted covering space. Since the Euler characteristic of D is 1, this covering space has 1-sheet so that $M_0 \simeq D$ is a disk bounded by the sphere S, as required. $\square$

It follows that if a compact 3-manifold M is not irreducible, then $\pi_2 M$ does not vanish. We might ask if the converse is true: if $\pi_2 M$ is nonvanishing, does M fail to be irreducible? The answer is not obvious: the nonvanishing of $\pi_2 M$ guarantees a nontrivial homotopy class of map $i : S^2 \to M$, but the map i need not be an embedding. However, it turns out that the existence of nontrivial homotopy class guarantees the existence of an embedded 2-sphere with a nontrivial homotopy class, at least when M is oriented.

Theorem 3 (The Sphere Theorem). *Let M be an oriented 3-manifold, and suppose that $\pi_2 M$ is nontrivial. Then there exists an embedded 2-sphere $S \hookrightarrow M$ representing a nontrivial class in $\pi_2 M$. More generally, given any $\pi_1 M$-invariant normal subgroup $N \subset \pi_2 M$, there exists an embedded 2-sphere $S \hookrightarrow M$ representing an element of $\pi_2 M$ which does not belong to N.*

We will prove this theorem over the course of the next few lectures. The idea is to begin with an arbitrary map $i : S \to M$ representing a homotopy class which does not belong to N, and to adjust this map to make it an embedding. The same techniques will be used to prove the following companion to the sphere theorem:

Theorem 4 (The Loop Theorem). *Let M be a 3-manifold with boundary and let X be a boundary component of M. If N is a normal subgroup of $\pi_1 X$ which does not contain the kernel of the map $\pi_1 X \to \pi_1 M$, then there exists an embedding $(D^2, S^1) \to (M, X)$ such that the loop $S^1 \hookrightarrow X$ represents a class in $\pi_1 X$ which does not belong to N.*

Remark 5. The hypothesis of orientability in the sphere theorem is essential. If P denotes the 2-dimensional real projective space, then $P \times S^1$ is a nonorientable 3-manifold with $\pi_2(P \times S^1) \simeq \mathbf{Z}$, yet $P \times S^1$ does not contain any nontrivial embedded 2-spheres (it contains many *immersed* 2-spheres, given by the double covering $S^2 \to P$).

We now begin to pave the way for our proofs of the loop and sphere theorems by discussing the notion of a *general position* map from a surface S into a 3-manifold M. We will treat this notion informally and not give a precise definition: roughly speaking, a map $i : S \to M$ is in general position if the behavior of i satisfies all of the conditions we like that can be guaranteed by moving the map i by a small amount. In particular, any "singularities" of the map i can be assumed to appear in the expected codimension, which means they do not appear at all if the expected codimension is ≥ 3 (in S) or ≥ 4 (in M):

Assume therefore that we are given a smooth map $i : S \to M$. How can this map fail to be an embedding? There are essentially two things that can go wrong:

(i) The map i can be fail to be an immersion at a point $s \in S$. In other words, the derivative Di can fail to have rank 2 at s. The derivative Di_s takes values in the 6-dimensional space of linear maps $T_{S,s} \to T_{M,i(s)}$. A linear map of rank 1 is determined by specifying a 1-dimensional quotient Q of $T_{S,s}$ (the set of such choices forms a 1-dimensional space), a 1-dimensional subspace Q' of $T_{M,i(s)}$ (where we have a 2-dimensional space of choices), and a linear isomorphism $Q \simeq Q'$ (for which we have 1-dimensional space of choices); in total, we find that the space of maps having rank 1 is a manifold of dimension $1+2+1 = 4$. Including the zero map does not increase the dimension: we conclude that Di_s should be expected to have rank ≤ 2 in on a subset of S having codimension 2. Since S is a surface, the map i should fail to be an immersion at a discrete set of points of S. The images of these points in M are called *branch points* of the map i.

(ii) The map i can fail to be injective, so that $i(x) = i(y)$ for $x \neq y$. Since $i(x)$ and $i(y)$ take values in the 3-manifold M, we should expect the relation $i(x) = i(y)$ to hold with codimension 3 among $(x, y) \in S^2$. We will say that $x \in M$ is a *double point* of i if $i^{-1}\{x\}$ has cardinality 2. If i is in general position, then we expect the set of double points to be a smooth submanifold of codimension 1 in M. We can also arrange that this set does not intersect the set of branch points (although, as we will see in a moment, every branch point lies in the *closure* of the set of double points).

(*iii*) The map i can fail to be injective more drastically: we can have $i(x_1) = i(x_2) = \ldots = i(x_n)$. This behavior is to be expected in codimension $3(n-1)$ in the space S^n of dimension $2n$. If $n > 3$, then $3(n-1) > 2n$ so that a generic map i will have not exhibit this behavior. If $n = 3$, then we expect this to happen for a discrete subset of S^3: in other words, we expect an isolated set of points $x \in M$ where $i^{-1}\{x\}$ has cardinality 3. We will call such points *triple points* of the map i.

What does the map i look like near a branch point? If we work in the piecewise linear category, then the local structure of a PL map $i : D^2 \to D^3$ is given by taking the cone over some PL map $i_0 : S^1 \to S^2$. If i_0 is an embedding, then so is i, and we do not have any branching. We may therefore assume that i_0 fails to be an embedding and therefore has some double points. It follows that every branch point of i lies at the endpoint of a curve of double points of i. (For a generic choice of i, the curve $i_0 : S^1 \to S^2$ will have only a single self-intersection so that this double curve is unique. However, we will not need to know this.)

The Loop Theorem: Reduction to a Special Case (Lecture 28)

April 17, 2009

Let us continue our analysis of a general position map $f : S \to M$, where Σ is a compact surface and M is a 3-manifold. We will allow S and M to have boundary, but insist that f be proper: that is, f carries the boundary of S into the boundary of M. In the previous lecture, we saw that $f(S)$ is a smooth submanifold away from three types of points:

(1) Double points of the map f: that is, points $x \in M$ such that $f^{-1}\{x\}$ has exactly two elements, near each of which f is an immersion. These form a locally closed smooth submanifold of M having codimension 2 (which may intersect the boundary of M).

(2) Triple points of the map f: points $x \in M$ such that $f^{-1}\{x\}$ has exactly three points, near each of which f is an immersion. These form a submanifold of M having codimension 3: that is, a finite collection of points in M.

(3) Branch points of the map f: that is, points $x \in M$ such that $f^{-1}\{x\}$ has a single point over which f fails to be an immersion. The number of branch points of M is also finite, and every branch point lies in the closure of the set of double points of f.

We define the *singular locus* of f to be the subset of $f(M)$ consisting of these three types of points. Our analysis shows that the singular locus of f is a finite graph, which can be written as a union of the following constituents:

(i) *Double loops* of f: that is, closed loops in M contained in the locus of double points.

(ii) *Double arcs* of f: that is, closed arcs in M whose interior consists of double points, and whose endpoints are either triple points, branch points, or double points contained in ∂M.

We now turn to the proof of the loop theorem.

Theorem 1 (Loop Theorem)**.** *Let M be a connected 3-manifold with boundary, let X be a connected 2-manifold with boundary contained in ∂M, and let N be a normal subgroup of $\pi_1 X$. Suppose that N does not contain the kernel of the map $\pi_1 X \to \pi_1 M$. Then there exists an embedding $f : (D^2, S^1) \to (M, X)$ such that the underlying loop $S^1 \to X$ represents a class in $\pi_1 X$ which does not belong to N.*

By assumption, we can choose a loop $S^1 \to X$ representing a class which does not belong to N, but does belong to the kernel of the map $\pi_1 X \to \pi_1 M$. We can therefore extend the loop to a map $f_0 : (D^2, S^1) \to (M, X)$. This map f_0 need not be an embedding. However, we can assume without loss of generality that f_0 is piecewise linear and in general position (our discussion of general position maps above was in the smooth category, but there is a parallel discussion in the PL category; we will not worry about the details). The image $f_0(D^2)$ is a compact polyhedron in M. Let M_0 be a closed regular neighborhood of $f_0(D^2)$, and let $X_0 = M_0 \cap X$. Then M_0 is a *compact* 3-manifold with boundary which contains $f_0(D^2)$ as a deformation retract, and X_0 is a regular neighborhood of $f_0(S^1)$ (and therefore connected). Let N_0 be the inverse image of N in $\pi_1 X_0$. Note that $f_0|S^1$ represents a class in $\pi_1 X_0$ (well-defined up to conjugacy) which does not belong to N_0.

Suppose that M_0 admits a connected double cover $\widetilde{M}_0$. Since D^2 is simply connected, we can lift f_0 to a map $f_1 : D^2 \to \widetilde{M}_0$. Let M_1 be a regular neighborhood of $f_1(D^2)$, and X_1 its intersection with the inverse image of X_0 (so that X_1 is a regular neighborhood of $f_1(S^1)$), and let N_1 denote the inverse image of N_0 in $\pi_1 X_1$. Then f_1 can be regarded as a map $(D^2, S^1) \to (M_1, X_1)$, representing a class in $\pi_1 X_1$ which does not belong to N_1.

Iterating this procedure, we can produce a sequence of maps $f_i : (D^2, S^1) \to (M_i, X_i)$. We claim that this process must eventually terminate, meaning that eventually the compact 3-manifold M_i does not admit any connected double covers. To see this, we observe that if $f_{i+1}D^2 \to f_i D^2$ is a homeomorphism, then the map $M_{i+1} \to M_i$ must be a homotopy equivalence (since each M_j contains $f_j(D^2)$ as a deformation retract), so that $\phi : \pi_1 M_{i+1} \to \pi_1 M_i$ is surjective; this is a contradiction, since by construction M_{i+1} is a subset of a connected double cover of M_i so that the image of ϕ has index at least 2. Consequently, each of the maps $f_{i+1}D^2 \to f_i D^2$ must fail to be a homeomorphism. This implies that some double point of f_i fails to be a double point of f_{i+1}. Thus f_{i+1} has fewer double curves than f_i. Since the number of double curves of f_0 is finite, our process must halt after finitely many steps.

Suppose therefore that we have constructed a map $f_m : (D^2, S^1) \to (M_m, X_m)$ where M_m does not admit any connected double covers. In other words, the cohomology group $\mathrm{H}^1(M_m)$ vanishes (here and in what follows, we will assume that all cohomology is taken with coefficients in $\mathbf{Z}/2\mathbf{Z}$). Then $\mathrm{H}_1(M_m) \simeq 0$ and, by Poincare duality, $\mathrm{H}_2(M_m, \partial M_m) \simeq 0$. Using the long exact sequence

$$\mathrm{H}_2(M_m, \partial M_m) \to \mathrm{H}_1(\partial M_m) \to \mathrm{H}_1(M_m),$$

we deduce that $\mathrm{H}_1(\partial M_m) \simeq 0$. Consequently, every boundary component of M_m must be a sphere.

The space X_m is a connected compact surface with (nonempty) boundary equipped with an embedding into some boundary component of X_m, which must be a 2-sphere S^2. It follows that the group $\pi_1 X_m$ is generated by the conjugacy classes of elements which are represented by loops in the boundary ∂X_m. Since N_m does not contain the class of the loop $f_m | S^1$, we have $N_m \neq \pi_1 X_m$. It follows that N_m does not contain the class of some embedded loop $S^1 \hookrightarrow \partial X_m$. This embedded loop bounds an embedded disk in the 2-sphere $S^2 \subseteq \partial M_m$, and therefore bounds an embedded disk in M_m. We can therefore choose an embedding $g_m : (D^2, S^1) \to (M_m, X_m)$ such that the underlying map $S^1 \to X_m$ represents a class in $\pi_1 X_m$ not belonging to N_m.

Of course, the composition of g_m with the projection $M_m \to M$ need not be an embedding. However, we will "descend" g_m to a sequence of embeddings $g_i : (D^2, S^1) \to (M_i, X_i)$ (representing a loop in $\pi_1 X_i$ not belonging to N_i) using descending induction on i. Assuming that g_{i+1} has been constructed, consider the composite map

$$g'_i : D^2 \stackrel{g_{i+1}}{\to} M_{i+1} \subseteq \widetilde{M}_i \to M_i.$$

This map need not be an embedding. However, it is the composition of an embedding with a 2-fold covering map. It follows that g'_i is an immersion and that g'_i has no triple points. Moving g'_i by a small isotopy, we can assume that g'_i is a general position map with the same properties. To construct g'_i from g_i, it suffices to prove the following special case of the Loop Theorem:

Theorem 2. *Let M be a connected 3-manifold with boundary, let X be a connected 2-manifold with boundary contained in ∂M, and let N be a normal subgroup of $\pi_1 X$. Suppose we are given a map $g' : (D^2, S^1) \to (M, X)$ with the following properties:*

(1) *The map g' is an immersion without triple points (consequently, the singular locus of g' consists of closed double loops and double arcs which join double points belonging to X).*

(2) *The restriction $g' | S^1$ represents a class in $\pi_1 X$ which does not belong to N.*

Then there exists an embedding $g : (D^2, S^1) \to (M, X)$ which satisfies (2).

We will prove Theorem 2 in the next lecture.

The Loop Theorem: Special Case (Lecture 29)

April 22, 2009

In the last lecture, we reduced the proof of the Loop Theorem to the following special case:

Theorem 1. *Let M be a connected 3-manifold with boundary, let X be a connected 2-manifold with boundary contained in ∂M, and let N be a normal subgroup of $\pi_1 X$. Suppose we are given a map $g' : (D^2, S^1) \rightarrow (M, X)$ with the following properties:*

(1) *The map g' is an immersion without triple points (consequently, the singular locus of g' consists of closed double loops and double arcs which join double points belonging to X).*

(2) *The restriction $g'|S^1$ represents a class in $\pi_1 X$ which does not belong to N.*

Then there exists an embedding $g : (D^2, S^1) \rightarrow (M, X)$ which satisfies (2).

Our goal in this lecture is to prove Theorem 1. Let $X \subseteq D^2$ denote the locus consisting of points where g' is not an embedding. Since g' has only double points, X is a submanifold having codimension 1 in D^2: it therefore consists of finitely many closed curves and finitely many arcs whose endpoints lie in the boundary of the disk. Moreover, for every point $x \in X$ there is a unique point $y \neq x$ such that $f(x) = f(y)$. The construction $x \mapsto y$ is a fixed-point-free involution on X; let Y denote the quotient of X by this involution (it is a smooth submanifold of M).

Let $k = k(g')$ be the number of connected components of Y. We will prove Theorem 1 using induction on k. If $k = 0$, then the map g' is an embedding and we can take $g' = g$. Assume therefore that $k > 0$, so that X is nonempty. Our goal is to replace g' by another map g'' with $k(g'') < k(g')$ (in other words, g'' has fewer double curves than g').

