An introduction to algebraic surgery

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Introduction

Surgery theory investigates the homotopy types of manifolds, using a combination of algebra and topology. It is the aim of these notes to provide an introduction to the more algebraic aspects of the theory, without losing sight of the geometric motivation.

0.1 Historical background

A closed m-dimensional topological manifold M has Poincaré duality isomorphisms

$$H^{m-*}(M) \;\cong\; H_*(M) \;.$$

In order for a space X to be homotopy equivalent to an m-dimensional manifold it is thus necessary (but not in general sufficient) for X to be an m-dimensional Poincaré duality space, with $H^{m-*}(X) \cong H_*(X)$. The *topological structure set* $\mathcal{S}^{TOP}(X)$ is defined to be the set of equivalence classes of pairs

$$(m\text{-dimensional manifold } M \;,\; \text{homotopy equivalence } h : M \to X)$$

subject to the equivalence relation

$$\begin{aligned}(M,h) \sim (M',h') \;\;& \text{if there exists a homeomorphism} \\ & f : M \to M' \text{ such that } h'f \simeq h : M \to X.\end{aligned}$$

The basic problem of surgery theory is to decide if a Poincaré complex X is homotopy equivalent to a manifold (i.e. if $\mathcal{S}^{TOP}(X)$ is non-empty), and if so to compute $\mathcal{S}^{TOP}(X)$ in terms of the algebraic topology of X.

Surgery theory was first developed for differentiable manifolds, and then extended to PL and topological manifolds.

The classic Browder–Novikov–Sullivan–Wall obstruction theory for deciding if a Poincaré complex X is homotopy equivalent to a manifold has two stages :

(i) the primary topological K-theory obstruction $\nu_X \in [X, B(G/TOP)]$ to a TOP reduction $\widetilde{\nu}_X : X \to BTOP$ of the Spivak normal fibration $\nu_X : X \to BG$, which vanishes if and only if there exists a manifold M with a normal map $(f, b) : M \to X$, that is a degree 1 map $f : M \to X$ with a bundle map $b : \nu_M \to \widetilde{\nu}_X$,

(ii) a secondary algebraic L-theory obstruction

$$\sigma_*(f, b) \in L_m(\mathbb{Z}[\pi_1(X)])$$

in the surgery obstruction group of Wall [29], which is defined if the obstruction in (i) vanishes, and which depends on the choice of TOP reduction $\widetilde{\nu}_X$, or equivalently on the bordism class of the normal map $(f, b) : M \to X$. The surgery obstruction is such that $\sigma_*(f, b) = 0$ if (and for $m \geq 5$ only if) (f, b) is normal bordant to a homotopy equivalence.

There exists a TOP reduction $\widetilde{\nu}_X$ of ν_X for which the corresponding normal map $(f, b) : M \to X$ has zero surgery obstruction if (and for $m \geq 5$ only if) the structure set $\mathcal{S}^{TOP}(X)$ is non-empty. A relative version of the theory gives a two-stage obstruction for deciding if a homotopy equivalence $M \to X$ from a manifold M is homotopic to a homeomorphism, which is traditionally formulated as the *surgery exact sequence*

$$\ldots \to L_{m+1}(\mathbb{Z}[\pi_1(X)]) \to \mathcal{S}^{TOP}(X) \to [X, G/TOP] \to L_m(\mathbb{Z}[\pi_1(X)]) \ .$$

See the paper by Browder [2] for an account of the original Sullivan-Wall surgery exact sequence in the differentiable category in the case when X has the homotopy type of a differentiable manifold

$$\ldots \to L_{m+1}(\mathbb{Z}[\pi_1(X)]) \to \mathcal{S}^{O}(X) \to [X, G/O] \to L_m(\mathbb{Z}[\pi_1(X)]) \ .$$

The algebraic L-groups $L_*(\Lambda)$ of a ring with involution Λ are defined using quadratic forms over Λ and their automorphisms, and are 4-periodic

$$L_m(\Lambda) \ = \ L_{m+4}(\Lambda) \ .$$

The surgery classification of exotic spheres of Kervaire and Milnor [7] included the first computation of the L-groups, namely

$$L_m(\mathbb{Z}) \ = \ \begin{cases} \mathbb{Z} & \text{if } m \equiv 0 \pmod 4 \\ 0 & \text{if } m \equiv 1 \pmod 4 \\ \mathbb{Z}_2 & \text{if } m \equiv 2 \pmod 4 \\ 0 & \text{if } m \equiv 3 \pmod 4 \ . \end{cases}$$

The relationship between topological and PL manifolds was investigated using surgery methods in the 1960's by Novikov, Casson, Sullivan, Kirby and Siebenmann [8] (cf. Ranicki [23]), culminating in a disproof of the manifold Hauptvermutung: there exist homeomorphisms of PL manifolds which are not homotopic to PL homeomorphisms, and in fact there exist topological manifolds without PL structure. The surgery exact sequence for the PL manifold structure set $\mathcal{S}^{PL}(M)$ for a PL manifold M was related to the exact sequence for $\mathcal{S}^{TOP}(M)$ by a commutative braid of exact sequences

$$
\begin{tikzcd}
H^3(M;\mathbb{Z}_2) \arrow[dr] \arrow[rr, bend left] & & {[M,G/PL]} \arrow[dr] \arrow[rr, bend left] & & L_m(\mathbb{Z}[\pi_1(M)]) \\
& \mathcal{S}^{PL}(M) \arrow[ur] \arrow[dr] & & {[M,G/TOP]} \arrow[ur] \arrow[dr] & \\
L_{m+1}(\mathbb{Z}[\pi_1(M)]) \arrow[ur] \arrow[rr, bend right] & & \mathcal{S}^{TOP}(M) \arrow[ur] \arrow[rr, bend right] & & H^4(M;\mathbb{Z}_2)
\end{tikzcd}
$$

with

$$\pi_*(G/TOP) \ = \ L_*(\mathbb{Z}) \ .$$

Quinn [17] gave a geometric construction of a spectrum of simplicial sets for any group π

$$\mathbb{L}_\bullet(\mathbb{Z}[\pi]) \ = \ \{\mathbb{L}_n(\mathbb{Z}[\pi]) \,|\, \Omega\mathbb{L}_n(\mathbb{Z}[\pi]) \simeq \mathbb{L}_{n+1}(\mathbb{Z}[\pi])\}$$

with homotopy groups

$$\pi_n(\mathbb{L}_\bullet(\mathbb{Z}[\pi])) \ = \ \pi_{n+k}(\mathbb{L}_{-k}(\mathbb{Z}[\pi])) \ = \ L_n(\mathbb{Z}[\pi]) \ ,$$

and

$$\mathbb{L}_0(\mathbb{Z}) \ \simeq \ L_0(\mathbb{Z}) \times G/TOP \ .$$

The construction included an assembly map

$$A \ : \ H_*(X;\mathbb{L}_\bullet(\mathbb{Z})) \to L_*(\mathbb{Z}[\pi_1(X)])$$

and for a manifold X the surgery obstruction function is given by

$$[X,G/TOP] \subset [X,L_0(\mathbb{Z}) \times G/TOP] \ \cong \ H_n(X;\mathbb{L}_\bullet(\mathbb{Z})) \xrightarrow{A} L_m(\mathbb{Z}[\pi_1(X)]) \ .$$

The surgery classifying spectra $\mathbb{L}_\bullet(\Lambda)$ and the assembly map A were constructed algebraically in Ranicki [22] for any ring with involution Λ, using quadratic Poincaré complex n-ads over Λ. The spectrum $\mathbb{L}_\bullet(\mathbb{Z})$ is appropriate for the surgery classification of homology manifold structures (Bryant, Ferry, Mio and Weinberger [3]); for topological manifolds it is necessary to work with the 1-connective spectrum $\mathbb{L}_\bullet = \mathbb{L}_\bullet(\mathbb{Z})\langle 1\rangle$, such that $\mathbb{L}_n$ is n-connected with $\mathbb{L}_0 \simeq G/TOP$. The relative homotopy groups of the spectrum-level assembly map

$$\mathcal{S}_m(X) \ = \ \pi_m(A : X_+ \wedge \mathbb{L}_\bullet \to \mathbb{L}_\bullet(\mathbb{Z}[\pi_1(X)]))$$

fit into the *algebraic surgery exact sequence*

$$\begin{aligned}\dots \to L_{m+1}(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_{m+1}(X)& \\ \to H_m(X;\mathbb{L}_\bullet) \xrightarrow{A} L_m(\mathbb{Z}[\pi_1(X)]) \to \dots\ .\end{aligned}$$

The algebraic surgery theory of Ranicki [20], [22] provided one-stage obstructions:

(i) An m-dimensional Poincaré duality space X has a *total surgery obstruction* $s(X) \in \mathbb{S}_m(X)$ such that $s(X) = 0$ if (and for $m \geq 5$ only if) X is homotopy equivalent to a manifold.