First suppose that Y contains a closed curve $C \simeq S^1$. Let $\widetilde{C}$ denote the inverse image of C in X. There are two possibilities to consider:

(1) The curve $\widetilde{C}$ is connected. Without loss of generality, we may assume that $\widetilde{C}$ is a circle of radius $\frac{1}{2}$ in D^2, and that the involution on X restricts to the antipodal map on $\widetilde{C}$. We can then define a new map $g'' : D^2 \rightarrow M$ by the formula

$$g''(x) = \begin{cases} g(x) & \text{if } |x| \geq \frac{1}{2} \\ g(-x) & \text{if } |x| \leq \frac{1}{2}. \end{cases}$$

Modifying g'' by a small perturbation, we can arrange that g'' is injective along $\widetilde{C}$ (and elsewhere has the same singularities as g'). Since g'' and g' have the same restriction to ∂D^2, and that $k(g'') < k(g')$, we can conclude by the inductive hypothesis.

Remark 2. The analogue of case (1) will prove more troublesome in our proof of the sphere theorem. Consequently, it is worth noting now that (1) is impossible if the manifold M is orientable. More precisely, we have the following:

(∗) Let Σ be an oriented surface, M an oriented 3-manifold, and $f : \Sigma \rightarrow M$ a general position map. Suppose that $C \subseteq M$ is a closed double curve of f. Then the inverse image $\widetilde{C} \subseteq \Sigma$ of C is disconnected.

For suppose that $\widetilde{C}$ is connected. Since C is a circle, it has trivial tangent bundle; let v be a nowhere vanishing vector field on $\widetilde{C}$. Since Σ is orientable, the normal bundle N to $\widetilde{C}$ in Σ must also be trivial, so it has a nonzero section w over $\widetilde{C}$. Let σ denote the involution on $\widetilde{C}$. At every point $x \in \widetilde{C}$, the vectors $v_{f(x)}$, $df(w_x)$, and $df(w_{\sigma(x)})$ form an ordered basis for the tangent space $T_{M,f(x)}$, which depends continuously on x. However, if we replace x by $\sigma(x)$, then this ordered basis changes by an odd permutation. It follows that the orientation obstruction $w_1(M) \in \mathrm{H}^1(M; \mathbf{Z}/2\mathbf{Z})$ is nontrivial on $[C] \in \mathrm{H}_1(M; \mathbf{Z}/2\mathbf{Z})$.

Assume now that $\widetilde{C}$ has two connected components C_0 and C_1. These components bound disks D_0 and D_1. We next consider the special case:

(2) One of the disks D_i contains the other. Without loss of generality, we may assume that D_0 contains D_1. Choose a homeomorphism $h : D_0 \to D_1$ extending the homeomorphism $C_0 \simeq C \simeq C_1$. We can then define a new map $g'' : D^2 \to M$ by the formula

$$g''(x) = \begin{cases} g'(x) & \text{if } x \notin D_0 \\ g'(hx) & \text{if } x \in D_0. \end{cases}$$

It is easy to see that $k(g'') < k(g')$, and g'' has the same restriction to the boundary as g'; we may therefore conclude by the inductive hypothesis.

There is one other case to consider:

(3) Suppose that the disks D_0 and D_1 are disjoint. Choose a homeomorphism $h : D_0 \to D_1$ extending the homeomorphism $C_0 \simeq C \simeq C_1$ of their boundaries. We define a new map $g'' : D^2 \to M$ by the formula

$$g''(x) = \begin{cases} g'(hx) & \text{if } x \in D_0 \\ g'(h^{-1}x) & \text{if } x \in D_1 \\ g'(x) & \text{otherwise.} \end{cases}$$

Modifying g'' by a small perturbation, we again have $k(g'') < k(g')$, while g'' agrees with g' on ∂D^2, so we can conclude by induction.

Now suppose that g' has no closed double curves. Since $k(g') > 0$, g' must have a double arc $C \subseteq M$, which is doubly covered by a pair of arcs $C_0, C_1 \subseteq D^2$. We will identify D^2 with the product $[0,1] \times [-1,1]$. Without loss of generality, we may assume that $C_0 = \frac{1}{3} \times [0,1]$ and that $C_1 = \frac{2}{3} \times [-1,1]$. We have an identification $C_0 \simeq C \simeq C_1$, which we may assume without loss of generality is given by $(\frac{1}{3}, t) \mapsto (\frac{2}{3}, \pm t)$.

We define two new maps $g_0'', g_1'' : D^2 \to M$ by the following formulae:

$$g_0''(s,t) = \begin{cases} g'(\frac{2}{3}s, t) & \text{if } s \leq \frac{1}{2} \\ g'(\frac{2}{3}s + \frac{1}{3}, \pm t) & \text{if } s \geq \frac{1}{2} \end{cases}$$

$$g_1''(s,t) = \begin{cases} g'(s,t) & \text{if } s \leq \frac{1}{3} \\ g'(\frac{2}{3} - s, \pm t) & \text{if } \frac{1}{3} \leq s \leq \frac{2}{3} \\ g'(s,t) & \text{if } \frac{2}{3} \leq s \leq 1. \end{cases}$$

Note that $k(g_0'') < k(g')$ (since we have eliminated at least one double arc), and we will have $k(g_1'') < k(g')$ after replacing g_1'' by a small perturbation to ensure that it is in general position. To complete the inductive step, it will suffice to show that either g_0'' or g_1'' represents a class not belonging to the normal subgroup $N \subseteq \pi_1 X$. To prove this, it suffices to observe that $[g'|S^1]$ belongs to the normal subgroup of $\pi_1 X$ generated by $[g_0''|S^1]$ and $[g_1''|S^1]$ (this is clear if we draw some pictures which are not included in the notes).

The Sphere Theorem: Part 1 (Lecture 30)

April 23, 2009

In this lecture, we will begin to prove the following result:

Theorem 1 (The Sphere Theorem)**.** *Let M be an oriented connected 3-manifold and let $N \subset \pi_2 M$ be a $\pi_1 M$-invariant proper subgroup. Then there exists an embedded 2-sphere $S \hookrightarrow M$ whose homotopy class does not belong to N. In particular, M is not irreducible.*

Since N is a proper subgroup of $\pi_2 M$, we can choose a map $f : S^2 \to M$ representing a homotopy class which does not belong to N. We will follow a basic strategy similar to that of the loop theorem: we will repeatedly modify the map f until it becomes an embedding. To begin with, we may assume that f is in general position. The proof now proceeds in several stages:

(1) We may reduce to the case where f is an immersion.

To see this, we construct a tower similar to that appearing in our proof of the loop theorem. Namely, we define a sequence of maps $f_n : S^2 \to M_n$ by induction as follows:

- Set $M_0 = M$, and $f_0 = f$.
- Assume that we have constructed $f_n : S^2 \to M_n$. Let U_n be a regular neighborhood of $f_n(S^2)$ in M_n (a compact 3-manifold with boundary) If $\pi_1 f_n(S^2) \simeq \pi_1 U_n$ is finite, then we terminate the process. Otherwise, let M_{n+1} be the universal cover of U_n, and let $f_{n+1} : S^2 \to M_{n+1}$ be any map lifting f_n (such a map exists, since S^2 is simply connected).

As in the proof of the loop theorem, this process must eventually terminate at some stage n, so that $\pi_1 U_n$ is finite. It follows that $\mathrm{H}_1(U_n, \mathbf{Q}) = 0$. By Poincare duality, we have $\mathrm{H}_2(U_n, \partial U_n; \mathbf{Q}) = 0$. Using the long exact sequence

$$\mathrm{H}_2(U_n, \partial U_n; \mathbf{Q}) \to \mathrm{H}_1(\partial U_n; \mathbf{Q}) \to \mathrm{H}_1(U_n; \mathbf{Q})$$

we deduce that $\mathrm{H}_1(\partial U_n; \mathbf{Q}) = 0$, so that the boundary ∂U_n (which is an orientable 2-manifold) is a union of finitely many spheres. Let W be the universal cover of ∂U_n and let $\widehat{W}$ be the the 3-manifold obtained by capping off its boundary spheres. Since $\pi_1 U_n$ is finite, W is compact, so that $\widehat{W} \simeq S^3$ by the Poincare conjecture. It follows that W is obtained from S^3 by removing finitely many open disks, so that $\pi_2 W$ is generated by the classes represented by its boundary spheres. We deduce that $\pi_2 U_n \simeq \pi_2 W$ is generated (as a $\pi_1 U_n$-module) by the classes represented by boundary spheres.

Let N' be the inverse image of N in $\pi_2 U_n$. Since the homotopy class of f_n does not belong to N', we deduce that N' is a proper $\pi_1 U_n$-invariant subgroup of $\pi_2 U_n$. It follows that N' does not contain the class of some embedding $g : S^2 \hookrightarrow \partial U_n \subseteq U_n$. Let f' denote the composite map $S^2 \xrightarrow{g} U_n \to M$. Since g is an embedding, f' is an immersion. Replacing f by f', we can reduce to the case where f is itself an immersion.

Modifying f slightly, we may assume also that f is in general position: it may therefore have both double and triple points (but no branch points). Let $\Sigma(f)$ denote the singular locus of f (the subset of M consisting of those points $x \in M$ for which $f^{-1}(x)$ contains at least two points). Then $\Sigma(f)$ is a 1-dimensional subset of M, which is a submanifold except at a set of isolated points (the triple points of f). The inverse image

$f^{-1}\Sigma(f)$ is a 1-dimensional submanifold of S^2, which can be written as the union of finitely many circles. We will call the images of these circles under f *double curves* of M.

We now proceed by induction on the pair $(t(f), d(f))$, where $t(f)$ denotes the number of triple points of f and $d(f)$ the number of double curves of f. We order these pairs lexicographically: we consider another general position map $f' : S^2 \to M$ to be simpler than f if $t(f') < t(f)$ or if $t(f') = t(f)$ and $d(f') < d(f)$.

(2) Suppose that f has a simple double curve (i.e., there is a component of $f^{-1}\Sigma_f$ which embeds into M). Then we can replace f by a simpler map $f' : S^2 \to M$ which again represents a class in $\pi_2 M$ not belonging to N.

To see this, let $C \subseteq M$ be a simple double curve of M. Then $f^{-1}C$ consists of a few isolated points together with a double cover $\widetilde{C}$ of C. Since M and S^2 are oriented, the argument of the previous lecture shows that $\widetilde{C}$ must be disconnected, consisting of two circles $C_1, C_2 \subseteq S^2$. These circles bound disjoint disks $D_1, D_2 \subseteq S^2$. Let $h : D_1 \to D_2$ be a homeomorphism extending the identification $C_1 \simeq C \simeq C_2$. Let $f'_0 : S^2 \to M$ be the map given by the formula

$$f'_0(x) = \begin{cases} f(hx) & \text{if } x \in D_1 \\ f(h^{-1}x) & \text{if } x \in D_2 \\ f(x) & \text{otherwise,} \end{cases}$$

and let $f'_1 : D_1 \coprod_C D_2 \to M$ be the map given by amalgamating $f|D_1$ and $f|D_2$. Then:

(*i*) After replacing f'_0 by a small perturbation, we can arrange that f'_0 and f'_1 are general position maps, both simpler than the original map f (in both cases, we have either eliminated all triple points along the double curve C, or left the number of triple points constant while eliminating at least one double curve).

(*ii*) The homotopy class of f in $\pi_2 M$ belongs to the $\pi_1 M$-invariant subgroup generated by the homotopy classes of f'_0 and f'_1. Consequently, either $[f'_0]$ or $[f'_1]$ will not belong to the subgroup N.

This completes the proof of (2). Unfortunately, this is not yet enough to prove the sphere theorem, because the double curves of the map f will generally intersect themselves.

Lemma 2. *Let $q : \widetilde{M} \to M$ be a local homeomorphism of 3-manifolds, let $f : S^2 \to M$ be a general position map without branch points, and let $\widetilde{f} : S^2 \to \widetilde{M}$ be a lift of f. If $\widetilde{f}$ has a simple double curve C, then $q(C)$ is a simple double curve of f.*

Proof. It suffices to show that $q|C$ is injective. If not, then there exist points $x, y \in C$ such that $q(x) = q(y) = z \in M$. Then $f^{-1}M = \widetilde{f}^{-1}\{x\} \cup \widetilde{f}^{-1}\{y\}$ has at least four points, contradicting our assumption that f is in general position. □

We now try to exploit Lemma 2 using the tower

$$U_n \subseteq M_n \to U_{n-1} \to \cdots \to M_0 = M$$

constructed in (1) (for our given map f).

(3) Suppose that $f_n : S^2 \to M$ is an embedding. Then f has a simple double curve, and we may conclude by applying (2).

To prove (3), we first consider the group $\mathrm{H}_1(U_{n-1}; \mathbf{Z})$. If this group is finite, then the reasoning of step (1) implies that every boundary component of U_{n-1} is a sphere, so that the map $\pi_1 U_{n-1} \to \pi_1 M_{n-1}$ is injective by van Kampen's theorem. Since M_{n-1} is simply connected, we conclude that U_{n-1} is also simply connected, which contradicts our choice of n. Thus $\mathrm{H}_1(U_{n-1}, \mathbf{Z})$ is infinite. Let T denote the torsion subgroup of $\mathrm{H}_1(U_{n-1}, \mathbf{Z})$, and let $\widetilde{T}$ denote the inverse image of T in $\pi_1 U_{n-1}$; note that $\widetilde{T} \neq \pi_1 U_{n-1}$. Since the inclusion $f_{n-1}(S^2) \subseteq U_{n-1}$ is a homotopy equivalence, the inverse image of $f_{n-1}(S^2)$ in M_n is connected. This inverse image consists of all translates of the 2-sphere $S = f_n(S^2)$ by elements of $\pi_1 U_{n-1}$. It follows that the intersection

$$(\bigcup_{g \in \widetilde{T}} g(S)) \cap (\bigcup_{g' \notin \widetilde{T}} g'(S))$$

is nonempty, so that there exists an element $\tau \in \pi_1 U_{n-1} - \widetilde{T}$ such that $\tau(S) \cap S \neq \emptyset$.

By construction, the group element τ has infinite order. Let k be the largest integer such that $\tau^k(S) \cap S \neq \emptyset$, let Z denote the cyclic subgroup of $\pi_1 U_{n-1}$ generated by τ^k, and let $\widetilde{M} = M_n/Z$. We have a local homeomorphism $\widetilde{M} \to M$. Consequently, by Lemma 2, it will suffice to show that the composite map $\widetilde{f} : S^2 \xrightarrow{f_n} M_n \to \widetilde{M}$ has a simple double curve.

Since the map f is in general position, the spheres $\tau^k(S)$ and S must meet transversely in M_n. Let C be a connected component of their intersection. We claim that the image of C is a simple double curve of $\widetilde{f}$. To prove this, it suffices to show that the map $C \to \widetilde{M}$ is injective. Suppose otherwise: then there exist points $x, y \in C$ such that $x = \tau^{nk} y$ for some integer $n \geq 0$. Then $x \in S \cap \tau^{(n+1)k} S \neq \emptyset$, contradicting our choice of k. This completes the proof of (3).