(ii) A homotopy equivalence of m-dimensional manifolds $h : M' \to M$ has a *total surgery obstruction* $s(h) \in \mathbb{S}_{m+1}(M)$ such that $s(h) = 0$ if (and for $m \geq 5$ only if) h is homotopic to a homeomorphism.

Moreover, if X is an m-dimensional manifold and $m \geq 5$ the geometric surgery exact sequence is isomorphic to the algebraic surgery exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & L_{m+1}(\mathbb{Z}[\pi_1(X)]) \arrow[r] \arrow[d, equal] & \mathbb{S}^{TOP}(X) \arrow[r] \arrow[d, "\cong"] & {[X, G/TOP]} \arrow[r] \arrow[d, "\cong"] & L_m(\mathbb{Z}[\pi_1(X)]) \arrow[d, equal] \\
\cdots \arrow[r] & L_{m+1}(\mathbb{Z}[\pi_1(X)]) \arrow[r] & \mathbb{S}_{m+1}(X) \arrow[r] & H_m(X;\mathbb{L}_\bullet) \arrow[r, "A"] & L_m(\mathbb{Z}[\pi_1(X)])
\end{tikzcd}$$

with

$$\mathbb{S}^{TOP}(X) \to \mathbb{S}_{m+1}(X)\ ;\ (M, h : M \to X) \longmapsto s(h)\ .$$

Given a normal map $(f, b) : M^m \to X$ it is possible to *kill* an element $x \in \pi_r(f)$ by surgery if and only if x can be represented by an embedding $S^{r-1} \times D^{n-r+1} \hookrightarrow M$ with a null-homotopy in X, in which case the *trace* of the surgery is a normal bordism

$$((g, c); (f, b), (f', b'))\ :\ (N; M, M') \to X \times ([0, 1]; \{0\}, \{1\})$$

with

$$\begin{aligned} N^{m+1} &= M \times I \cup D^r \times D^{m-r+1}\ , \\ M'^m &= (M \backslash S^{r-1} \times D^{m-r+1}) \cup D^r \times S^{m-r}\ . \end{aligned}$$

The normal map $(f', b') : M' \to X$ is the *geometric effect* of the surgery on (f, b). Surgery theory investigates the extent to which a normal map can be made bordant to a homotopy equivalence by killing as much of $\pi_*(f)$ as possible. The original definition of the non-simply-connected surgery obstruction $\sigma_*(f, b) \in L_m(\mathbb{Z}[\pi_1(X)])$ (Wall [29]) was obtained after preliminary surgeries below the middle dimension, to kill the relative homotopy groups

$\pi_r(f)$ for $2r \leq m$. It could thus be assumed that $(f,b) : M \to X$ is $[m/2]$-connected, with $\pi_r(f) = 0$ for $2r \leq m$, and $\sigma_*(f,b)$ was defined using the Poincaré duality structure on the middle-dimensional homotopy kernel(s). The surgery obstruction theory is much easier in the even-dimensional case $m = 2n$ when $\pi_r(f)$ can be non-zero at most for $r = m+1$ than in the odd-dimensional case $m = 2n+1$ when $\pi_r(f)$ can be non-zero for $r = m+1$ and $r = m+2$.

Wall [29,§18G] asked for a chain complex formulation of surgery, in which the obstruction groups $L_m(\Lambda)$ would appear as the cobordism groups of chain complexes with m-dimensional quadratic Poincaré duality, by analogy with the cobordism groups of manifolds Ω_*. Mishchenko [15] initiated such a theory of "m-dimensional symmetric Poincaré complexes" (C,ϕ) with C an m-dimensional f. g. free Λ-module chain complex

$$C \ : \ C_m \xrightarrow{d} C_{m-1} \xrightarrow{d} C_{m-2} \to \ldots \to C_1 \xrightarrow{d} C_0$$

and ϕ a quadratic structure inducing m-dimensional Poincaré duality isomorphisms $\phi_0 : H^*(C) \to H_{m-*}(C)$. The cobordism groups $L^m(\Lambda)$ (which are covariant in Λ) are such that for any m-dimensional geometric Poincaré complex X there is defined a symmetric signature invariant

$$\sigma^*(X) \ = \ (C(\widetilde{X}),\phi) \in L^m(\mathbb{Z}[\pi_1(X)]) \ .$$

The corresponding quadratic theory was developed in Ranicki [19]; the m-dimensional quadratic L-groups $L_m(\Lambda)$ for any $m \geq 0$ were obtained as the groups of equivalence classes of "m-dimensional quadratic Poincaré complexes" (C,ψ). The surgery obstruction of an m-dimensional normal map $(f,b) : M^m \to X$ was expressed as a cobordism class

$$\sigma_*(f,b) \ = \ (C,\psi) \in L_m(\mathbb{Z}[\pi_1(X)])$$

with

$$H_*(C) \ = \ K_*(M) \ = \ H_{*+1}(\widetilde{f} : \widetilde{M} \to \widetilde{X}) \ .$$

The symmetrization maps $1+T : L_*(\Lambda) \to L^*(\Lambda)$ are isomorphisms modulo 8-torsion, and the symmetrization of the surgery obstruction is the difference of the symmetric signatures

$$(1+T)\sigma_*(f,b) \ = \ \sigma^*(M) - \sigma^*(X) \in L^m(\mathbb{Z}[\pi_1(X)]) \ .$$

However, the theory of [19] is fairly elaborate. The algebra of [18] and [19] is used in these notes to simplify the original theory of Wall [29] in the odd-dimensional case, without invoking the full theory of [19]. Ranicki [25] is a companion paper to this one, which provides an introduction to the use of algebraic Poincaré complexes in surgery theory.

0.2 What is in these notes

These notes give an elementary account of the construction of the L-groups L_* and the surgery obstruction σ_* for differentiable manifolds. For the more computational aspects of the L-groups see the papers by Hambleton and Taylor [4] and Stark [26].

The even-dimensional L-groups $L_{2n}(\Lambda)$ are the Witt groups of nonsingular $(-1)^n$-quadratic forms over Λ. It is relatively easy to pass from an n-connected $2n$-dimensional normal map $(f,b): M^{2n} \to X$ to a $(-1)^n$-quadratic form representing $\sigma_*(f,b)$, and to see how the form changes under a surgery on (f,b). This will be done in §§1–5 of these notes.

The odd-dimensional L-groups $L_{2n+1}(\Lambda)$ are the stable automorphism groups of nonsingular $(-1)^n$-quadratic forms over Λ. It is relatively hard to pass from an n-connected $(2n+1)$-dimensional normal map (f,b) : $M^{2n+1} \to X$ to an automorphism of a $(-1)^n$-quadratic form representing $\sigma_*(f,b)$, and even harder to follow through in algebra the effect of a surgery on (f,b). Novikov [16] suggested the reformulation of the odd-dimensional theory in terms of the language of hamiltonian physics, and to replace the automorphisms by ordered pairs of lagrangians (= maximal isotropic subspaces). This reformulation was carried out in Ranicki [18], where such pairs were called 'formations', but it was still hard to follow the algebraic effects of individual surgeries. This became easier after the further reformulation of Ranicki [19] in terms of chain complexes with Poincaré duality – see §§8,9 for a description of how the kernel formation changes under a surgery on (f,b).

The original definition of $L_*(\Lambda)$ in Wall [29] was for the category of based f. g. free Λ-modules and simple isomorphisms, for surgery up to simple homotopy equivalence. Here, f. g. stands for finitely generated and simple means that the Whitehead torsion is trivial, as in the hypothesis of the s-cobordism theorem. These notes will only deal with free L-groups $L_*(\Lambda) = L^h_*(\Lambda)$, the obstruction groups for surgery up to homotopy equivalence.

The algebraic theory of ϵ-quadratic forms (K,λ,μ) over a ring Λ with an involution $\Lambda \to \Lambda; a \mapsto \bar{a}$ is developed in §§1,2, with $\epsilon = \pm 1$ and

$$\lambda \;:\; K \times K \to \Lambda \;;\; (x,y) \;\longmapsto \lambda(x,y)$$

an ϵ-symmetric pairing on a Λ-module K such that

$$\lambda(x,y) \;=\; \epsilon\overline{\lambda(y,x)} \in \Lambda \quad (x,y \in K)$$

and

$$\mu \;:\; K \to Q_\epsilon(\Lambda) \;=\; \Lambda/\{a - \epsilon\bar{a} \,|\, a \in \Lambda\}$$

an ϵ-quadratic refinement of λ such that

$$\lambda(x,x) \;=\; \mu(x)+\epsilon\overline{\mu(x)} \in \Lambda \quad (x\in K)\ .$$

For an n-connected $2n$-dimensional normal map $(f,b) : M^{2n} \to X$ geometric (intersection, self-intersection) numbers define a $(-1)^n$-quadratic form $(K_n(M),\lambda,\mu)$ on the kernel stably f. g. free $\mathbb{Z}[\pi_1(X)]$-module

$$K_n(M) \;=\; \ker(\widetilde{f}_* : H_n(\widetilde{M}) \to H_n(\widetilde{X}))$$

with $\widetilde{X}$ the universal cover of X and $\widetilde{M}=f^*\widetilde{X}$ the pullback of $\widetilde{X}$ to M.

The hyperbolic ϵ-quadratic form on a f. g. free Λ-module Λ^k

$$H_\epsilon(\Lambda^k) \;=\; (\Lambda^{2k},\lambda,\mu)$$

is defined by

$$\begin{aligned}
&\lambda \;:\; \Lambda^{2k}\times\Lambda^{2k}\to\Lambda\ ;\\
&\qquad ((a_1,a_2,\dots,a_{2k}),(b_1,b_2,\dots,b_{2k})) \;\longmapsto\; \sum_{i=1}^{k}(b_{2i-1}\bar{a}_{2i}+\epsilon b_{2i}\bar{a}_{2i-1})\ ,\\
&\mu \;:\; \Lambda^{2k}\to Q_\epsilon(\Lambda)\ ;\ (a_1,a_2,\dots,a_{2k}) \longmapsto \sum_{i=1}^{k} a_{2i-1}\bar{a}_{2i}\ .
\end{aligned}$$

The even-dimensional L-group $L_{2n}(\Lambda)$ is defined in §3 to be the abelian group of stable isomorphism classes of nonsingular $(-1)^n$-quadratic forms on (stably) f. g. free Λ-modules, where stabilization is with respect to the hyperbolic forms $H_{(-1)^n}(\Lambda^k)$. A nonsingular $(-1)^n$-quadratic form (K,λ,μ) represents 0 in $L_{2n}(\Lambda)$ if and only if there exists an isomorphism

$$(K,\lambda,\mu)\oplus H_{(-1)^n}(\Lambda^k) \;\cong\; H_{(-1)^n}(\Lambda^{k'})$$

for some integers $k,k'\geq 0$. The surgery obstruction of an n-connected $2n$-dimensional normal map $(f,b) : M^{2n}\to X$ is defined by

$$\sigma_*(f,b) \;=\; (K_n(M),\lambda,\mu)\in L_{2n}(\mathbb{Z}[\pi_1(X)])\ .$$

The algebraic effect of a geometric surgery on an n-connected $2n$-dimensional normal map (f,b) is given in §5. Assuming that the result of the surgery is still n-connected, the effect on the kernel form of a surgery on $S^{n-1}\times D^{n+1}\hookrightarrow M$ (resp. $S^n\times D^n\hookrightarrow M$) is to add on (resp. split off) a hyperbolic $(-1)^n$-quadratic form $H_{(-1)^n}(\mathbb{Z}[\pi_1(X)])$.

§6 introduces the notion of a "$(2n+1)$-complex" (C,ψ), which is a f. g. free Λ-module chain complex of the type

$$C \;:\; \dots\to 0\to C_{n+1} \xrightarrow{\;d\;} C_n \;\to 0\to\dots$$

together with a quadratic structure ψ inducing Poincaré duality isomorphisms $(1+T)\psi \;:\; H^{2n+1-*}(C)\to H_*(C)$. (This is just a $(2n+1)$-

dimensional quadratic Poincaré complex (C,ψ) in the sense of [19], with $C_r=0$ for $r\neq n,n+1$.) An n-connected $(2n+1)$-dimensional normal map $(f,b):M^{2n+1}\to X$ determines a kernel $(2n+1)$-complex (C,ψ) (or rather a homotopy equivalence class of such complexes) with

$$H_*(C) \;=\; K_*(M) \;=\; \ker(\widetilde{f}_*:H_*(\widetilde{M})\to H_*(\widetilde{X}))\ .$$

The cobordism of $(2n+1)$-complexes is defined in §7. The odd-dimensional L-group $L_{2n+1}(\Lambda)$ is defined in §8 as the cobordism group of $(2n+1)$-complexes. The surgery obstruction of an n-connected normal map $(f,b):M^{2n+1}\to X$ is the cobordism class of the kernel complex

$$\sigma_*(f,b) \;=\; (C,\psi)\in L_{2n+1}(\mathbb{Z}[\pi_1(X)])\ .$$

The odd-dimensional L-group $L_{2n+1}(\Lambda)$ was originally defined in [29] as a potentially non-abelian quotient of the stable unitary group of the matrices of automorphisms of hyperbolic $(-1)^n$-quadratic forms over Λ

$$L_{2n+1}(\Lambda) \;=\; U_{(-1)^n}(\Lambda)/EU_{(-1)^n}(\Lambda)$$

with

$$U_{(-1)^n}(\Lambda) \;=\; \bigcup_{k=1}^{\infty}\mathrm{Aut}_\Lambda H_{(-1)^n}(\Lambda^k)$$

and $EU_{(-1)^n}(\Lambda)\triangleleft U_{(-1)^n}(\Lambda)$ the normal subgroup generated by the elementary matrices of the type

$$\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{*-1}\end{pmatrix}\ ,\ \begin{pmatrix}1 & 0\\ \beta+(-1)^{n+1}\beta^* & 1\end{pmatrix}\ ,\ \begin{pmatrix}0 & 1\\ (-1)^n & 0\end{pmatrix}$$

for any invertible matrix α, and any square matrix β. The group $L_{2n+1}(\Lambda)$ is abelian, since

$$[U_{(-1)^n}(\Lambda),U_{(-1)^n}(\Lambda)]\subseteq EU_{(-1)^n}(\Lambda)\ .$$

The surgery obstruction $\sigma_*(f,b)\in L_{2n+1}(\mathbb{Z}[\pi_1(X)])$ of an n-connected $(2n+1)$-dimensional normal map $(f,b):M^{2n+1}\to X$ is represented by an automorphism of a hyperbolic $(-1)^n$-quadratic form obtained from a high-dimensional generalization of the Heegaard decompositions of 3-dimensional manifolds as twisted doubles.