It remains to treat the case where $f_n : S^2 \to M$ fails to be an embedding. We will return to this case in the next lecture.

The Sphere Theorem: Part 2 (Lecture 31)

April 27, 2009

In this lecture, we will complete the proof of the sphere theorem.

Let us recall the situation. We are given an oriented, connected 3-manifold M and a $\pi_1 M$-invariant proper subgroup $N \subset \pi_2 M$. Our goal is to prove that there exists an embedded 2-sphere $S \subseteq M$ whose homotopy class does not belong to N.

Since $N \neq \pi_2 M$, there exists a map $f : S^2 \to M$ whose homotopy class does not belong to N. We may assume that f is in general position and (as we saw in the last lecture) an immersion. We will suppose that f has been chosen so as to minimize the number $t(f)$ of triple points of f.

In the last lecture, we argued as follows:

(1) If the map f has a simple double curve, then we can modify f so as to obtain a new map f' (whose homotopy class again does not belong to N) which either has fewer triple points ($t(f') < t(f)$) or the same number of triple points and fewer double curves. Since $t(f)$ is minimal, f' must have fewer double curves. Applying this procedure repeatedly, we can reduce to the case where f does not have any double curves.

(2) There exists a 3-manifold with boundary $\widetilde{M}$ (namely, the 3-manifold V_n at the top of the tower that we constructed in the last lecture) and an immersion $q : \widetilde{M} \to M$ with the following properties:

- (i) The map f lifts to a map $\widetilde{f} : S^2 \to \widetilde{M}$.
- (ii) The 3-manifold $\widetilde{M}$ is a regular neighborhood of $\widetilde{f}(S^2)$.
- (iii) The fundamental group $\pi_1 \widetilde{M}$ is finite. As we saw last time, this guarantees that the universal cover of $\widetilde{M}$ is a punctured sphere, so that $\pi_2 \widetilde{M}$ is generated (as a $\pi_1 \widetilde{M}$-module) by its boundary components.
- (iv) The map $\widetilde{f}$ is not an embedding (otherwise we were able to produce a simple double curve of f.

Let $\Sigma(\widetilde{f})$ denote the singular locus of the map $\widetilde{f}$. Condition (iv) guarantees that $\Sigma(\widetilde{f})$ is nonempty. Let X be a small neighborhood of $\Sigma(\widetilde{f})$ in $\widetilde{M}$. Since f is in general position, no point of M has more than 3 preimages under f. It follows that q must be injective on $\Sigma(\widetilde{f})$. Shrinking X, we may assume that q is injective on X. Let T denote the closure of $\widetilde{f}(S^2) - X$.

Let $x \in \Sigma(\widetilde{f})$. Since f is a general position map, $q(x)$ has at most 3 preimages under f. At least two of these are preimages of x under $\widetilde{f}$. There are two possibilities:

(a) The inverse image $f^{-1}(q(x)) = \widetilde{f}^{-1}(x)$. Then $q(x)$ does not intersect $q(T)$, so we can choose a neighborhood V_x of x such that $q(V_x) \cap q(T) = \emptyset$.

(b) The inverse image $f^{-1}(q(x))$ consists of $\widetilde{f}^{-1}(x)$ together with one additional point $s \in S^2$. Let $y = \widetilde{f}(s)$. Since q is injective on X, we must have $y \notin X$, so that $y \in T$. Since q is an immersion, there exists a neighborhood U of y in T on which q is injective. Then $q(x)$ does not intersect $q(T - U)$, so there is a neighborhood of V_x of x such that $q(V_x) \cap q(T) \subseteq q(U)$.

Let X_0 be a regular neigborhood of $\Sigma(\widetilde{f})$ which is contained in the open set $\bigcup V_x$. By construction, if $x \in X_0$ then there is at most one element $y \in \widetilde{f}(S^2)$ such that $x \neq y$ but $q(x) = q(y)$.

Let $X_1 \subset X_0$ be a slightly smaller regular neighborhood of $\Sigma(\widetilde{f})$. The map $\widetilde{f}$ is an embedding outside of X_1; let $S_1, \ldots, S_m$ be the connected components of its image. Then $\widetilde{f}(S^2)$ has a regular neighborhood of the form $X_1 \cup (S_1 \times [-1,1]) \cup \ldots \cup (S_m \times [-1,1])$. Shrinking $\widetilde{M}$ if necessary, we may assume that it coincides with this regular neighborhood.

Let $\widetilde{N}$ denote the inverse image of N in $\pi_2 \widetilde{M}$. Since $\widetilde{N}$ does not contain the homotopy class of $\widetilde{f}$, it is a proper subgroup $\pi_1 \widetilde{M}$-invariant subgroup of $\pi_2 \widetilde{M}$. Using (iii), we deduce that $\widetilde{N}$ does not contain the homotopy class of some boundary component S of $\widetilde{M}$. Let $f' : S^2 \to \widetilde{M}$ be the inclusion of this boundary component. Then the image of f' is contained in

$$X_1 \cup (S_1 \times \{-1,1\}) \cup \ldots \cup (S_m \times \{-1,1\}).$$

Claim 1. *For each index $1 \leq i \leq m$, the image of f' cannot intersect both $S_i \times \{-1\}$ and $S_i \times \{1\}$.*

Proof. Otherwise, there exists a simple arc α on $f'(S^2)$ joining points $(x,-1)$ and $(y,1)$, where $x, y \in S_i$. Choose a path joining y to x in S_i, which determines a path β from $(y,1)$ to $(x,-1)$ in $S_i \times [-1,1]$. The composition $\alpha \circ \beta$ is a simple loop which meets $\widetilde{f}(S^2)$ transversely at exactly one point (belonging to S_i). It follows that $\alpha \circ \beta$ represents a nontorsion homology class in $\mathrm{H}_1(\widetilde{M}, \mathbf{Z})$, which contradicts our assumption that $\pi_1 \widetilde{M}$ is finite. □

Using Claim 1, we can modify the map f' by an isotopy to obtain an embedding $f'' : S^2 \to \widetilde{M}$ whose image is contained in $X_0 \cup S_1 \cup S_2 \cup \ldots \cup S_m$. By construction, the homotopy class of f'' does not belong to $\widetilde{N}$, so the homotopy class of $q \circ f''$ does not belong to N. We will obtain a contradiction by showing that $t(q \circ f'')$ has fewer triple points than f''.

Let $x \in M$ be a triple point for $q \circ f''$. Since f'' is an embedding, we must have three distinct points $x_1, x_2, x_3 \in f''(S^2)$ such that $q(x_1) = q(x_2) = q(x_3) = x$. Note that $f''(S^2) \subseteq T \cup X_0$. Since q is injective on X_0, at most one element of $\{x_1, x_2, x_3\}$ belongs to X_0. However, if $x_i \in X_0$, then there is at most one element $y \in T$ distinct from x_i such that $q(y) = q(x_i)$. It follows that none of the elements x_1, x_2, and x_3 belong to X_0. Thus $x_1, x_2, x_3 \in T \subseteq \widetilde{f}(S^2)$, so that x is also a triple point of f. This proves that $t(q \circ f'') \leq t(f)$. To prove that the equality is strict, it suffices to show that f has at least one triple point x such that $q^{-1}\{x\}$ is not contained in T. For this, it suffices to show that the map $\widetilde{f}$ has a triple point. Assume otherwise. Then the singular locus $\Sigma(\widetilde{f})$ is a 1-dimensional submanifold of $\widetilde{M}$. This singular locus is nonempty (by (iv)), and therefore contains a circle C. This circle is a simple double curve of $\widetilde{f}$, so that $q(C)$ is a simple double curve of f, which contradicts (1).

Incompressible Surfaces (Lecture 32)

April 29, 2009

In this lecture, we will describe some applications of the loop theorem to the study of a 3-manifold M. For simplicity, we will restrict our attention to the case where M is connected, closed and oriented, though the ideas below generalize to the case of nonorientable manifolds with boundary.

Definition 1. An embedded two-sided surface $\Sigma \subseteq M$ is *compressible* if one of the following conditions holds:

(1) There exists an embedded loop $L \subseteq \Sigma$ which does not bound an embedded disk in Σ, but does bound an embedded disk D in M such that $D \cap \Sigma = \partial D$.

(2) The surface Σ is a 2-sphere which bounds a disk in M.

If Σ is not compressible, then we say that Σ is *incompressible*.

Lemma 2. *Let $\Sigma \subseteq M$ be a 2-sided surface of genus $g > 0$. Then Σ is incompressible if and only if the map $\pi_1 \Sigma \to \pi_1 M$ is injective.*

Proof. The "if" direction is clear: if $\pi_1 \Sigma \to \pi_1 M$ is injective, then any loop in Σ which bounds a disk in M (embedded or not) must be nullhomotopic in Σ, and therefore bound a disk.

Conversely, suppose that Σ is incompressible. If $\pi_1 \Sigma \to \pi_1 M$ is not injective, then there exists a nontrivial loop L in Σ which is the boundary restriction of a map $f : D^2 \to M$. We may assume without loss of generality that the map f is transverse to Σ, so that $f^{-1}\Sigma$ is a union of k circles for $k > 0$. We will assume that f has been chosen to minimize k.

Suppose first that $k = 1$, so that $f^{-1}\Sigma = \partial D^2$. Let M' be the 3-manifold with boundary obtained by cutting M along Σ. Then f lifts to a map $f' : D^2 \to M'$. Applying the loop theorem, we deduce that there exists an embedding $\widetilde{f'} : (D^2, S^1) \to (M', \partial M')$ representing a nontrivial homotopy class on the boundary. Then the composite map $D^2 \xrightarrow{\widetilde{f'}} M' \to M$ is an embedded disk in M, contradicting our assumption that Σ is incompressible.

If $k > 1$, then $f^{-1}\Sigma$ includes a circle C in the interior of D^2. We may assume that C is chosen innermost, so that it bounds a disk D' with $f^{-1}\Sigma \cap D' = C$. If $f|C$ is a nontrivial loop in Σ, then we can replace f by $f|D$ and thereby contradict the minimality of k. Otherwise, we may assume that $f|C$ is nullhomotopic, so that there exists another map $f_0 : D^2 \to M$ which agrees with f outside of D' and carries D' into Σ. Moving f_0 by a small homotopy on D', we obtain a new map $f_1 : D^2 \to M$ which agrees with f on the boundary and such that $f_1^{-1}\Sigma$ consists of $k - 1$ circles, again contradicting the minimality of k. □

Our next result guarantees the existence of a good supply of incompressible surfaces:

Proposition 3. *Let M be a closed connected oriented 3-manifold. Let X be a topological space containing an open subset homeomorphic to $Y \times (-1, 1)$, for some simply connected space Y (which we identify with $Y \times \{0\}$), and let $f : M \to X$ be a map. Assume that $\pi_2 Y \simeq 0$, and that π_2 vanishes for each component of $X - Y$. Then there exists a map $f' : M \to X$ satisfying the following conditions:*

(1) *The maps f and f' are homotopic when restricted to $M - F$, where F is a finite set (in fact, we can choose F to consist of only one point). In particular, f and f' induce the same map $\pi_1 M \to \pi_1 X$.*

(2) *The map f' is transverse to Y, and ${f'}^{-1}Y$ is a union of incompressible surfaces of M.*

Proof. Adjusting f by a small homotopy, we may assume that f is transverse to Y, so that $f^{-1}Y$ is a union of finitely many two-sided surfaces Σ_i in M, each having genus g_i. We will assume that these surfaces have been chosen to minimize $c(f) = \sum_i 3^{g_i}$. If each of these surfaces is incompressible, we are done. Otherwise, we will explain how to modify the map f to obtain a new map f' satisfying (1) with $c(f') < c(f)$; this will contradict the minimality of f.

Let Σ be a compressible component of $f^{-1}Y$. If Σ is a 2-sphere, then Σ bounds a disk D. Since $\pi_2 Y$ is equal to zero, there exists a map $f_0 : M \to X$ which agrees with f outside of D, and carries D into Y (moreover, this map is homotopic to f after removing a single point of D). Adjusting f_0 by a small homotopy, we obtain a map $f' : M \to X$ such that ${f'}^{-1}Y = f^{-1}Y - \Sigma$, so that $c(f') < c(f)$ as desired.

Suppose now that Σ is not a 2-sphere, so there exists a 2-disk $D \subseteq M$ such that $D \cap \Sigma = \partial D$ is a nontrivial loop in Σ. Choose a tubular neighborhood $D \times [-1, 1] \subseteq M$ such that $(D \times [-1, 1]) \cap M \subseteq \Sigma$. We may assume that $f(x, t) \in Y \times [0, 1]$ for $(x, t) \in D \times [-1, 1]$ near $\partial D \times [-1, 1]$.

We define a new map $f' : M \to X$ as follows:

(*i*) We let f' coincide with f outside of the interior of $D \times [-1, 1]$ (so that f' will be homotopic to f after removing a point of $D \times [-1, 1]$).

(*ii*) Since Y is simply connected, the loop $f| \partial D \times \frac{1}{2}$ extends to a map $g_+ : D \times \frac{1}{2} \to Y$; we let $f'|D \times \frac{1}{2} = g_+$. Define $f'|D \times \frac{-1}{2}$ similarly.

(*iii*) Using the assumption that each component of $X - Y$ has vanishing π_2, we can extend f' over $D \times [\frac{1}{2}, 1]$ and over $D \times [-1, \frac{-1}{2}]$, carrying the complement of $(D \times \{\pm\frac{1}{2}\}) \cup (\partial D \times [-1, 1])$ into $X - Y$.

(*iv*) Using the assumption that $\pi_2 Y = 0$, we can extend f' over $D \times [\frac{-1}{2}, \frac{1}{2}]$ so that $f'(D \times [\frac{-1}{2}, \frac{1}{2}] \subseteq Y$.

Adjust f' by a small homotopy which pushes $f'(D \times (\frac{-1}{2}, \frac{1}{2})$ into $Y \times (-1, 0)$. Then the inverse image ${f'}^{-1}Y$ can be identified with the surface obtained from $f^{-1}Y$ by doing surgery along the loop $L : \partial D$. There are two possibilities:

(*a*) The curve L is separating in Σ. Since L is nontrivial, we deduce that L surgery along L cuts Σ into two surfaces of positive genus g and g', where Σ has genus $g + g'$. Since $3^g + 3^{g'} < 3^{g+g'}$, we deduce that $c(f') < c(f)$.

(*b*) The curve L is nonseparating in Σ. Then surgery along L replaces Σ by a curve having smaller genus. Since $3^{g-1} < 3^g$ we deduce that $c(f') < c(f)$.

□

We now describe some applications of Proposition 3.