§8, §9 and §10 describe three equivalent ways of defining $L_{2n+1}(\Lambda)$, using unitary matrices, formations and chain complexes. In each case it is necessary to make some choices in passing from the geometry to the algebra, and to verify that the equivalence class in the L-group is independent of the choices.

The definition of $L_{2n+1}(\Lambda)$ using complexes given in §8 is a special case of the general theory of chain complexes with Poincaré duality of Ranicki

[19]. The 4-periodicity in the quadratic L-groups

$$L_m(\Lambda) \;=\; L_{m+4}(\Lambda)$$

(given geometrically by taking product with $\mathbb{C}\mathbb{P}^2$, as in Chapter 9 of Wall [29]) was proved in [19] using an algebraic analogue of surgery below the middle dimension: it is possible to represent every element of $L_m(\Lambda)$ by a quadratic Poincaré complex (C,ψ) which is "highly-connected", meaning that

$$C_r \;=\; 0 \text{ for } \begin{cases} r \neq n & \text{if } m = 2n \\ r \neq n, n+1 & \text{if } m = 2n+1 \ . \end{cases}$$

In these notes only the highly-connected $(2n+1)$-dimensional quadratic Poincaré complexes are considered, namely the "$(2n+1)$-complexes" of §6.

I am grateful to the referee for suggesting several improvements.

The titles of the sections are:

§1. Duality
§2. Quadratic forms
§3. The even-dimensional L-groups
§4. Split forms
§5. Surgery on forms
§6. Short odd complexes
§7. Complex cobordism
§8. The odd-dimensional L-groups
§9. Formations
§10. Automorphisms

§1. Duality

§1 considers rings Λ equipped with an "involution" reversing the order of multiplication. An involution allows right Λ-modules to be regarded as left Λ-modules, especially the right Λ-modules which arise as the duals of left Λ-modules. In particular, the group ring $\mathbb{Z}[\pi_1(M)]$ of the fundamental group $\pi_1(M)$ of a manifold M has an involution, which allows the Poincaré duality of the universal cover $\widetilde{M}$ to be regarded as $\mathbb{Z}[\pi_1(M)]$-module isomorphisms.

Let X be a connected space, and let $\widetilde{X}$ be a regular cover of X with group of covering translations π. The action of π on $\widetilde{X}$ by covering translations

$$\pi \times \widetilde{X} \to \widetilde{X} \ ; \ (g,x) \longmapsto gx$$

induces a left action of the group ring $\mathbb{Z}[\pi]$ on the homology of $\widetilde{X}$

$$\mathbb{Z}[\pi] \times H_*(\widetilde{X}) \to H_*(\widetilde{X}) \ ; \ (\sum_{g\in\pi} n_g g, x) \longmapsto \sum_{g\in\pi} n_g gx$$

so that the homology groups $H_*(\widetilde{X})$ are left $\mathbb{Z}[\pi]$-modules. In dealing with cohomology let

$$H^*(\widetilde{X}) \ = \ H^*_{cpt}(\widetilde{X})$$

be the compactly supported cohomology groups, regarded as left $\mathbb{Z}[\pi]$-modules by

$$\mathbb{Z}[\pi] \times H^*(\widetilde{X}) \to H^*(\widetilde{X}) \ ; \ (\sum_{g\in\pi} n_g g, x) \ \longmapsto \ \sum_{g\in\pi} n_g x g^{-1} \ .$$

(For finite π $H^*(\widetilde{X})$ is just the ordinary cohomology of $\widetilde{X}$.) Cap product with any homology class $[X] \in H_m(X)$ defines $\mathbb{Z}[\pi]$-module morphisms

$$[X] \cap - \ : \ H^*(\widetilde{X}) \to H_{m-*}(\widetilde{X}) \ .$$

DEFINITION 1.1 An *oriented m-dimensional geometric Poincaré complex* (Wall [28]) is a finite CW complex X with a fundamental class $[X] \in H_m(X)$ such that cap product defines $\mathbb{Z}[\pi_1(X)]$-module isomorphisms

$$[X] \cap - \ : \ H^*(\widetilde{X}) \xrightarrow{\cong} H_{m-*}(\widetilde{X})$$

with $\widetilde{X}$ the universal cover of X. □

See 1.14 below for the general definition of a geometric Poincaré complex, including the nonorientable case.

EXAMPLE 1.2 A compact oriented m-dimensional manifold is an oriented m-dimensional geometric Poincaré complex. □

In order to also deal with nonorientable manifolds and Poincaré complexes it is convenient to have an involution:

DEFINITION 1.3 Let Λ be an associative ring with 1. An *involution* on Λ is a function

$$\Lambda \to \Lambda \ ; \ a \longmapsto \overline{a}$$

satisfying

$$\overline{(a+b)} \ = \ \overline{a} + \overline{b} \ , \ \overline{(ab)} \ = \ \overline{b}.\overline{a} \ , \ \overline{\overline{a}} \ = \ a \ , \ \overline{1} \ = \ 1 \in \Lambda \ . \qquad □$$

EXAMPLE 1.4 A commutative ring Λ admits the identity involution

$$\Lambda \to \Lambda \ ; \ a \longmapsto \overline{a} \ = \ a \ . \qquad □$$

DEFINITION 1.5 Given a group π and a group morphism

$$w \ : \ \pi \to \mathbb{Z}_2 \ = \ \{\pm 1\}$$

define the *w-twisted involution* on the integral group ring $\Lambda = \mathbb{Z}[\pi]$

$$\Lambda \to \Lambda \; ; \; a \;=\; \sum_{g\in\pi} n_g g \longmapsto \bar{a} \;=\; \sum_{g\in\pi} w(g) n_g g^{-1} \; (n_g \in \mathbb{Z}) \; . \qquad \square$$

In the topological application π is the fundamental group of a space with $w : \pi \to \mathbb{Z}_2$ an orientation character. In the oriented case $w(g) = +1$ for all $g \in \pi$.

EXAMPLE 1.6 Complex conjugation defines an involution on the ring of complex numbers $\Lambda = \mathbb{C}$

$$\mathbb{C} \to \mathbb{C} \; ; \; z \;=\; a + ib \longmapsto \overline{z} = a - ib \; . \qquad \square$$

A “hermitian” form is a symmetric form on a (finite-dimensional) vector space over $\mathbb{C}$ with respect to this involution. The study of forms over rings with involution is sometimes called “hermitian K-theory”, although “algebraic L-theory” seems preferable.

The dual of a left Λ-module K is the right Λ-module

$$K^* \;=\; \mathrm{Hom}_\Lambda(K, \Lambda)$$

with

$$K^* \times \Lambda \to K^* \; ; \; (f, a) \longmapsto (x \longmapsto f(x).a) \; .$$

An involution $\Lambda \to \Lambda; a \longmapsto \bar{a}$ determines an isomorphism of categories

$$\{\text{right } \Lambda\text{-modules}\} \xrightarrow{\cong} \{\text{left } \Lambda\text{-modules}\} \; ; \; L \longmapsto L^{op} \; ,$$

with L^{op} the left Λ-module with the same additive group as the right Λ-module L and Λ acting by

$$\Lambda \times L^{op} \to L^{op} \; ; \; (a, x) \longmapsto x\overline{a} \; .$$

From now on we shall work with a ring Λ which is equipped with a particular choice of involution $\Lambda \to \Lambda$. Also, Λ-modules will always be understood to be left Λ-modules.

For any Λ-module K the Λ-module $(K^*)^{op}$ is written as K^*. Here is the definition of K^* all at once:

DEFINITION 1.7 The *dual* of a Λ-module K is the Λ-module

$$K^* \;=\; \mathrm{Hom}_\Lambda(K, \Lambda) \; ,$$

with Λ acting by

$$\Lambda \times K^* \to K^* \; ; \; (a, f) \longmapsto (x \longmapsto f(x).\overline{a})$$

for all $a \in \Lambda$, $f \in K^*$, $x \in K$. □

There is a corresponding notion for morphisms:

Definition 1.8 The *dual* of a Λ-module morphism $f : K \to L$ is the Λ-module morphism

$$f^* \; : \; L^* \to K^* \; ; \; g \longmapsto \big(x \longmapsto g(f(x))\big) \; . \qquad \square$$

Thus duality is a contravariant functor

$$* \; : \; \{\Lambda\text{-modules}\} \to \{\Lambda\text{-modules}\} \; ; \; K \longmapsto K^* \; .$$

Definition 1.9 For any Λ-module K define the Λ-module morphism

$$e_K \; : \; K \to K^{**} \; ; \; x \longmapsto (f \longmapsto \overline{f(x)}) \; . \qquad \square$$

The morphism e_K is natural in the sense that for any Λ-module morphism $f : K \to L$ there is defined a commutative diagram

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "e_K"'] & L \arrow[d, "e_L"] \\ K^{**} \arrow[r, "f^{**}"] & L^{**} \end{tikzcd}$$

Definition 1.10 (i) A Λ-module K is *f. g. projective* if there exists a Λ-module L such that $K \oplus L$ is isomorphic to the f. g. free Λ-module Λ^n, for some $n \geq 0$.
(ii) A Λ-module K is *stably f. g. free* if $K \oplus \Lambda^m$ is isomorphic to Λ^n, for some $m, n \geq 0$. □

In particular, f. g. free Λ-modules are stably f. g. free, and stably f. g. free Λ-modules are f. g. projective.

Proposition 1.11 *The dual of a f. g. projective Λ-module K is a f. g. projective Λ-module K^*, and $e_K : K \to K^{**}$ is an isomorphism. Moreover, if K is stably f. g. free then so is K^*.*
Proof: For any Λ-modules K, L there are evident identifications

$$(K \oplus L)^* \; = \; K^* \oplus L^* \; ,$$

$$e_{K \oplus L} \; = \; e_K \oplus e_L \; : \; K \oplus L \to (K \oplus L)^{**} \; = \; K^{**} \oplus L^{**} \; ,$$

so it suffices to consider the special case $K = \Lambda$. The Λ-module isomorphism

$$f \; : \; \Lambda \xrightarrow{\cong} \Lambda^* \; ; \; a \longmapsto (b \longmapsto b\overline{a}) \; ,$$

can be used to construct an explicit inverse for e_Λ

$$(e_\Lambda)^{-1} ~:~ \Lambda^{**} \to \Lambda ~;~ g \longmapsto g(f(1)) ~. \qquad \square$$

In dealing with f. g. projective Λ-modules K use the natural isomorphism $e_K : K \cong K^{**}$ to identify $K^{**} = K$. For any morphism $f : K \to L$ of f. g. projective Λ-modules there is a corresponding identification

$$f^{**} ~=~ f ~:~ K^{**} ~=~ K \to L^{**} ~=~ L ~.$$

Remark 1.12 The additive group $\mathrm{Hom}_\Lambda(\Lambda^m, \Lambda^n)$ of the morphisms $\Lambda^m \to \Lambda^n$ between f. g. free Λ-modules Λ^m, Λ^n may be identified with the additive group $M_{m,n}(\Lambda)$ of $m \times n$ matrices $(a_{ij})_{1 \le i \le m, 1 \le j \le n}$ with entries $a_{ij} \in \Lambda$, using the isomorphism

$$\begin{aligned} & M_{m,n}(\Lambda) \xrightarrow{\cong} \mathrm{Hom}_\Lambda(\Lambda^m, \Lambda^n) ~; \\ & (a_{ij}) \longmapsto ((x_1, x_2, \dots, x_m) \longmapsto (\sum_{i=1}^m x_i a_{i1}, \sum_{i=1}^m x_i a_{i2}, \dots, \sum_{i=1}^m x_i a_{in})) ~. \end{aligned}$$

The composition of morphisms

$$\begin{aligned} \mathrm{Hom}_\Lambda(\Lambda^m, \Lambda^n) \times \mathrm{Hom}_\Lambda(\Lambda^n, \Lambda^p) \to \mathrm{Hom}_\Lambda(\Lambda^m, \Lambda^p) ~; \\ (f, g) \longmapsto \big(gf : x \longmapsto (gf)(x) = g(f(x))\big) \end{aligned}$$

corresponds to the multiplication of matrices

$$\begin{aligned} M_{m,n}(\Lambda) \times M_{n,p}(\Lambda) \to M_{m,p}(\Lambda) ~;~ ((a_{ij}), (b_{jk})) \longmapsto (c_{ik}) \\ (\, c_{ik} ~=~ \sum_{j=1}^n a_{ij} b_{jk} ~~ (1 \le i \le m,~ 1 \le k \le p)\,) ~. \end{aligned}$$

Use the isomorphism of f. g. free Λ-modules

$$\Lambda^m \xrightarrow{\cong} (\Lambda^m)^* ~;~ (x_1, x_2, \dots, x_m) \longmapsto ((y_1, y_2, \dots, y_m) ~\longmapsto \sum_{i=1}^m y_i \overline{x}_i)$$

to identify

$$(\Lambda^m)^* ~=~ \Lambda^m ~.$$

The duality isomorphism

$$\begin{aligned} * ~:~ \mathrm{Hom}_\Lambda(\Lambda^m, \Lambda^n) \xrightarrow{\cong} \mathrm{Hom}_\Lambda((\Lambda^n)^*, (\Lambda^m)^*) ~=~ \mathrm{Hom}_\Lambda(\Lambda^n, \Lambda^m) ~; \\ f \longmapsto f^* \end{aligned}$$

can be identified with the isomorphism defined by conjugate transposition of matrices

$$M_{m,n}(\Lambda) \xrightarrow{\cong} M_{n,m}(\Lambda) ~;~ \alpha ~=~ (a_{ij}) \longmapsto \alpha^* ~=~ (b_{ji}) ~,~ b_{ji} = \overline{a}_{ij} ~. \qquad \square$$

Example 1.13 A 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2,2}(\Lambda)$$

corresponds to the Λ-module morphism

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \Lambda \oplus \Lambda \to \Lambda \oplus \Lambda \; ; \; (x,y) \longmapsto (xa + yb, xc + yd) \; .$$

The conjugate transpose matrix

$$\begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} \in M_{2,2}(\Lambda)$$

corresponds to the dual Λ-module morphism

$$f^* = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} : (\Lambda \oplus \Lambda)^* = \Lambda \oplus \Lambda \to (\Lambda \oplus \Lambda)^* = \Lambda \oplus \Lambda \; ;$$

$$(x,y) \longmapsto (x\overline{a} + y\overline{c}, x\overline{b} + y\overline{d}) \; . \qquad \square$$