Corollary 4. *Let M be a closed connected oriented 3-manifold, and suppose that $\mathrm{H}_1(M; \mathbf{Q}) \neq 0$. Then M contains a two-sided incompressible surface.*

Proof. If $\mathrm{H}_1(M; \mathbf{Q}) \neq 0$, then $\mathrm{H}^1(M; \mathbf{Z}) \neq 0$. Choose a nontrivial cohomology class represented by a map $f : M \to S^1$. Applying Proposition 3, we may suppose that the inverse image of a point $x \in S^1$ is a union of incompressible surfaces in M. If $f^{-1}(x) = \emptyset$, then f is nullhomotopic. Otherwise, some component of $f^{-1}\{x\}$ is incompressible. □

Remark 5. If M is irreducible, then Corollary 4 must produce an incompressible surface Σ of positive genus. Let M' be the 3-manifold with boundary obtained by cutting M along Σ. Since not every boundary component of M' is a sphere, we must have $\mathrm{H}_1(M'; \mathbf{Q}) \neq 0$. Applying an analogue of Corollary 4 for 3-manifolds with boundary, we can produce another incompressible surface in M'. By repeatedly cutting M along incompressible surfaces in this way, it is possible to obtain a very good understanding of the 3-manifold M.

Corollary 6. *Let M be a closed oriented connected 3-manifold and suppose that $\pi_1 M \simeq G \star H$ is a free product of nontrivial groups G and H. Then M can be written as a connected sum $M_1 \# M_2$ where $\pi_1 M_1 \simeq G$ and $\pi_1 M_2 \simeq H$.*

Proof. Let BG and BH denote classifying spaces for G and H, and let X be the space $BG \coprod_{\{-1\}} [-1,1] \coprod_{\{1\}} BH$. Then X is a classifying space for $G \star H$, so there exists a map $f : M \to X$ which is the identity on $\pi_1 M$. Applying Proposition 3, we may suppose that f is transverse to $\{0\} \subseteq X$ and that $f^{-1}\{0\}$ is a union of incompressible surfaces Σ of M. If any such surface Σ has positive genus, then the map

$$\pi_1 \Sigma \to \pi_1 M \simeq G \star H$$

is injective (Lemma 2) which is a contradiction. Thus $f^{-1}\{0\}$ is a union of k spheres, for some k. Since G and H are both nontrivial, we must have $k > 0$. If $k = 1$, we obtain the desired connect sum decomposition of M. We will assume that f has been chosen to as to minimize k.

Assume that $k > 1$, and let α be a path in M joining two components of $f^{-1}\{0\}$. Then $f(\alpha)$ is a loop in X. Since $\pi_1 M \simeq \pi_1 X$, we can adjust the path α by composing with a loop in M to guarantee that $f(\alpha)$ is nullhomotopic. Adjusting α by a homotopy, we may assume that $\alpha : [0,1] \to M$ is transverse to $f^{-1}\{0\}$, so that α can be written as a composition

$$\alpha = \alpha_1 \circ \ldots \circ \alpha_m$$

where α_i lies in $f^{-1}(BG \coprod_{\{-1\}} [-1,0])$ for i odd (without loss of generality) and α_i lies in $f^{-1}([0,1] \coprod_{\{1\}} BH)$ for i even. We assume that m has been chosen as small as possible. Since $[f(\alpha)]$ vanishes, the structure of free products of groups guarantees that some $f([\alpha_i])$ must vanish. If α_i connects two different components of $f^{-1}\{0\}$, then we can replace α by α_i and reduce to the case $m = 1$. If α_i connects two points in the same component Σ of $f^{-1}M$, then we can replace α_i by a path α_i' in Σ. Adjusting the composite path

$$\alpha_1 \circ \ldots \circ \alpha_{i-1} \circ \alpha_i' \circ \alpha_{i+1} \circ \ldots \circ \alpha_m$$

by a small homotopy, obtain a new path having fewer intersections with $f^{-1}\{0\}$, again contradicting the minimality of M.

We may therefore assume that α is a path intersecting $f^{-1}\{0\}$ only in its endpoints. Let $K \simeq D^2 \times [0,1]$ be a tubular neighborhood of the image of α so that $K \cap f^{-1}\{0\} = D^2 \times \{0,1\}$. Using the assumption that $f(\alpha)$ is nullhomotopic, we can construct a new map $f' : M \to X$ which agrees with f outside of K (and therefore induces the same isomorphism $\pi_1 M \to \pi_1 X$) and carries $D' \times [0,1]$ into $\{0\}$, where D' is a slightly smaller disk in D^2. Adjusting f' by a small homotopy, we obtain a map such that ${f'}^{-1}\{0\}$ is obtained from $f^{-1}\{0\}$ by a surgery along the 0-sphere $\alpha| \, \partial([0,1])$: this surgery reduces the number of connected components which contradicts the minimality of k. □

Classification of Surfaces (Lecture 33)

May 1, 2009

In this lecture, we will (belatedly) discuss the classification of 2-manifolds, which we have frequently used in our discussion of 3-manifolds. We begin with the oriented case.

Theorem 1. *Let Σ be a connected compact oriented surface. Then Σ can be obtained as a connected sum $T \# T \# \cdots \# T$ of g copies of the torus T, for some $g \geq 0$.*

The integer g is called the *genus* of the surface Σ. It is a topological invariant of Σ: a simple calculation shows that $\chi(\Sigma) = 2 - 2g$.

The proof will require a few preliminaries.

Lemma 2. *Let Σ be a connected compact surface. Then $\chi(\Sigma) \leq 2$, and equality holds if and only if Σ is a 2-sphere.*

Proof. We have $\chi(\Sigma) = b_0 - b_1 + b_2$, where b_i denotes the ith Betti number of Σ. Since Σ is connected, we have $b_0 = 1$, and b_2 is either 1 or 0 depending on whether Σ is orientable or nonorientable. It follows that

$$\chi(\Sigma) = \begin{cases} 2 - b_1 & \text{if } \Sigma \text{ is orientable} \\ 1 - b_1 & \text{if } \Sigma \text{ is nonorientable.} \end{cases}$$

This proves the inequality. If equality holds, then Σ must be orientable, and therefore admits a complex structure. As we explained in a previous lecture, a Riemann surface with $\chi(\Sigma) = 2$ must be biholomorphic to the Riemann sphere, and in particular is a topological sphere. □

The following can be regarded as a baby version of the loop theorem:

Lemma 3. *Let Σ be a connected surface and let $N \subset \pi_1 \Sigma$ be a proper normal subgroup. Then there is an embedded loop $f : S^1 \hookrightarrow \Sigma$ such that $[f] \notin N$.*

Proof. Since N is proper, we can choose a closed loop $f : S^1 \to \Sigma$ such that $[f]$ (which is well-defined up to conjugacy) does not belong to N. Without loss of generality, we may assume that f is in general position. Then f is an immersion with a finite number k of double points. We will assume that f has been chosen minimally. If $k = 0$, then f is an embedding and we are done. Otherwise, there exist $x, y \in S^1$ with $x \neq y$ but $f(x) = f(y)$. The points x and y partition S^1 into two intervals I_0 and I_1. The restrictions of f to I_0 and I_1 give two other loops $f_0, f_1 : S^1 \to \Sigma$. Since each of these loops has a smaller number of double points, the minimality of k guarantees that $[f_0], [f_1] \in N$. We now conclude by observing that $[f]$ belongs to the normal subgroup of $\pi_1 \Sigma$ generated by $[f_0]$ and $[f_1]$, and therefore also belongs to N, which contradicts our assumption. □

We now prove Theorem 1. We proceed by descending induction on $\chi(\Sigma)$. If $\chi(\Sigma) \geq 2$, then Lemma 2 implies that $\chi(\Sigma) = 2$ and Σ is a 2-sphere. We may therefore assume that $\chi(\Sigma) = 2 - b_1 < 2$, so that $\mathrm{H}_1(\Sigma; \mathbf{Z}) \neq 0$. It follows that the commutator subgroup $[\pi_1 \Sigma, \pi_1 \Sigma]$ is a proper subgroup of $\pi_1 \Sigma$. Using Lemma 3, we can choose an embedded loop $f : S^1 \hookrightarrow \Sigma$ which represents a nontrivial class in $\mathrm{H}_1(\Sigma; \mathbf{Z})$. It follows that f must be nonseparating, so that the surface Σ' obtained by cutting Σ along f is connected.

Let Σ'' be the closed surface obtained by capping off the boundary circles of Σ'. A simple calculation shows that

$$\chi(\Sigma'') = 2 + \chi(\Sigma') = 2 + \chi(\Sigma).$$

By the inductive hypothesis, Σ'' can be realized as a connected sum $T\#T\#\ldots\#T$.

The surface Σ can be obtained from Σ'' by removing small disks D_x and D_y around two points $x, y \in \Sigma''$ (to obtain Σ'), and then gluing the boundary of these disks together. Without loss of generality, we may assume that x and y are close to one another, so that D_x and D_y are contained in a larger disk D. Let K_0 be the surface with boundary obtained from Σ'' by removing the interior of D, and let K_1 be the surface obtained from D by removing the interiors of D_x and D_y and identifying their boundary. Then $\Sigma = K_0 \coprod_{S^1} K_1$, so we can identify Σ with the connected sum of two surfaces $\widehat{K}_0$ and $\widehat{K}_1$ obtained by capping off the boundary circles of K_0 and K_1. We note that $\widehat{K}_0 \simeq \Sigma''$, and a simple calculation shows that $\widehat{K} = T$ (if we like, we can take this to be a definition of the 2-manifold T). We then obtain

$$\Sigma \simeq \Sigma''\#T \simeq T\#T\#\ldots\#T$$

as desired.

We now treat the case of a nonorientable 2-manifold.

Theorem 4. *Let Σ be a closed connected nonorientable 2-manifold. Then Σ can be obtained as a connected sum $\mathbf{R}P^2 \simeq \mathbf{R}P^2\#\ldots\#\mathbf{R}P^2$ for some $k \geq 1$.*

Remark 5. In the situation of Theorem 4, the integer k is uniquely determined: a simple calculation of Euler characteristics shows that $\chi(\Sigma) = 2 - k$.

Warning 6. A priori, the connected sum $X\#Y$ of two surfaces X and Y is not well-defined: it depends on a choice of identification of the boundary circles of punctured copies of X and Y. This issue did not arise in the statement of Theorem 1, because in the orientable case there is a unique choice of identification which allows us to orient $X\#Y$ in a manner compatible with given orientations of X and Y (which we were implicitly using). It also does not matter in the case of Theorem 4, for a different reason: there exists an diffeomorphism of $\mathbf{R}P^2$ which fixes a point x and induces an orientation reversing automorphism of the tangent space at x. Namely, we observe that $\mathbf{R}P^2 = (\mathbb{R}^3 - \{0\})/\mathbb{R}^\times$ carries an action of the orthogonal group $O(3)$: any reflection in $O(3)$ will do the job.

We now prove Theorem 4. The proof proceeds by descending induction on $\chi(\Sigma)$ (which is at most 1, by virtue of Lemma 2). Since Σ is nonorientable, the 1st Stiefel-Whitney class $w_1 \in \mathrm{H}^1(\Sigma; \mathbf{Z}/2\mathbf{Z})$ induces a nontrivial map $\pi_1\Sigma \to \mathbf{Z}/2\mathbf{Z}$. Let N be the kernel of this map, so that N is a proper normal subgroup of $\pi_1\Sigma$. Using Lemma 3, we obtain an embedded loop $f : S^1 \to \Sigma$ such that $[f] \notin N$. Consequently, the restriction of w_1 to S^1 is nontrivial: this means that the normal bundle to the embedding $S^1 \hookrightarrow \Sigma$ is nontrivial, so that S^1 is a one-sided loop in Σ. Let K be a tubular neighborhood of S^1: then K is a Mobius band, whose boundary is another circle C. Let Σ' be the surface obtained from Σ by removing the interior of K, and let $\widehat{\Sigma}'$ and $\widehat{K}$ be the closed surfaces obtained by capping off the boundary circles of K and Σ'. Then $\widehat{K} = \mathbf{R}P^2$ (if you like, you can take this to be the definition of $\mathbf{R}P^2$, and we have $\Sigma \simeq \widehat{\Sigma}'\#\mathbf{R}P^2$. A simple calculation with Euler characteristics shows that $\chi(\Sigma) = \chi(\widehat{\Sigma}') + \chi(\mathbf{R}P^2) - 2 = \chi(\widehat{\Sigma}') - 1$.

There are now two cases to consider. If $\widehat{\Sigma}'$ is nonorientable, then the inductive hypothesis implies that $\widehat{\Sigma}'$ is a connected sum of finitely many copies of $\mathbf{R}P^2$: it then follows that Σ is a connected sum of finitely many copies of $\mathbf{R}P^2$. If $\widehat{\Sigma'}$ is orientable, then we apply Theorem 1 to deduce that $\widehat{\Sigma'}$ is a connected sum of g copies of the torus T, for some $g \geq 0$. If $g = 0$, then $\widehat{\Sigma}' \simeq S^2$, so that $\Sigma \simeq S^2\#\mathbf{R}P^2 \simeq \mathbf{R}P^2$. The case $g > 0$ is handled through repeated application of the following Lemma:

Lemma 7. *There is a diffeomorphism*

$$\mathbf{R}P^2\#\mathbf{R}P^2\#\mathbf{R}P^2 \simeq T\#\mathbf{R}P^2.$$

Proof. Choose a pair of embedded circles $C, C' \subset T$ which meet transversely in one point x. Let us identify $T \# \mathbf{R}P^2$ with the 2-manifold obtained from T by removing a small disk D around x, and gluing on a Mobius band K along the boundary ∂D. Then $C - C \cap D$ and $C' - C' \cap D$ can be extended to *nonintersecting* embedded loops $\overline{C}$ and $\overline{C}'$ on $T \# \mathbf{R}P^2$, both of which are one-sided. Using the preceding arguments, we deduce that there exists a decomposition

$$T \# \mathbf{R}P^2 \simeq (\mathbf{R}P^2 \# \mathbf{R}P^2) \# \Sigma,$$

where Σ is the surface obtained by removing tubular neighborhoods of $\overline{C}$ and $\overline{C}'$ and capping of their boundary components. A simple calculation shows that $\chi(\Sigma) = 1$, so that Σ must be nonorientable: we therefore have $\Sigma \simeq \mathbf{R}P^2 \# \Sigma'$. Then $\chi(\Sigma') = 2$, so that Σ' is a 2-sphere (Lemma 2). It follows that $\Sigma \simeq \mathbf{R}P^2$ so that

$$T \# \mathbf{R}P^2 \simeq \mathbf{R}P^2 \# \mathbf{R}P^2 \# \mathbf{R}P^2$$

as desired. □

Remark 8. In the next few lectures, we will need to understand not only closed 2-manifolds, but also 2-manifolds with boundary. However, it is easy to extend the above classification: the boundary of a (compact) 2-manifold Σ is a compact 1-manifold, hence a union of finitely many circles. If we let Σ' be the 2-manifold obtained by capping off these boundary circles, then Σ' is diffeomorphic to a 2-manifold of the form

$$T \# T \# \ldots \# T \qquad \mathbf{R}P^2 \# \mathbf{R}P^2 \# \ldots \# \mathbf{R}P^2,$$

and Σ is obtained from Σ' by removing small disks around finitely many points.