The dual of a chain complex of modules over a ring with involution Λ

$$C : \ldots \longrightarrow C_{r+1} \xrightarrow{d} C_r \xrightarrow{d} C_{r-1} \longrightarrow \ldots$$

is the cochain complex

$$C^* : \ldots \longrightarrow C^{r-1} \xrightarrow{d^*} C^r \xrightarrow{d^*} C^{r+1} \longrightarrow \ldots$$

with

$$C^r = (C_r)^* = \mathrm{Hom}_\Lambda(C_r, \Lambda) \; .$$

Definition 1.14 An *m-dimensional geometric Poincaré complex* (Wall [28]) is a finite CW complex X with an orientation character $w(X) : \pi_1(X) \to \mathbb{Z}_2$ and a $w(X)$-twisted fundamental class $[X] \in H_m(X; \mathbb{Z}^{w(X)})$ such that cap product defines $\mathbb{Z}[\pi_1(X)]$-module isomorphisms

$$[X] \cap - : H^*_{w(X)}(\widetilde{X}) \xrightarrow{\cong} H_{m-*}(\widetilde{X})$$

with $\widetilde{X}$ the universal cover of X. The $w(X)$-twisted cohomology groups are given by

$$H^*_{w(X)}(\widetilde{X}) = H^*(C(\widetilde{X})^*)$$

with $C(\widetilde{X})$ the cellular $\mathbb{Z}[\pi_1(X)]$-module chain complex, using the $w(X)$-twisted involution on $\mathbb{Z}[\pi_1(X)]$ (1.5) to define the left $\mathbb{Z}[\pi_1(X)]$-module structure on the dual cochain complex

$$C(\widetilde{X})^* = \mathrm{Hom}_{\mathbb{Z}[\pi_1(X)]}(C(\widetilde{X}), \mathbb{Z}[\pi_1(X)]) \; . \qquad \square$$

The orientation character $w(X) : \pi_1(X) \to \mathbb{Z}_2$ sends a loop $g : S^1 \to X$ to $w(g) = +1$ (resp. $= -1$) if g is orientation-preserving (resp. orientation-reversing).

An oriented Poincaré complex X (1.1) is just a Poincaré complex (1.14) with $w(X) = +1$.

Example 1.15 A compact m-dimensional manifold is an m-dimensional geometric Poincaré complex. □

§2. Quadratic forms

In the first instance suppose that the ground ring Λ is commutative, with the identity involution $\bar{a} = a$ (1.4). A symmetric form (K, λ) over Λ is a Λ-module K together with a bilinear pairing

$$\lambda \ : \ K \times K \to \Lambda \ ; \ (x, y) \longmapsto \lambda(x, y)$$

such that for all $x, y, z \in K$ and $a \in \Lambda$

$$\begin{aligned} \lambda(x, ay) &= a\lambda(x, y) \ , \\ \lambda(x, y + z) &= \lambda(x, y) + \lambda(x, z) \ , \\ \lambda(x, y) &= \lambda(y, x) \in \Lambda \ . \end{aligned}$$

A quadratic form (K, λ, μ) over Λ is a symmetric form (K, λ) together with a function

$$\mu \ : \ K \to Q_{+1}(\Lambda) \ = \ \Lambda \ ; \ x \longmapsto \mu(x)$$

such that for all $x, y \in K$ and $a \in \Lambda$

$$\begin{aligned} \mu(x + y) &= \mu(x) + \mu(y) + \lambda(x, y) \ , \\ \mu(ax) &= a^2\mu(x) \in Q_{+1}(\Lambda) \ . \end{aligned}$$

In particular, for every $x \in K$

$$2\mu(x) \ = \ \lambda(x, x) \in Q^{+1}(\Lambda) \ = \ \Lambda \ .$$

If $2 \in \Lambda$ is invertible (e.g. if Λ is a field of characteristic $\neq 2$, such as $\mathbb{R}, \mathbb{C}, \mathbb{Q}$) there is no difference between symmetric and quadratic forms, with μ determined by λ according to $\mu(x) = \lambda(x, x)/2$.

A symplectic form (K, λ) over a commutative ring Λ is a Λ-module K together with a bilinear pairing $\lambda : K \times K \to \Lambda$ such that for all $x, y, z \in K$ and $a \in \Lambda$

$$\begin{aligned} \lambda(x, at) &= a\lambda(x, y) \ , \\ \lambda(x, y + z) &= \lambda(x, y) + \lambda(x, z) \ , \\ \lambda(x, y) &= -\lambda(y, x) \in \Lambda \ . \end{aligned}$$

A (-1)-quadratic form (K,λ,μ) over Λ is a symplectic form (K,λ) together with a function

$$\mu\ :\ K\to Q_{-1}(\Lambda)\ =\ \Lambda/\{2a\,|\,a\in\Lambda\}\ ;\ x\longmapsto \mu(x)$$

such that for all $x,y\in K$ and $a\in\Lambda$

$$\begin{aligned}\mu(x+y)\ &=\ \mu(x)+\mu(y)+\lambda(x,y)\ ,\\ \mu(ax)\ &=\ a^2\mu(x)\in Q_{-1}(\Lambda)\ .\end{aligned}$$

In particular, for every $x\in K$

$$2\mu(x)\ =\ \lambda(x,x)\in Q^{-1}(\Lambda)\ =\ \{a\in\Lambda\,|\,2a=0\}\ .$$

If $2\in\Lambda$ is invertible then $Q_{-1}(\Lambda)=0$ and there is no difference between symplectic and (-1)-quadratic forms, with $\mu=0$.

In the applications of forms to surgery theory it is necessary to work with quadratic and (-1)-quadratic forms over noncommutative group rings with the involution as in 1.5. §2 develops the general theory of forms over rings with involution, taking account of these differences.

Let X be an m-dimensional geometric Poincaré complex with universal cover $\widetilde{X}$ and fundamental group ring $\Lambda=\mathbb{Z}[\pi_1(X)]$, with the $w(X)$-twisted involution. The Poincaré duality isomorphism

$$\phi\ =\ [X]\cap-\ :\ H^{m-r}_{w(X)}(\widetilde{X})\xrightarrow{\cong} H_r(\widetilde{X})$$

and the evaluation pairing

$$H^r_{w(X)}(\widetilde{X})\to H_r(\widetilde{X})^*\ =\ \mathrm{Hom}_\Lambda(H_r(\widetilde{X}),\Lambda)\ ;y\longmapsto (x\longmapsto\langle y,x\rangle)$$

can be combined to define a sesquilinear pairing

$$\lambda\ :\ H_r(\widetilde{X})\times H_{m-r}(\widetilde{X})\to\Lambda\ ;\ (x,\phi(y))\longmapsto\langle y,x\rangle$$

such that

$$\lambda(y,x)\ =\ (-1)^{r(m-r)}\overline{\lambda(x,y)}$$

with $\Lambda\to\Lambda; a\longmapsto\bar{a}$ the involution of 1.5.

If M is an m-dimensional manifold with fundamental group ring $\Lambda=\mathbb{Z}[\pi_1(M)]$ the pairing $\lambda\ :\ H_r(\widetilde{M})\times H_{m-r}(\widetilde{M})\to\Lambda$ can be interpreted geometrically using the geometric intersection numbers of cycles. For any two immersions $x:S^r\looparrowright\widetilde{M}$, $y:S^{m-r}\looparrowright\widetilde{M}$ in general position

$$\lambda(x,y)\ =\ \sum_{g\in\pi_1(M)}n_g g\in\Lambda$$

with $n_g\in\mathbb{Z}$ the algebraic number of intersections in $\widetilde{M}$ of x and gy. In particular, for $m=2n$ there is defined a $(-1)^n$-symmetric pairing

$$\lambda\ :\ H_n(\widetilde{M})\times H_n(\widetilde{M})\to\Lambda$$

which is relevant to surgery in the middle dimension n. An element $x \in \pi_n(M)$ can be killed by surgery if and only if it is represented by an embedding $S^n \times D^n \hookrightarrow M^{2n}$. The condition that the Hurewicz image $x \in H_n(\widetilde{M})$ be such that $\lambda(x,x) = 0 \in \Lambda$ is necessary but not sufficient to kill $x \in \pi_n(M)$ by surgery. The theory of forms developed in §2 is required for an algebraic formulation of the necessary and sufficient condition for an element in the kernel $K_n(M)$ of an n-connected $2n$-dimensional normal map $(f,b) : M \to X$ to be killed by surgery, assuming $n \geq 3$.

As in §1 let Λ be a ring with involution, not necessarily a group ring.