Remark 9. Let Σ be a compact connected 2-manifold (possibly nonorientable or with boundary). The properties of Σ depend strongly on the sign of the Euler characteristic $\chi(\Sigma)$. It is therefore convenient to list the possibilities for Σ when χ is nonnegative:

- If $\chi(\Sigma) = 2$, then $\Sigma \simeq S^2$ (Lemma 2).
- If $\chi(\Sigma) = 1$, then either $\Sigma \simeq \mathbf{R}P^2$ or $\Sigma \simeq D^2$.
- If $\chi(\Sigma) = 0$, there are several possibilities. If Σ is orientable, then either $\Sigma \simeq T$ or Σ is a twice-punctured sphere (an annulus $S^1 \times [0,1]$). Each of these possibilties has a nonorientable analogue: if Σ is nonorientable and has boundary, then it is diffeomorphic to a punctured copy of $\mathbf{R}P^2$: this is a Mobius band, given by a nonorientable $[0,1]$-bundle over S^1. If Σ is nonorientable and closed, then it is diffeomorphic to the Klein bottle $\mathbf{R}P^2 \# \mathbf{R}P^2$. This 2-manifold can be viewed as obtained by gluing together two Mobius bands along their boundary, which realizes it as a nonorientable S^1-bundle over S^1 (alternatively, one can start with the surface Σ which is a nonorientable S^1-bundle over S^1; then $\chi(\Sigma) = 0$ so that Theorem 4 guarantees a diffeomorphism $\Sigma \simeq \mathbf{R}P^2 \# \mathbf{R}P^2$.
- If $\chi < 0$, then we are in the "generic case".

Surfaces and Complex Analysis (Lecture 34)

May 3, 2009

Let Σ be a smooth surface. We have seen that Σ admits a conformal structure (which is unique up to a contractible space of choices). If Σ is oriented, then a conformal structure on Σ allows us to view Σ as a Riemann surface: that is, as a 1-dimensional complex manifold. In this lecture, we will exploit this fact together with the following important fact from complex analysis:

Theorem 1 (Riemann uniformization). *Let Σ be a simply connected Riemann surface. Then Σ is biholomorphic to one of the following:*

(i) *The Riemann sphere $\mathbf{C}\mathrm{P}^1$.*

(ii) *The complex plane $\mathbf{C}$.*

(iii) *The open unit disk $D = \{z \in \mathbf{C} : z < 1\}$.*

If Σ is an arbitrary surface, then we can choose a conformal structure on Σ. The universal cover $\widehat{\Sigma}$ then inherits the structure of a simply connected Riemann surface, which falls into the classification of Theorem 1. We can then recover Σ as the quotient $\widehat{\Sigma}/\Gamma$, where $\Gamma \simeq \pi_1 \Sigma$ is a group which acts freely on $\widehat{\Sigma}$ by holmorphic maps (if Σ is orientable) or holomorphic and antiholomorphic maps (if Σ is nonorientable). For simplicity, we will consider only the orientable case.

If $\widehat{\Sigma} \simeq \mathbf{C}\mathrm{P}^1$, then the group Γ must be trivial: every orientation preserving automorphism of S^2 has a fixed point (by the Lefschetz trace formula). Because Γ acts freely, we must have $\Gamma \simeq 0$, so that $\Sigma \simeq S^2$.

To see what happens in the other two cases, we need to understand the holomorphic automorphisms of $\mathbf{C}$ and D.

Theorem 2. *Let $f : \mathbf{C} \to \mathbf{C}$ be a holomorphic homeomorphism. Then f has the form $f(z) = az + b$.*

Proof. Since f is a homeomorphism, it extends continuously to the one-point compactification by setting $f(\infty) = \infty$. We can therefore regard f as a map from $\mathbf{C}\mathrm{P}^1$ to itself. We claim that this map is holomorphic. Without loss of generality, we may assume $f(0) = 0$. To prove this, consider the behavior of f in a neighborhood of ∞: we have a map $g : \mathbf{C} \to \mathbf{C}$ defined by

$$g(z) = \begin{cases} \frac{1}{f(\frac{1}{z})} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$$

We claim that g is holomorphic. It is clearly holomorphic away from 0. The function

$$h(z) = \frac{1}{2\pi i} \int \frac{g(z)}{z} dz$$

is holomorphic and coincides with g away from the origin (by the Cauchy integral formula), and therefore coincides with g everywhere by continuity. The space of meromorphic functions on $\mathbf{C}\mathrm{P}^1$ having at most a simple pole at ∞ has dimension 2 (by the Riemann-Roch formula), and so consists of exactly those functions of the form $f(z) = az + b$. $\square$

Note that a homeomorphicm of the form $f(z) = az + b$ has a fixed point $\frac{b}{1-a}$ if $a \neq 1$. Consequently, if Γ is a group acting freely on $\mathbf{C}$ by holomorphic homeomorphisms, then Γ must act by translations $z \mapsto z + b$. We can then identify Γ with a subgroup of $\mathbf{C}$ (regarded additively). The action of Γ on $\mathbf{C}$ is properly discontinuous if and only if Γ is discrete: then Γ is a lattice in $\mathbf{C}$, which has rank 2 if and only if $\mathbf{C}/\Gamma$ is compact. In this case, the quotient $\mathbf{C}/\Gamma$ is a torus. We have proven:

Proposition 3. *Let Σ be a surface equipped with a conformal structure whose universal cover is $\mathbf{C}$ (as a Riemann surface). Then Σ is a torus.*

(We will prove the converse in a moment.)

Let us now consider the most interesting case: the unit disk D.

Theorem 4 (The Schwarz Lemma)**.** *Let $f : D \to D$ be a biholomorphic map such that $f(0) = 0$. Then $f(z) = az$ for some unit complex number a.*

Proof. Since $f(0) = 0$, we can write $f(z) = zg(z)$ for some holomorphic function g. By the maximum principle, if $|z| \leq r < 1$, then

$$|g(z)| \leq |g(y)| = \frac{|f(y)|}{y} \leq \frac{1}{r}$$

for some y satisfying $|y| = r$. It follows that $|g(z)| \leq 1$, so that $|f(z)| \leq |z|$. Applying the same argument to f^{-1}, we deduce that $|f(z)| = |z|$, so that $|g(z)| = 1$ everywhere. Since g is holomorphic, it must be constant, so that f is a linear map given by multiplication by some unit complex number $a = g(0)$. □

Corollary 5. *Every biholomorphic map from D to itself has the form*

$$z \mapsto a\frac{z-b}{1-\bar{b}z}$$

where a is a unit complex number and $|b| < 1$.

Proof. It is an easy exercise to see that the collection of such transformations forms a group G which maps D to itself. It is therefore a subgroup of the group G' of holomorphic automorphisms of D. We claim that $G = G'$. Fix $g' \in G'$, we wish to show that $g' \in G$. Composing g' with a transformation of the form $z \mapsto \frac{z-b}{1-\bar{b}z}$, we can assume that $g'(0) = 0$. Theorem 4 now implies that $g' \in G$. □

The group G has another description: it is precisely the group of orientation-preserving isometries of the unit disk D with respect to the hyperbolic metric $|ds|_{hyp} = \frac{2|ds|}{1-|z|^2}$. Consequently, if Σ is a conformal surface whose universal cover is D, then Σ is the quotient D/Γ, where Γ is a subgroup of G acting by hyperbolic isometries of D. It follows that Σ admits a hyperbolic metric: that is, a Riemannian metric of constant curvature -1. More precisely, Σ admits a unique hyperbolic metric compatible with the given conformal structureo n Σ.

Proposition 6. *Let Σ be as above. Then $\widetilde{\Sigma} \simeq D$ if and only if Σ has genus at least 2: that is, if and only if $\chi(\Sigma) < 0$.*

Proof. If Σ has genus ≥ 2, then we have already seen that $\widetilde{\Sigma}$ cannot be S^2 or $\mathbf{C}$, so the desired result follows from the Riemann uniformization theorem. Conversely, suppose that $\widetilde{\Sigma} = D$. Then Σ admits a hyperbolic metric. The Gauss-Bonnet theorem allows us to compute $\chi(\Sigma)$ as an integral of the curvature of this metric: since the curvature is everywhere negative, we get $\chi(\Sigma) < 0$. □

In the next few lectures, we will exploit the existence of hyperbolic metrics to understand the diffeomorphism groups of surfaces of genus $g \geq 2$. We conclude this lecture by explaining how these ideas carry over to the case of surfaces with boundary and nonorientable surfaces. There are two rather different ways to use hyperbolic metrics in this case.

Suppose first that Σ is a surface with boundary. Each boundary component of Σ is a circle. We can therefore view Σ as the real blow-up of a closed surface Σ' obtained by collapsing each boundary circle of Σ. Choose a conformal structure on Σ', and identify $\Sigma - \partial\Sigma$ with the punctured Riemann surface obtained by removing finitely many points from Σ. Then $\Sigma - \partial\Sigma$ has a universal cover X, which is the unit disk if and only if $\chi(\Sigma) < 0$. The arguments sketched above show that $\Sigma - \partial\Sigma$ inherits the structure of a hyperbolic manifold. This manifold is not compact, but there is a good replacement: it is a hyperbolic surface of finite area. In fact, one can show that this construction establishes an equivalence of categories between *punctured* Riemann surfaces of negative Euler characteristic and (oriented) hyperbolic surfaces finite volume.

There is another very different way to apply these ideas to a compact connected surface Σ, which we need not assume to be closed or oriented. Let $\overline{\Sigma}$ be the orientation double cover of Σ, and let σ be its canonical involution. The *double* $d(\Sigma)$ of Σ is the quotient of $\overline{\Sigma}$ obtained by identifying x with $\sigma(x)$ for $x \in \partial\Sigma$. Then $d(\Sigma)$ is a compact closed oriented surface (which is connected if and only if Σ is either nonorientable or has boundary). It is equipped with an orientation reversing involution, which we will continue to denote by σ. We can recover the original surface σ by forming the quotient $d(\Sigma)/\sigma$, and we can recover $\partial\Sigma$ as the fixed point locus of σ.

Remark 7. The preceding construction establishes a correspondence between compact surfaces with boundary and compact closed oriented surface with an orientation-reversing involution.

We can now apply all of our preceding methods to the double $d(\Sigma)$, but keeping track of the orientation reversing involution σ. First, choose an arbitrary Riemannian metric on $d(\Sigma)$. Averaging under σ, we can assume that this metric is σ-equivariant. The metric determines a complex structure on $d(\Sigma)$, with respect to which σ is an antiholomorphic involution. This lets us think of $d(\Sigma)$ as a Riemann surface with a real structure: in other words, as an algebraic curve over $\mathbb{R}$. The original surface Σ can be recovered as the set of closed points of the underlying $\mathbb{R}$-scheme, while the boundary $\partial\Sigma$ can be identified with the set of $\mathbb{R}$-points of this scheme.

Assuming $d(\Sigma)$ is connected (for simplicity), the universal cover $\widetilde{d(\Sigma)}$ is biholomorphic to either $\mathbf{C}\mathrm{P}^1$, $\mathbf{C}$, or the unit disk D. Note that $\chi(d(\Sigma)) = 2\chi(\Sigma)$, so this universal cover is the unit disk D if and only if $\chi(\Sigma) < 0$. In this case, the surface $d(\Sigma)$ inherits a canonical hyperbolic structure, and the map σ is an orientation-reversing isometry. It follows that $\Sigma = d(\Sigma)/\sigma$ again inherits a hyperbolic metric, this time as a manifold with boundary. Moreover, we understand what happens to the metric as we approach the boundary: namely, the boundary consists of geodesics.

We can summarize the discussion as follows: let Σ be a compact surface such that each connected component of Σ has negative Euler characteristic. The above construction determines a bijection between conformal structures on Σ which behave well at the boundary (these form a contractible space) and hyperbolic metrics on Σ with respect to which the boundary is geodesic.

Conjugacy Classes and Geodesic Loops (Lecture 35)

May 5, 2009

Let X be a path connected topological space and let $f : S^1 \to X$ be a map. Then f determines a conjugacy class $[f]$ in the fundamental group $\pi_1 X$. Our goal in this lecture is to show any nonzero conjugacy class is represented by an essentially *canonical* map f in the case where X is a hyperbolic surface.

Lemma 1. *Assume that X is a compact Riemannian manifold. Then any conjugacy class $\gamma \in \pi_1 X$ can be represented by a closed geodesic $f : S^1 \to X$.*

Proof. Endow the circle S^1 with its standard Riemannian metric, normalized so that the circle has total length 1. Define the *Lipschitz constant* $L(f)$ of a loop f to be the supremum of

$$\frac{d(f(x), f(y))}{d(x, y)}$$

. This supremum may be infinite: however, for a smooth path f it is finite (and coincides with maximum length of the derivative f' on S^1). Let c be the infimum of the set $\{L(f)\}$, where f varies over all representatives of γ. We will show that this infimum is achieved: that is, there exists a loop f with $L(f) = c$. Then f must be a smooth geodesic (of speed c) if it fails to be a geodesic near some point t, we can obtain a shorter loop representing γ by modifying f near t (and then changing our parametrization).

To prove that c is achieved, choose a sequence of loops $\{f_i\}_{i \geq 0}$ such that the real numbers $L(f_i)$ converge to c from above. Passing to a subsequence, we may assume that $L(f_i) < c + 1$. Choose a countable dense subset $\{t_j\} \subseteq S^1$. Since X is compact, we can pass to a subsequence and thereby assume that $f_0(t_0), f_1(t_0), \ldots$ converges to some point $x_0 \in X$. Similarly, we can pass to a subsequence of $\{f_1, f_2, \ldots\}$ and thereby guarantee that the sequence $f_1(t_1), f_2(t_1), \ldots$ converges to a point $x_1 \in X$. Proceeding in this way, we obtain a refinement of the original sequence such that $\{f_i(t_j)\}_{i \geq 0}$ converges to some $x_j \in X$. We define a new map $f : \{t_j\} \to X$ by the formula $f(t_j) = x_j$. We claim that f extends to a continuous map $S^1 \to X$ having $L(f) \leq c$. To prove this, it suffices to show that

$$d(f(t_i), f(t_j)) \leq c d(t_i, t_j)$$

for each pair of integers $i \neq j$. This is clear:

$$d(f(t_i), f(t_j)) \leq d(f(t_i), f_n(t_i)) + d(f(t_j), f_n(t_j) + d(f_n(t_i), f_n(t_j)) \leq \epsilon + L(f_n) d(t_i, t_j)$$

where ϵ can be made arbitrarily small (by choosing n large enough) and $L(f_n)$ can be made arbitrarily close to c.

Choose $\epsilon > 0$ small enough that every pair of points of X within a distance ϵ are connected by a unique geodesic. For $n \gg 0$, we have $d(f(t), f_n(t)) < \epsilon$ for all t, so that f and f_n can be connected by a geodesic homotopy; it follows that f is homotopic to f_n and therefore represents the free homotopy class γ. □

Let us now suppose that X is a hyperbolic surface, so that X can be represented as H/Γ where H is the upper half place $\{x + iy : y > 0\}$ and Γ is a group which acts on H by hyperbolic isometries. Then

$\Gamma \simeq \pi_1 X$, and we can identify Γ with a subgroup of the group $P\,\mathrm{SL}_2(\mathbb{R})$ of linear fractional transformations of the form

$$z \mapsto \frac{az+b}{cz+d}.$$

It is traditional to decompose elements of $P\,\mathrm{SL}_2(\mathbb{R})$ into three types:

(i) An element $A \in \mathrm{SL}_2(\mathbb{R})$ is called *elliptic* if $|\operatorname{tr}(A)| < 2$. In this case, the eigenvalues of A are unit complex numbers (and complex conjugate to one another); the transformation A itself is given by $z \mapsto \frac{\cos(\theta)z - \sin(\theta)}{\sin(\theta)z + \cos(\theta)}$ for some real number θ. Elliptic elements never appear in the discrete groups Γ under consideration, because they always have fixed points in the upper half plane (the above transformation has the complex number $z = i$ as a fixed point).

(ii) An element $A \in \mathrm{SL}_2(\mathbb{R})$ is called *parabolic* if $|\operatorname{tr}(A)| = 2$; in this case, the eigenvalues of A are both ± 1 but A is generally not semisimple: it is conjugate to a transformation of the form $z \mapsto z + t$ for some real number t. Nontrivial transformations of this kind cannot appear in Γ when the quotient $X = H/\Gamma$ is compact. For suppose otherwise: then, by Lemma 1, we would have a geodesic loop $f : S^1 \to X$ representing the conjugacy class of a parabolic transformation $z \mapsto z + t$. Then f lifts to a geodesic path $\widetilde{f}$ with the translation-invariance property $\widetilde{f}(x+1) = \widetilde{f}(t)$. There is no geodesic in the upper half plane with this property: the unique geodesic passing through $\widetilde{f}(0)$ and $\widetilde{f}(0) + t$ does not pass through $\widetilde{f}(0) + 2t$.

This argument does not apply if the quotient H/Γ is noncompact. In fact, a finite volume quotient H/Γ is compact if and only if Γ contains no parabolic elements: in fact, there is a bijection between cusps of H/Γ and conjugacy classes of maximal parabolic subgroups of Γ.

(iii) An element $A \in \mathrm{SL}_2(\mathbb{R})$ is called *hyperbolic* if $|\operatorname{tr}(A)| > 2$ (modifying A by a sign, we may assume that $\operatorname{tr}(A) > 2$). In this case, A has distinct real eigenvalues $\lambda, \frac{1}{\lambda}$ for some $\lambda > 1$. Then A is conjugate to the transformation $z \mapsto \lambda z$. In this case, there is a unique geodesic path $\widetilde{f} : \mathbb{R} \to H$ satisfying $\widetilde{f}(t+1) = A\widetilde{f}(t)$: namely, the path given by the formula $\widetilde{f}(t) = \lambda^t i$. This path descends to a geodesic loop $f : S^1 \to H/\Gamma$ representing the conjugacy class of $\pm A$ in $\Gamma \simeq \pi_1 H/\Gamma$.

The above analysis proves the following result:

Theorem 2. *Let $X = H/\Gamma$ be a compact hyperbolic surface. Then every nontrivial element γ of $\pi_1 X \simeq \Gamma \subseteq P\,\mathrm{SL}_2(\mathbb{R})$ is hyperbolic. Moreover, the conjugacy class of γ can be represented by a geodesic loop $f : S^1 \to X$ which is* unique *up to reparametrization.*

In other words, if X is a hyperbolic surface, then every conjugacy class in $\pi_1 X$ has a canonical representative. We now show that these representatives are well-behaved:

Theorem 3. *Let X be a hyperbolic surface, and suppose we are given distinct nontrivial conjugacy classes $\gamma_1, \ldots, \gamma_n \in \pi_1 X$. The following conditions are equivalent:*

(1) *The conjugacy classes γ_i can be represented by simple closed curves $C_i \subseteq X$ such that $C_i \cap C_j = \emptyset$ for $i \neq j$.*

(2) *The canonical geodesic representatives for $\gamma_1, \ldots, \gamma_n$ are simple closed curves $C_i \subseteq X$ such that $C_i \cap C_J = \emptyset$ for $i \neq j$.*

Proof. It is clear that (2) $\Rightarrow$ (1). Suppose that (1) is satisfied. Let $\{f_i : S^1 \to X\}_{1 \leq i \leq n}$ be a parametrizations of the curves C_i which satisfy condition of (1), and let $\{g_i : S^1 \to X\}_{1 \leq i \leq n}$ be the geodesic representatives of the conjugacy classes γ_i. We wish to prove that each g_i is a simple curve, and that $g_i(S^1) \cap g_j(S^1) = \emptyset$ for $i \neq j$. We will prove the latter; the former follows by the same argument.

Choose a lifting of g_i to a geodesic path $\widetilde{g}_i : \mathbb{R} \to D$, where D is the unit disk. If $g_i(S^1) \cap g_j(S^1) \neq \emptyset$, then we can lift g_j to a geodesic path $\widetilde{g}_j : \mathbb{R} \to D$ such that $\widetilde{g}_i(\mathbb{R})$ and $\widetilde{g}_j(\mathbb{R})$ intersect. Let $a, b \in \partial D$ be the endpoints of $\widetilde{g}_i$ on the circle at infinity, and let a', b' be the endpoints of $\widetilde{g}_j$. Note that $\widetilde{g}_i(\mathbb{R})$ and $\widetilde{g}_j(\mathbb{R})$ intersect if and only if the sets $\{a, b\}$ and $\{a', b'\}$ are disjoint, and the points a' and b' belong to different components of $\partial D - \{a, b\}$.

Since f_i and g_i represent the same conjugacy class in $\pi_1 X$, there is a homotopy h from f_i to g_i. Lifting this homotopy to the universal cover, we get a lift $\widetilde{f}_i : \mathbb{R} \to D$ of f_i and a homotopy from $\widetilde{f}_i$ to $\widetilde{g}_i$. This homotopy moves points by a bounded amount with respect to the hyperbolic metric on D. Consequently, it moves points which are close to the boundary ∂D by very small amounts with respect to the Euclidean metric on the closure of D. It follows that $\widetilde{f}_i$ has the same endpoints a and b as $\widetilde{g}_i$.

A similar argument shows that we can lift f_j to a path $\widetilde{f}_j : \mathbb{R} \to D$ having endpoints $a', b' \in \partial D$. If a' and b' belong to different components of $\partial D - \{a, b\}$, then $\widetilde{f}_i(\mathbb{R})$ and $\widetilde{f}_j(\mathbb{R})$ must have a point of intersection $\widetilde{x} \in D$. The image of $\widetilde{x}$ is a point $x \in f_i(S^1) \cap f_j(S^1) \subseteq X$, contradicting our assumptions. $\square$

Diffeomorphisms of Hyperbolic Surfaces (Lecture 36)

May 7, 2009

Let Σ be a compact, connected, oriented surface with $\chi(\Sigma) < 0$. Our goal in this lecture (and the next) is to describe the homotopy type of the diffeomorphism group $\mathrm{Diff}(\Sigma)$. We begin by observing that the universal cover $\widetilde{\Sigma}$ of $\Sigma - \partial\,\Sigma$ can be identified with the hyperbolic plane. It follows that Σ is an Eilenberg-MacLane space $K(\Gamma, 1)$, where Γ is a subgroup of $P\,\mathrm{SL}_2(\mathbb{R})$.

Lemma 1. *Let g be a nontrivial element of Γ. Then the centralizer of g is an infinite cyclic group, generated by an nth root of Γ for some $n \geq 1$.*

Proof. If Σ is closed, then g must be a hyperbolic element of $P\,\mathrm{SL}_2(\mathbb{R})$: without loss of generality, Σ corresponds to a fractional linear transformation of the form $z \mapsto \lambda z$. The centralizer of g in $P\,\mathrm{SL}_2(\mathbb{R})$ consists of linear fractional transformations of the form $z \mapsto \mu z$, where μ is a positive real number. It follows that the centralizer of g in Γ can be identified with a discrete subgroup of $(\mathbb{R}_{>0}, \times) \simeq (\mathbb{R}, +)$, and is therefore infinite cyclic.

If Σ has boundary, then g might be a parabolic element of $P\,\mathrm{SL}_2(\mathbb{R})$: in this case, we may assume without loss of generality that g is the linear fractional transformation $z \mapsto z + 1$. The centralizer of g in $P\,\mathrm{SL}_2(\mathbb{R})$ consists of linear fractional transformations of the form $z \mapsto z + t$. Consequently, the centralizer of g in Γ is a discrete subgroup of $(\mathbb{R}, +)$, and therefore infinite cyclic. $\square$

Corollary 2. *The center of Γ is trivial.*

Proof. Let g be a nonzero element of the center of Γ. Lemma 1 implies that the centralizer of g is cyclic, so that Γ is cyclic (genereated by either a hyperbolic element of the form $z \mapsto \lambda z$ or a parabolic transformation of the form $z \mapsto z + 1$). In either case, $\Sigma - \partial\,\Sigma \simeq D/\Gamma$ is homeomorphic to an annulus, and has Euler characteristic zero. $\square$

Let $\mathrm{Aut}(\Sigma)$ denote the monoid of self-homotopy equivalences of Σ, and let $\mathrm{Aut}_*(\Sigma)$ denote the monoid of self-homotopy equivalences of Σ that preserve a base point. Since Σ is a $K(\Gamma, 1)$, we deduce that $\mathrm{Aut}_*(\Sigma)$ is homotopy equivalent to the discrete space $\mathrm{Aut}(\Sigma)$ of automorphisms of the group Σ. We have a fiber sequence

$$\mathrm{Aut}_*(\Sigma) \to \mathrm{Aut}(\Sigma) \to \Sigma.$$

The long exact sequence of homotopy groups shows that $\pi_0 \,\mathrm{Aut}(\Sigma)$ can be identified with the group $\mathrm{Out}(\Gamma)\,\mathrm{Aut}(\Gamma)/\Gamma$ of outer automorphisms of Γ, the group $\pi_1 \,\mathrm{Aut}(\Sigma)$ can be identified with kernel of the map $\Gamma \to \mathrm{Aut}(\Sigma)$ (which vanishes by Corollary 2), and the groups $\pi_i \,\mathrm{Aut}(\Sigma)$ vanish for $i > 1$. In other words, $\mathrm{Aut}(\Sigma)$ homotopy equivalent to the discrete space $\mathrm{Out}(\Gamma)$.

Our goal in this lecture (and the next) is to prove the following:

Theorem 3. *Assume that Σ is closed. Then the obvious map $\mathrm{Diff}(\Sigma) \to \mathrm{Aut}(\Sigma) \simeq \mathrm{Out}(\Gamma)$ is a homotopy equivalence.*

We now describe the analogue of Theorem 3 in the case where Σ has boundary $C_1 \cup C_2 \cup \ldots \cup C_n$. Let $\gamma_1, \ldots, \gamma_n$ denote representatives for these loops in $\pi_1 \Sigma$. Let $\mathrm{Diff}_\partial(\Sigma)$ denote the group of diffeomorphisms of Σ that fix each C_i pointwise. Similarly, we let $\mathrm{Aut}_\partial(\Sigma)$ be the monoid of self-homotopy equivalences of the pair $(\Sigma, \partial\,\Sigma)$ which are the identity on the boundary. We have a fiber sequence

$$\mathrm{Aut}_\partial(\Sigma) \to \mathrm{Aut}(\Sigma) \to \mathrm{Map}(\partial\,\Sigma, \Sigma).$$

The base of this fibration can be identified with the nth power of $\mathrm{Map}(S^1, \Sigma)$, whose connected components can be identified with conjugacy classes in Γ where each connected component is a classifying space for the centralizer of the corresponding element of Γ. We obtain a group-theoretic description of $\mathrm{Aut}_\partial(\Sigma)$: it is homotopy equivalent to the discrete set $\mathrm{Out}_\partial(\Gamma)$ consisting sequences $(\phi; \phi_1, \ldots, \phi_n)$, where ϕ is an outer automorphism of Γ, and each ϕ_i is an automorphism of Γ representing ϕ such that $\phi_i(\gamma_i) = \gamma_i$.

Remark 4. To obtain this identification more precisely, we should be more careful about base points. Fix a point x_i on each C_i. A homotopy equivalence f of Σ which is the identity on $\partial\,\Sigma$ induces well-defined maps $\phi_i : \pi_1(\Sigma, x_i) \to \pi_1(\Sigma, x_i)$, each of which fixes the class γ_i represented by the loop C_i.

The analogue of Theorem 3 is the following:

Theorem 5. *Let Σ be a compact connected oriented surface with $\chi(\Sigma) < 0$. Then the obvious map $\mathrm{Diff}_\partial(\Sigma) \to \mathrm{Aut}_\partial(\Sigma) \simeq \mathrm{Out}_\partial(\Sigma)$ is a homotopy equivalence.*

We can break the assertion of Theorem 5 into two parts. Let $\mathrm{Diff}^0_\partial(\Sigma)$ denote the inverse image of the identity element of $\mathrm{Out}_\partial(\Sigma)$. We must show:

(1) The space $\mathrm{Diff}^0_\partial(\Sigma)$ is contractible.

(2) The map $\mathrm{Diff}_\partial(\Sigma) \to \mathrm{Out}_\partial(\Sigma)$ is surjective.

We will begin the proof of (1) in this lecture. The proof proceeds by induction on the complexity of Σ: we consider another surface Σ' to be simpler than Σ if either it has a smaller genus, or has the same genus and a smaller number of boundary components. The base case for the induction is when Σ is a pair of pants: a surface of genus zero with exactly three boundary components. We will treat this case (and assertion (2)) in the next lecture.

Assume therefore that Σ is more complicated than a pair of pants. If Σ has positive genus, then we can choose a simple nonseparating closed curve C in Σ such that cutting Σ along C decreases the genus. If Σ has genus 0 but $n > 3$ boundary components, then there exists a separating simple closed curve C which decomposes Σ into two components, each of which has fewer than n boundary components. In either case, we can choose the curve C to be smooth.