DEFINITION 2.1 A *sesquilinear pairing* (K,L,λ) on Λ-modules K,L is a function

$$\lambda \ : \ K \times L \to \Lambda \ ; \ (x,y) \longmapsto \lambda(x,y)$$

such that for all $w,x \in K$, $y,z \in L$, $a,b \in \Lambda$

(i) $\lambda(w+x,y+z) \ = \ \lambda(w,y)+\lambda(w,z)+\lambda(x,y)+\lambda(x,z) \in \Lambda$,

(ii) $\lambda(ax,by) \ = \ b\lambda(x,y)\overline{a} \in \Lambda$.

The *dual* (or *transpose*) sesquilinear pairing is

$$T\lambda \ : \ L \times K \to \Lambda \ ; \ (y,x) \longmapsto T\lambda(y,x) \ = \ \overline{\lambda(x,y)} \ . \qquad \square$$

DEFINITION 2.2 Given Λ-modules K,L let $S(K,L)$ be the additive group of sesquilinear pairings $\lambda : K \times L \to \Lambda$. Transposition defines an isomorphism

$$T \ : \ S(K,L) \ \cong \ S(L,K)$$

such that

$$T^2 \ = \ \text{id.} \ : \ S(K,L) \xrightarrow{\cong} S(L,K) \xrightarrow{\cong} S(K,L) \ . \qquad \square$$

PROPOSITION 2.3 *For any Λ-modules K,L there is a natural isomorphism of additive groups*

$$S(K,L) \xrightarrow{\cong} \mathrm{Hom}_\Lambda(K,L^*) \ ;$$
$$(\lambda \ : \ K \times L \to \Lambda) \longmapsto (\lambda \ : \ K \to L^* \, ; \, x \longmapsto (\, y \longmapsto \lambda(x,y)\,)\,) \, .$$

For f. g. projective K,L the transposition isomorphism $T : S(K,L) \cong S(L,K)$ corresponds to the duality isomorphism

$$* : \mathrm{Hom}_\Lambda(K,L^*) \xrightarrow{\cong} \mathrm{Hom}_\Lambda(L,K^*) \, ;$$
$$(\lambda : K \to L^*) \longmapsto (\lambda^* \ : \ L \ \to \ K^* \, ; \, y \longmapsto (x \ \longmapsto \ \overline{\lambda(y,x)})) \, . \qquad \square$$

Use 2.3 to identify

$$\begin{aligned} S(K,L) &= \mathrm{Hom}_\Lambda(K,L^*)\ ,\ S(K)\ =\ \mathrm{Hom}_\Lambda(K,K^*)\ ,\\ Q^\epsilon(K) &= \ker(1-T_\epsilon:\mathrm{Hom}_\Lambda(K,K^*)\to\mathrm{Hom}_\Lambda(K,K^*))\ ,\\ Q_\epsilon(K) &= \mathrm{coker}(1-T_\epsilon:\mathrm{Hom}_\Lambda(K,K^*)\to\mathrm{Hom}_\Lambda(K,K^*)) \end{aligned}$$

for any f. g. projective Λ-modules K,L.

Remark 2.4 For f. g. free Λ-modules Λ^m, Λ^n it is possible to identify $S(\Lambda^m,\Lambda^n)$ with the additive group $M_{m,n}(\Lambda)$ of $m\times n$ matrices (a_{ij}) with entries $a_{ij}\in\Lambda$, using the isomorphism

$$M_{m,n}(\Lambda)\xrightarrow{\cong} S(\Lambda^m,\Lambda^n)\ ;\ (a_{ij})\longmapsto\lambda$$

defined by

$$\lambda((x_1,x_2,\dots,x_m),(y_1,y_2,\dots,y_n))\ =\ \sum_{i=1}^m\sum_{j=1}^n y_ja_{ij}\overline{x}_i\ .$$

The transposition isomorphism $T:S(\Lambda^m,\Lambda^n)\cong S(\Lambda^n,\Lambda^m)$ corresponds to the isomorphism defined by conjugate transposition of matrices

$$T\ :\ M_{m,n}(\Lambda)\xrightarrow{\cong} M_{n,m}(\Lambda)\ ;\ (a_{ij})\longmapsto(b_{ji})\ ,\ b_{ji}\ =\ \overline{a}_{ij}\ .\qquad\square$$

The group $S(K,L)$ is particularly significant in the case $K=L$:

Definition 2.5 (i) Given a Λ-module K let

$$S(K)\ =\ S(K,K)$$

be the abelian group of sesquilinear pairings $\lambda:K\times K\to\Lambda$.
(ii) The *ϵ-transposition involution* is given for $\epsilon=\pm1$ by

$$T_\epsilon\ :\ S(K)\xrightarrow{\cong} S(K)\ ;\ \lambda\longmapsto T_\epsilon\lambda\ =\ \epsilon(T\lambda)\ ,$$

such that

$$T_\epsilon\lambda(x,y)\ =\ \epsilon\overline{\lambda(y,x)}\in\Lambda\ ,\ (T_\epsilon)^2\ =\ \mathrm{id}\ :\ S(K)\to S(K)\ .\qquad\square$$

Definition 2.6 The *ϵ-symmetric group* of a Λ-module K is the additive group

$$Q^\epsilon(K)\ =\ \ker(1-T_\epsilon:S(K)\to S(K))\ .$$

The *ϵ-quadratic group* of K is the additive group

$$Q_\epsilon(K)\ =\ \mathrm{coker}(1-T_\epsilon:S(K)\to S(K))\ .$$

The *ϵ-symmetrization morphism* is given by

$$1+T_\epsilon\ :\ Q_\epsilon(K)\to Q^\epsilon(K)\ ;\ \psi\longmapsto\psi+T_\epsilon\psi\ .\qquad\square$$

For $\epsilon = +1$ it is customary to refer to ϵ-symmetric and ϵ-quadratic objects as symmetric and quadratic, as in the commutative case.

For $K = \Lambda$ there is an isomorphism of additive groups with involution

$$\Lambda \xrightarrow{\cong} S(\Lambda) \; ; \; a \longmapsto ((x,y) \longmapsto ya\overline{x})$$

allowing the identifications

$$\begin{aligned} Q^{\epsilon}(\Lambda) &= \{a \in \Lambda \,|\, \epsilon\overline{a} = a\} \; , \\ Q_{\epsilon}(\Lambda) &= \Lambda/\{a - \epsilon\overline{a} \,|\, a \in \Lambda\} \; , \\ 1 + T_{\epsilon} &: \; Q_{\epsilon}(\Lambda) \to Q^{\epsilon}(\Lambda) \; ; \; a \longmapsto a + \epsilon\overline{a} \; . \end{aligned}$$

Example 2.7 Let $\Lambda = \mathbb{Z}$. The ϵ-symmetric and ϵ-quadratic groups of $K = \mathbb{Z}$ are given by

$$Q^{\epsilon}(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } \epsilon = +1 \\ 0 & \text{if } \epsilon = -1 \; , \end{cases}$$

$$Q_{\epsilon}(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } \epsilon = +1 \\ \mathbb{Z}/2 & \text{if } \epsilon = -1 \end{cases}$$

with generators represented by $1 \in \mathbb{Z}$, and with

$$1 + T_{+} = 2 \; : \; Q_{+1}(\mathbb{Z}) = \mathbb{Z} \to Q^{+1}(\mathbb{Z}) = \mathbb{Z} \; . \qquad \square$$

Definition 2.8 An *ϵ-symmetric form* (K, λ) over Λ is a Λ-module K together with an element $\lambda \in Q^{\epsilon}(K)$. Thus λ is a sesquilinear pairing

$$\lambda \; : \; K \times K \to \Lambda \; ; \; (x,y) \longmapsto \lambda(x,y)$$

such that for all $x, y \in K$

$$\lambda(x,y) = \epsilon\overline{\lambda(y,x)} \in \Lambda \; .$$

The *adjoint* of (K, λ) is the Λ-module morphism

$$K \to K^* \; ; \; x \longmapsto (y \longmapsto \lambda(x,y))$$

which is also denoted by λ. The form is *nonsingular* if $\lambda : K \to K^*$ is an isomorphism. $\square$

Unless specified otherwise, only forms (K, λ) with K a f. g. projective Λ-module will be considered.