Proposition 6. *Let $\mathrm{Diff}_\partial(\Sigma, C)$ be the subgroup of $\mathrm{Diff}_\partial(\Sigma)$ consisting of those diffeomorphisms restrict to an orientation-preserving diffeomorphism of C, and let $\mathrm{Diff}'_\partial(\Sigma)$ be the subgroup of $\mathrm{Diff}_\partial(\Sigma)$ consisting of those elements which fix the conjugacy class in Γ represented by C. Then the inclusion $\mathrm{Diff}_\partial(\Sigma, C) \hookrightarrow \mathrm{Diff}'_\partial(\Sigma)$ is a homotopy equivalence.*

Proof. Let $X(\Sigma)$ denote the collection of all hyperbolic metrics on Σ with respect to which each boundary component is geodesic. Let Y be the collection of all smooth simple closed curves in Σ which are freely homotopic to C. Given a hyperbolic metric on Σ, the class $[C]$ has a unique geodesic representative: this determines a fibration $X(\Sigma) \to Y$, whose fiber is the subspace $X_0(\Sigma) \subseteq X(\Sigma)$ of hyperbolic metrics with respect to which C is a geodesic loop.

Let Σ' be the surface obtained by cutting Σ along C; and let $M(C)$ denote the collection of smooth metrics on C. We have a (homotopy) pullback diagram

$$\begin{array}{ccc} X_0(\Sigma) & \longrightarrow & X(\Sigma') \\ \downarrow & & \downarrow \\ M(C) & \longrightarrow & M(C) \times M(C). \end{array}$$

The space $M(C)$ is contractible, so $X_0(\Sigma) \to X(\Sigma')$ is a homotopy equivalence. Since $X(\Sigma')$ is contractible, we deduce that $X_0(\Sigma)$ is contractible. Since $X(\Sigma)$ is contractible, we conclude that Y is contractible. Finally, we have a fiber sequence

$$\mathrm{Diff}_\partial(\Sigma, C) \to \mathrm{Diff}'_\partial(\Sigma) \to Y,$$

which shows that $\mathrm{Diff}_\partial(\Sigma, C) \to \mathrm{Diff}'_\partial(\Sigma)$ is a homotopy equivalence. $\square$

Let $\mathrm{Diff}_{\partial\cup C}(\Sigma)$ be the group of diffeomorphisms of Σ which fix $\partial\,\Sigma \cup C$ pointwise. If $f \in \mathrm{Diff}_\partial(\Sigma)$ fixes C pointwise, then f determines an automorphism ϕ_C of $\pi_1(\Sigma, x)$ which fixes the class $\gamma \in \pi_1(\Sigma, x)$ represented by C. Let $\mathrm{Out}_{\partial,C}(\Gamma)$ denote the set of quadruples $(\phi, \phi_1, \ldots, \phi_n, \phi_C)$ where $(\phi, \phi_1, \ldots, \phi_n) \in \mathrm{Out}_\partial(\Gamma)$ and ϕ_C is as above. Note that since the centralizer of γ is isomorphic to the cyclic group generated by γ, we have an exact sequence

$$\mathbf{Z} \to \mathrm{Out}_{\partial,C}(\Gamma) \to \mathrm{Out}_\partial(\Gamma).$$

Similarly, we have a fiber sequence

$$\mathrm{Diff}_{\partial,C}(\Sigma) \to \mathrm{Diff}_\partial(\Sigma, C) \to \mathrm{Diff}^+(C),$$

fitting into a map of fiber sequences

$$\begin{array}{ccccc} \Omega\,\mathrm{Diff}^+(C) & \longrightarrow & \mathrm{Diff}_{\partial,C}(\Sigma) & \longrightarrow & \mathrm{Diff}_\partial(\Sigma, C) \\ \downarrow & & \downarrow{\scriptstyle\psi} & & \downarrow \\ \mathbf{Z} & \longrightarrow & \mathrm{Out}_{\partial,C}(\Gamma) & \longrightarrow & \mathrm{Out}_\partial(\Gamma). \end{array}$$

Since the left map is a homotopy equivalence, we can identify fibers of the right map with fibers of the middle map. Consequently, to prove (1), it suffices to show that $\mathrm{Diff}^0_{\partial,C}(\Sigma) = \psi^{-1}\{e\}$ is contractible. Note that $\mathrm{Diff}_{\partial,C}(\Sigma)$ is homotopy equivalent to $\mathrm{Diff}_\partial(\Sigma')$. By the inductive hypothesis, $\mathrm{Diff}_\partial(\Sigma')$ is a union of contractible components. It therefore suffices to show that $\mathrm{Diff}^0_{\partial,C}(\Sigma)$ lies in a single one of these components. Unwinding the definitions, we must show that if f is a diffeomorphism of Σ fixing the boundary together with C and $\overline{f}$ is the corresponding diffeomorphism of Σ', then $\overline{f}$ induces the identity map on $\pi_1(\Sigma', x)$ for every point $x \in \partial\,\Sigma'$. To see this, it suffices to show that the map $\pi_1(\Sigma', x) \to \pi_1(\Sigma, x)$ is injective. There are two cases to consider:

(a) The curve C is separating, so that $\Sigma' = \Sigma_1 \cup \Sigma_2$. The van Kampen theorem allows us to compute that $\pi_1\Sigma' \simeq \pi_1\Sigma_1 \star_{\pi_1 C} \pi_1\Sigma_2$. Since $\pi_1 C \simeq \mathbf{Z}$ is a subgroup of both $\pi_1\Sigma_1$ and $\pi_1\Sigma_2$ (this follows from Lemma 1), we conclude that the maps $\pi_1\Sigma_1 \to \pi_1\Sigma \leftarrow \pi_1\Sigma_2$ are injective.

(b) We will discuss this case in the next lecture.

More on Mapping Class Groups (Lecture 37)

May 11, 2009

Let us begin with a recap of the previous lecture. Let Σ be a compact, connected, oriented surface with $\chi(\Sigma) < 0$, and let Γ denote the fundamental group of Γ. We let $\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\Gamma$ be the outer automorphism group of Γ. For any collection of embedded oriented loops $C_1, \ldots, C_n \subseteq \Gamma$, choose a base point x_i on each C_i, and let γ_i denote the homotopy class of C_i in $\pi_1(\Sigma, x_i) \simeq \Gamma$. We let $\mathrm{Out}_{C_1,\ldots,C_n}(\Sigma)$ denote the group of tuples $(\phi, \phi_1, \ldots, \phi_n)$ where $\phi \in \mathrm{Out}(\Gamma)$, and each ϕ_i is an automorphism of $\pi_1(\Sigma, x_i)$ which represents ϕ and fixes γ_i. The map

$$(\phi, \phi_1, \ldots, \phi_n) \to \phi$$

is a group homomorphism from $\mathrm{Out}_{C_1,\ldots,C_n}(\Sigma)$, whose image is the collection of outer automorphisms of Γ which fix the conjugacy classes of γ_i and whose kernel is the product of centralizers $\prod_{1 \leq i \leq n} Z(\gamma_i)$ Provided that each C_i is essential (that is, not nullhomotopic), these centralizers coincide with the cyclic group $\gamma_i^{\mathbf{Z}}$ generated by γ_i, and are canonically isomorphic to $\mathbf{Z}$.

In the special case where the collection C_i consist of all boundary components of Σ, we will denote $\mathrm{Out}_{C_1,\ldots,C_n}(\Gamma)$ by $\mathrm{Out}_\partial(\Gamma)$. If the collection C_i includes all boundary components together with one additional embedded loop C, we denote this group instead by $\mathrm{Out}_{\partial,C}(\Gamma)$.

Fix now an embedded loop C in Σ containing a point x, and let $\gamma \in \pi_1(\Sigma, x) \simeq \Gamma$ be the class represented by C. We let $\mathrm{Out}'_\partial(\Gamma)$ denote the subgroup of $\mathrm{Out}_\partial(\Gamma)$ consisting of outer automorphisms which fix the conjugacy class of γ. Let $\mathrm{Diff}_\partial(\Sigma)$ be the group of diffeomorphisms of Σ which fix the boundary pointwise, $\mathrm{Diff}_\partial(\Sigma, C)$ the subgroup consisting of diffeomorphisms which restrict to an orientation-preserving diffeomorphism of C, and $\mathrm{Diff}_{\partial,C}(\Sigma)$ the subgroup consisting of diffeomorphisms which fix C pointwise. In the last lecture, we saw that there is a homotopy pullback diagram

$$\begin{array}{ccccc}
\mathrm{Diff}_{\partial,C}(\Sigma) & \longrightarrow & \mathrm{Diff}_\partial(\Sigma, C) & \longrightarrow & \mathrm{Diff}_\partial(\Sigma) \\
\downarrow \psi & & \downarrow & & \downarrow \\
\mathrm{Out}_{\partial,C}(\Gamma) & \longrightarrow & \mathrm{Out}'_\partial(\Gamma) & \longrightarrow & \mathrm{Out}_\partial(\Gamma).
\end{array}$$

Moreover, $\mathrm{Diff}_{\partial,C}(\Sigma)$ is homotopy equivalent to $\mathrm{Diff}_\partial(\Sigma')$, where Σ' is the surface obtained by cutting Σ along C. Our ultimate goal is to prove that the vertical maps are homotopy equivalences. For the moment, we will be content to prove the following weaker statement:

($*$) In the above diagram, each of the vertical maps has a contractible kernel.

As we explained last time, the proof proceeds by induction. Since each square in the above diagram is a homotopy pullback, the kernels of the vertical maps are all homotopy equivalent. Consequently, it will suffice to show that the kernel of ψ is contractible. There are two cases to consider:

(1) The curve C is nonseparating. In this case, the surface Σ' is connected. Let $\psi' : \mathrm{Diff}_\partial(\Sigma') \to \mathrm{Out}_\partial(\Sigma')$ be the canonical map. Since Σ' is simpler than C, the inductive hypothesis guarantees that the kernel

$\ker(\psi')$ is contractible; in particular, the kernel of ψ' is the identity component of $\mathrm{Diff}_\partial(\Sigma')$. Since $\mathrm{Diff}_{\partial,C}(\Sigma)$ is homotopy equivalent to $\mathrm{Diff}_\partial(\Sigma')$, its identity component is also contractible. To prove that $\ker(\psi)$ is contractible, it suffices to show that $\ker(\psi)$ coincides with the identity component of $\mathrm{Diff}_{\partial,C}(\Sigma)$. Suppose otherwise: then there exists a diffeomorphism $f \in \mathrm{Diff}_{\partial,C}(\Sigma)$ which is not isotopic to the identity, such that f induces the identity map from $\pi_1(\Sigma, x_i)$ to itself, whenever x_i is a base point on C or some boundary component of Σ. Let f' be the induced diffeomorphism of Σ'. Then f' is not isotopic to the identity, so the image of $f' \in \mathrm{Out}_\partial(\Sigma')$ is nontrivial. It follows that for some base point y on some boundary component of Σ', f' induces a nontrivial automorphism $f'_* : \pi_1(\Sigma', y) \to \pi_1(\Sigma', y)$. We have a commutative diagram

$$\begin{array}{ccc} \pi_1(\Sigma', y) & \longrightarrow & \pi_1(\Sigma, y) \\ \downarrow f'_* & & \downarrow f_* \\ \pi_1(\Sigma', y) & \longrightarrow & \pi_1(\Sigma, y). \end{array}$$

Since f_* is the identity, we deduce that the horizontal maps are not injective.

On the other hand, we can compute $\pi_1\Sigma$ from $\pi_1\Sigma'$ using a generalization of van Kampen's theorem. Note that Σ is obtained from Σ' by gluing along a pair of boundary components B_0 and B_1 (having image C in Σ). Consider the following more general situation: let X' be a well-behaved connected topological space with a pair of disjoint, well-behaved connected closed subsets B_0 and B_1, and let X be the space obtained by gluing B_0 to B_1 along some homeomorphism h. The map h induces an isomorphism $\pi_1 B_0 \simeq \pi_1 B_1$; let us denote this common fundamental group by H. Let γ be a path in X' from a base point p of B_0 to the base point $h(p)$ of B_1, and take p to be a base point of X'. Then the inclusions of B_0 and B_1 into X' induce group homomorphisms $i, j : H \to G = \pi_1 X'$, where j is defined by carrying a loop α to $\gamma^{-1} \circ \alpha \circ \gamma$. Note that γ maps to a closed loop in X, and therefore determines a class $t \in \pi_1 X$. We have the following classical result:

Theorem 1. *The group $\pi_1 X$ is generated by $G = \pi_1 X'$ together with the element g, subject only to the relations $ti(h) = j(h)t$ for $h \in H$.*

In the special case where the maps i and j are injective, we say that $\pi_1 X$ is obtained from G by an *HNN-extension*. In this case, we can describe $\pi_1 X$ very explicitly. Choose a set C_+ of left coset representatives of $i(H)$ in G (including the identity) and set C_- of left coset representatives of $j(H)$ in G. Then every element of $\pi_1 X$ can be written uniquely in the form

$$g t^{n_1} c_1 t^{n_2} c_2 \dots t^{n_k} c_k$$

where the n_i are nonnegative integers, $c_i \in C_+$ if $n_i > 0$, $c_i \in C_-$ if $n_i < 0$, and c_i is nonzero unless $n = k$. The image of G corresponds to those elements for which $k = 0$. This description shows that G injects into $\pi_1 X$.

In our case, the subsets B_0 and B_1 are inclusions of boundary components in the surface Σ'. We therefore have $\pi_1 B_0 \simeq \pi_1 B_1 \simeq \mathbf{Z}$, and the inclusion maps $i, j : \mathbf{Z} \to \pi_1\Sigma'$ are both injective. It follows that $\pi_1\Sigma' \to \pi_1\Sigma$ is injective, as desired.

(2) The curve C is separating. In this case, we can write Σ' as a disjoint union of two connected components $\Sigma_0 \cup \Sigma_1$, each of which contains C as a boundary curve. Let Γ_0 and Γ_1 be their fundamental groups. We have a map $\psi' : \mathrm{Diff}_\partial(\Sigma') \to \mathrm{Out}_\partial(\Gamma_0) \times \mathrm{Out}_\partial(\Gamma_1)$. The inductive hypothesis guarantees that $\ker(\psi')$ is contractible; in particular, it is the identity component of $\mathrm{Diff}_\partial(\Sigma')$. We conclude again that the identity component of $\mathrm{Diff}_{\partial,C}(\Sigma)$ is contractible. To complete the proof, it will suffice to show that this identity component coincides with $\ker(\psi)$. Assume otherwise; then we have a diffeomorphism $f \in \mathrm{Diff}_{\partial C}(\Sigma)$ which is not isotopic to the identity, but induces the identity on $\pi_1(\Sigma, x_i)$ for any base point x_i in $\partial\Sigma$ or in C. Let f' be the induced diffeomorphism of Σ'. Since f' does not lie in the boundary component of $\mathrm{Diff}_\partial(\Sigma')$, its image is nontrivial in either $\mathrm{Out}_\partial(\Gamma_0)$ or $\mathrm{Out}_\partial(\Gamma_1)$. It follows

that there exists a point y in some boundary component of Σ' such that $f'_* : \pi_1(\Sigma', y) \to \pi_1(\Sigma', y)$ is nontrivial. Since f_* is trivial on $\pi_1(\Sigma, y)$, we deduce that $\pi_1(\Sigma', y) \to \pi_1(\Sigma, y)$ is not injective. We will obtain a contradiction.

By van Kampen's theorem (in its usual form), the fundamental group $\pi_1\Sigma$ can be recovered as an amalgamated product $\pi_1\Sigma_0 \star_{\pi_1 C} \pi_1\Sigma_1 = \Gamma_0 \star_{\mathbf{Z}} \Gamma_1$. Since the maps $\pi_1 C \to \pi_1\Sigma_i$ are injective, this free product admits an explicit description: if we chose sets of left coset representatives C_0 and C_1 (including the identity) for $\mathbf{Z}$ in Γ_0 and Γ_1, then every element of $\pi_1\Sigma$ can be written uniquely in the form

$$gc_0c_1c_2\ldots c_k,$$

where $g \in \mathbf{Z}$ and the c_i are nontrivial elements of $C_0 \coprod C_1$ which alternate between C_0 and C_1. The uniqueness guarantees that the maps $\Gamma_0 \to \Gamma \leftarrow \Gamma_1$ are injective.

The inductive mechanism above reduces the proof of the main theorem to the case where Σ is the simplest possible hyperbolic surface: namely, a pair of pants. In this case, we let $\mathrm{Diff}(\Sigma, \partial)$ be the group of diffeomorphisms of Σ which restrict to orientation preserving diffeomorphisms of each boundary component. We have a fiber sequence

$$\mathrm{Diff}^+_\partial(\Sigma) \to \mathrm{Diff}(\Sigma, \partial) \to \mathrm{Diff}^+(S^1)^3.$$

(Here the notation Diff^+ indicates that we are restricting our attention to orientation-preserving diffeomorphisms.) Since $\mathrm{Diff}^+(S^1)$ is homotopy equivalent to the circle group, the fiber sequence gives rise to another fiber sequence in the homotopy category.

$$\mathbf{Z}^3 \to \mathrm{Diff}^+_\partial(\Sigma) \to \mathrm{Diff}(\Sigma, \partial).$$

This sequence fits into a commutative diagram

$$\begin{array}{ccccc} \mathbf{Z}^3 & \longrightarrow & \mathrm{Diff}^+_\partial(\Sigma) & \longrightarrow & \mathrm{Diff}(\Sigma, \partial) \\ \downarrow & & \downarrow \psi & & \downarrow \psi_0 \\ \mathbf{Z}^3 & \longrightarrow & \mathrm{Out}_\partial(\Sigma) & \longrightarrow & \mathrm{Out}(\Sigma). \end{array}$$

It follows that the right square is a homotopy pullback, so that $\ker(\psi)$ is homotopy equivalent to $\ker(\psi_0)$, which is a union of connected components of $\mathrm{Diff}(\Sigma, \partial)$. To complete the proof in this case, it will suffice to show that $\mathrm{Diff}(\Sigma, \partial)$ is contractible.

Let S^2 denote the 2-sphere, so that Σ can be identified with the surface obtained from S^2 by performing a real blow-up at three points $\{x, y, z\}$. Let $\mathrm{Diff}^+(S^2, \{x, y, z\})$ be the group of diffeomorphisms of S^2 that fix the points x, y, and z. Then the construction of the real blow-up induces a map $\mathrm{Diff}^+(S^2, \{x, y, z\}) \to \mathrm{Diff}(\Sigma, \partial)$. This map is a homotopy equivalence: it has a homotopy inverse (in the PL category, say) given by coning off the boundary components. Consequently, it suffices to prove that $\mathrm{Diff}^+(S^2, \{x, y, z\})$ is contractible.

Let X denote the open subset of $(S^2)^3$ consisting of triples of *distinct* points of S^2. We have a homotopy fiber sequence

$$\mathrm{Diff}^+(S^2, \{x, y, z\}) \to \mathrm{Diff}^+(S^2) \xrightarrow{a} X.$$

Consequently, we are reduced to proving that the map a is a homotopy equivalence. In a previous lecture, we saw that the group $\mathrm{PGL}_2(\mathbf{C})$ of biholomorphisms of $S^2 \simeq \mathbf{C}\mathrm{P}^1$ is homotopy equivalent to $\mathrm{Diff}^+(S^2)$. It therefore suffices to show that the action of $\mathrm{PGL}_2(\mathbf{C})$ on X determines a homotopy equivalence $\mathrm{PGL}_2(\mathbf{C}) \to X$. But this map is actually a homeomorphism: for every triple of distinct points $x, y, z \in \mathbf{C}\mathrm{P}^1$, there is a unique linear fractional transformation which carries (x, y, z) to $(0, 1, \infty)$.

To complete our understanding of mapping class groups, we would also like to know that the map $\psi : \mathrm{Diff}_\partial(\Sigma) \to \mathrm{Out}_\partial(\Gamma)$ is *surjective*. This assertion can formulated in group theoretic terms: for example, it implies that if Γ is a surface group given as an amalgamated free product $\Gamma_0 \star_{\mathbf{Z}} \Gamma_1$, then any automorphism of Γ which is trivial on the subgroup $\mathbf{Z}$ arises from automorphisms of Γ_0 and Γ_1. However, we will give a more direct geometric argument in the next lecture.

The Dehn-Nielsen Theorem (Lecture 38)

May 13, 2009

In this lecture, we will complete our understanding of the homotopy types of diffeomorphism groups of hyperbolic surfaces by proving the following result:

Theorem 1 (Dehn-Nielsen)**.** *Let Σ be a compact oriented surface with $\chi(\Sigma) = -k < 0$. Then the map $\mathrm{Diff}_\partial(\Sigma) \rightarrow \mathrm{Out}_\partial(\Sigma)$ is surjective.*

Since $\mathrm{Out}_\partial(\Sigma)$ is the group of connected components of the space $\mathrm{Aut}_\partial(\Sigma)$ of self-homotopy equivalences of Σ which are fixed on the boundary, we can reformulate Theorem 1 as follows:

Theorem 2. *Let $f : \Sigma \rightarrow \Sigma'$ be a homotopy equivalence between compact oriented surfaces with $\chi(\Sigma) = \chi(\Sigma') = -k < 0$. Assume that f restricts to a diffeomorphism $\partial\,\Sigma \simeq \partial\,\Sigma'$. Then f is homotopic (relative to the boundary of Σ) to a diffeomorphism $\Sigma \simeq \Sigma'$.*

We may assume without loss of generality that f is a smooth map, and that $f^{-1}\,\partial\,\Sigma' = \partial\,\Sigma$. Choose a system of disjoint simple closed curves $C_1, C_2, \ldots, C_n$ in Σ' which cut Σ' into a union of finitely many pairs of pants (an Euler characteristic calculation shows that the number of pairs of pants must be exactly k, so that $\Sigma' = P_1 \cup \ldots \cup P_k$). Modifying f slightly, we may assume that f is transverse to the curves C_i. Let $T = f^{-1}(C_1 \cup \ldots \cup C_n)$, so that T is a smooth submanifold of Σ consisting of some finite number m of circles. We will assume that f has been chosen (in its homotopy class) so as to minimize m.

Let $Q_1, \ldots, Q_{k'}$ be the collection of components of the surface obtained by cutting Σ along T; we will identify each Q_i with a closed subset of Σ.

Claim 3. *Each Q_i has nonpositive Euler characteristic.*

Proof. If not, some Q_i must be a disk. Say f carries the boundary of Q_i into the circle C, and Q_i itself into a pair of pants P. Then f determines a class in the relative homotopy group $\pi_2(P, C)$, which is the fundamental group of the homotopy fiber of the inclusion $C \mapsto P$. Since $\pi_1 C \hookrightarrow \pi_1 P$, the relevant homotopy fiber is homotopy equivalent to the discrete space $\pi_1 P / \pi_1 C$, and has a trivial fundamental group. Consequently, $f|Q_i$ is homotopic to a map carrying Q_i into the circle C. Modifying this map by a small homotopy, we obtain a new map f' homotopic to the original f, such that ${f'}^{-1}(C_1 \cup \ldots \cup C_n)$ has fewer connected components than T. This contradicts the minimality of m. □

Claim 4. *Let T_i be a connected component of T, and suppose that f carries T_i into C_j. Then:*

(1) *The map $f|T_i : T_i \rightarrow C_j$ has degree ± 1.*

(2) *The loop T_i is not homotopic to any boundary loop of Σ.*

Proof. We first claim that T_i is not nullhomotopic in Σ. Otherwise, T_i would bound an embedded disk. Inside this disk we can find an "innermost" component $T_{i'}$ of T, which also bounds a disk, contradicting Claim 3. Thus $[T_i]$ is nontrivial in $\pi_1 \Sigma$. Since f is a homotopy equivalence, $f_*[T_i] = [C_j]^d$ is nontrivial in $\pi_1 \Sigma'$, where d is the degree of $f|T_i$. It follows that $d \neq 0$. If $|d| > 1$, then $f_*[T_i]$ is divisible in $\pi_1 \Sigma'$, so that $[T_i]$ is divisible in $\pi_1 \Sigma$; this contradicts our assumption that f is an embedded loop.

To prove (2), we note that if $[T_i]$ is conjugate to some boundary component of Σ, then $f_*[T_i] \simeq [C_j]^{\pm 1}$ is conjugate to some boundary component of Σ', which contradicts our choice of C_j. □

Adjusting f by a homotopy, we may assume that the restriction of f to each component of T is a diffeomorphism onto one of the circles C_i.

Claim 5. *Each Q_i has negative Euler characteristic.*

Proof. Assume that $\chi(Q_i) \geq 0$. It follows from Claim 3 that $\chi(Q_i) = 0$, so that Q_i is an annulus. Using Claim 4, we deduce that both boundary components of Q_i belong to T. Let us denote these boundary components by B and B'. Let P be the pair of pants containing $f(Q_i)$. Then $f(B)$ and $f(B')$ are boundary components of P. Since $f(B)$ and $f(B')$ are freely homotopic in P, they must be the same boundary component $P_0 \subseteq P$. Consider the map

$$\phi : \mathrm{Map}(S^1, P_0) \to \mathrm{Map}(S^1, P).$$

If we restrict attention to the connected component containing the isomorphism $S^1 \simeq P_0$, then ϕ is a homotopy equivalence: this follows from the observation that the centralizer of $[P_0]$ in $\pi_1 P$ is isomorphic to its centralizer in $\pi_0 P_0 \simeq \mathbf{Z}$. Consequently, the map $Q_i \to P$ is homotopic (relative to its boundary) to a map $Q_i \to P_0$. Modifying this map by a small homotopy, we obtain a new map $f' : \Sigma' \to \Sigma$ such that ${f'}^{-1}(C_1, \ldots, C_n)$ has fewer than m components, which is a contradiction. □

Since the map f has degree ± 1 (being a homotopy equivalence) it must be surjective. Consequently, the inverse image of each P_i is a finite union of Q_j's. According to Claim 5, each of these components has negative Euler characteristic. It follows that $\chi(f^{-1}(P_i)) \leq -1$. We have

$$-k = \chi(\Sigma) = \chi(f^{-1}P_1) + \ldots + \chi(f^{-1}P_k) \leq -1 + \ldots + -1 = -k.$$

It follows that each $f^{-1}P_i$ must consist of exactly one connected component (which we will denote by Q_i) having Euler characteristic -1. Since the map f is surjective, $Q_i \to P_i$ is surjective, so that Q_i has at least three boundary components. It follows that Q_i is also a pair of pants, and that f restricts to a map $f_i : Q_i \to P_i$ which is a diffeomorphism between their boundaries. To complete the proof, it will suffice to show that each f_i is homotopic to a diffeomorphism.

Choose disjoint smooth arcs D_1, D_2, D_3 which join the boundary components of P_i. We may assume without loss of generality that f_i is transverse to the arcs D_j, so that $f_i^{-1} D_j$ is a smooth submanifold (with boundary) of Q_i. The boundary of $f_i^{-1}(D_1 \cup D_2 \cup D_3)$ is $f_i^{-1}((D_1 \cup D_2 \cup D_3) \cap \partial P_i)$ which consists of six points (since f_i is a diffeomorphism on the boundaries). It follows that $f_i^{-1}(D_1 \cup D_2 \cup D_3)$ consists of three arcs together with m' circles, for some $m' \geq 0$. Let us denote these arcs by D'_1, D'_2, and D'_3; modifying f_i by a homotopy we may assume that D'_j maps homeomorphically onto D_j for $j \in \{1, 2, 3\}$.

We will assume that f_i has been chosen (in its homotopy class) to minimize m'. Cutting Q_i along the arcs D'_1, D'_2, and D'_3, we obtain a decomposition $Q_i \simeq Q_i^+ \cup Q_i^-$, where Q_i^+ and Q_i^- are disks.

Claim 6. *The integer m' is equal to zero.*

Proof. Let $\widetilde{C}$ be a circle component of $f_i^{-1}(D_1 \cup D_2 \cup D_3)$. Without loss of generality, $\widetilde{C} \subseteq Q_i^+$. Choosing a different circle component if necessary, we may assume that $\widetilde{C}$ is innermost: that is, $\widetilde{C}$ bounds a disk E such that $E \cap f^{-1}(D_1 \cup D_2 \cup D_3) = \partial E = \widetilde{C}$. Without loss of generality, the map f carries ∂E to the arc D_1. Then f determines a class in the relative homotopy group $\pi_2(E', D_1)$, where E' is one of the disks obtained by cutting P_i along $D_1 \cup D_2 \cup D_3$). Since E' and D_1 are both contractible, this homotopy group is trivial. It follows that $f_i | E$ is homotopic (relative to its boundary) to a map carrying E into D_1. Modifying this map by a small homotopy, we obtain a new map f'_i such that ${f'_i}^{-1}(D_1 \cup D_2 \cup D_3)$ has fewer circle components, contradicting the minimality of m'. □

The arcs $D_1 \cup D_2 \cup D_3$ cut P_i into two components, which we will denote by P_i^+ and P_i^-. Using Claim 6, we may assume without loss of generality that f_i restricts to a pair of maps

$$f_i^+ : Q_i^+ \to P_i^+ \qquad f_i^- : Q_i^- \to P_i^-,$$

each of which is a diffeomorphism on the boundary. Using the Alexander trick (remember that there is no essential difference between the smooth and PL categories in dimensison 2), we can assume that f_i^+ and f_i^- are diffeomorphisms.