Example 2.9 The symmetric form (Λ, λ) defined by

$$\lambda = 1 \; : \; \Lambda \to \Lambda^* \; ; \; a \longmapsto (b \longmapsto b\overline{a})$$

is nonsingular. $\square$

DEFINITION 2.10 For any f. g. projective Λ-module L define the nonsingular *hyperbolic ϵ-symmetric form*

$$H^{\epsilon}(L) \;=\; (L\oplus L^*,\lambda)$$

by

$$\lambda \;=\; \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix} \;:\; L\oplus L^* \to (L\oplus L^*)^* \;=\; L^*\oplus L \;;$$

$$(x,f) \longmapsto ((y,g) \longmapsto f(y)+\epsilon\overline{g(x)}) \;. \qquad \square$$

EXAMPLE 2.11 Let X be an m-dimensional geometric Poincaré complex, and let $\widetilde{X}$ be a regular oriented covering of X with group of covering translations π and orientation character $w:\pi\to\mathbb{Z}_2$. An element $g\in\pi$ has $w(g)=+1$ (resp. -1) if and only if the covering translation $g:\widetilde{X}\to\widetilde{X}$ is orientation-preserving (resp. reversing).
(i) Cap product with the fundamental class $[X]\in H_m(X;\mathbb{Z}^w)$ defines the Poincaré duality $\mathbb{Z}[\pi]$-module isomorphisms

$$[X]\cap - \;:\; H^{m-*}_w(\widetilde{X}) \xrightarrow{\cong} H_*(\widetilde{X}) \;.$$

If $m=2n$ and X is a manifold geometric intersection numbers define a $(-1)^n$-symmetric form $(H_n(\widetilde{X}),\lambda)$ over $\mathbb{Z}[\pi]$ with adjoint the composite

$$\lambda \;:\; H_n(\widetilde{X}) \xrightarrow{([X]\cap -)^{-1}} H^n_w(\widetilde{X}) \xrightarrow{\text{evaluation}} H_n(\widetilde{X})^* \;.$$

(ii) In general $H_n(\widetilde{X})$ is not a f. g. projective $\mathbb{Z}[\pi]$-module. If $H_n(\widetilde{X})$ is f. g. projective then the evaluation map is an isomorphism, and $(H_n(\widetilde{X}),\lambda)$ is a nonsingular form. $\square$

REMARK 2.12 (i) Let M be a $2n$-dimensional manifold, with universal cover $\widetilde{M}$ and intersection pairing $\lambda:H_n(\widetilde{M})\times H_n(\widetilde{M})\to\mathbb{Z}[\pi_1(M)]$. An element $x\in\operatorname{im}(\pi_n(M)\to H_n(\widetilde{M}))$ can be killed by surgery if and only if it can be represented by an embedding $x:S^n\times D^n\hookrightarrow M$, in which case the homology class $x\in H_n(\widetilde{M})$ is such that $\lambda(x,x)=0$. However, the condition $\lambda(x,x)=0$ given by the symmetric structure alone is not sufficient for the existence of such an embedding – see (ii) below for an explicit example.
(ii) The intersection form over $\mathbb{Z}$ for $M^{2n}=S^n\times S^n$ is the hyperbolic form (2.10)

$$(H_n(S^n\times S^n),\lambda) \;=\; H^{(-1)^n}(\mathbb{Z}) \;.$$

The element $x=(1,1)\in H_n(S^n\times S^n)$ is such that

$$\lambda(x,x) \;=\; \chi(S^n) \;=\; 1+(-1)^n\in\mathbb{Z} \;,$$

so that $\lambda(x,x)=0$ for odd n. The diagonal embedding $\Delta:S^n\hookrightarrow S^n\times S^n$ has normal bundle $\nu_\Delta=\tau_{S^n}:S^n\to BO(n)$, which is non-trivial for

$n \neq 1,3,7$, so that it is not possible to kill $x = \Delta_*[S^n] \in H_n(S^n \times S^n)$ by surgery in these dimensions. □

DEFINITION 2.13 An ϵ-*quadratic form* (K,λ,μ) over Λ is an ϵ-symmetric form (K,λ) together with a function

$$\mu \ : \ K \to Q_\epsilon(\Lambda) \ ; \ x \longmapsto \mu(x)$$

such that for all $x,y \in K$, $a \in \Lambda$

$$\begin{aligned}
&\text{(i)} \ \ \mu(x+y) - \mu(x) - \mu(y) \ = \ \lambda(x,y) \in Q_\epsilon(\Lambda) \ , \\
&\text{(ii)} \ \ \mu(x) + \epsilon\overline{\mu(x)} \ = \ \lambda(x,x) \in \mathrm{im}(1+T_\epsilon : Q_\epsilon(\Lambda) \to Q^\epsilon(\Lambda)) \ , \\
&\text{(iii)} \ \ \mu(ax) \ = \ a\mu(x)\overline{a} \in Q_\epsilon(\Lambda) \ .
\end{aligned}$$

□

DEFINITION 2.14 For any f. g. projective Λ-module L define the nonsingular *hyperbolic* ϵ-*quadratic form* over Λ by

$$H_\epsilon(L) \ = \ (L \oplus L^*, \lambda, \mu)$$

with

$$\begin{aligned}
\lambda \ = \ \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix} \ : \ & L \oplus L^* \to (L \oplus L^*)^* \ = \ L^* \oplus L \ ; \\
& (x,f) \longmapsto ((y,g) \longmapsto f(y) + \epsilon\overline{g(x)}) \ , \\
\mu \ : \ L \oplus L^* \to Q_\epsilon(\Lambda) \ ; \ & (x,f) \longmapsto f(x) \ .
\end{aligned}$$

$(L \oplus L^*, \lambda)$ is the hyperbolic ϵ-symmetric form $H^\epsilon(L)$ of 2.10. □

EXAMPLE 2.15 (Wall [29, Chapter 5]) An n-connected normal map $(f,b) : M^{2n} \to X$ from a $2n$-dimensional manifold with boundary $(M, \partial M)$ to a geometric Poincaré pair $(X, \partial X)$ with $\partial f = f| : \partial M \to \partial X$ a homotopy equivalence determines a $(-1)^n$-quadratic form $(K_n(M), \lambda, \mu)$ over $\Lambda = \mathbb{Z}[\pi_1(X)]$ with the $w(X)$-twisted involution (1.5), with

$$K_n(M) \ = \ \pi_{n+1}(f) \ = \ H_{n+1}(\widetilde{f}) \ = \ \ker(\widetilde{f}_* : H_n(\widetilde{M}) \to H_n(\widetilde{X}))$$

the stably f. g. free kernel Λ-module, and $\widetilde{f} : \widetilde{M} \to \widetilde{X}$ a $\pi_1(X)$-equivariant lift of f to the universal covers. Note that $K_n(M) = 0$ if (and for $n \geq 2$ only if) $f : M \to X$ is a homotopy equivalence, by the theorem of J.H.C. Whitehead.

(i) The pairing $\lambda : K_n(M) \times K_n(M) \to \Lambda$ is defined by geometric intersection numbers, as follows. Every element $x \in K_n(M)$ is represented by an X-nullhomotopic framed immersion $g : S^n \looparrowright M$ with a choice of path in $g(S^n) \subset M$ from the base point $* \in M$ to $g(1) \in M$. Any two elements $x, y \in K_n(M)$ can be represented by such immersions $g, h : S^n \looparrowright M$ with transverse intersections and self-intersections. The intersection of g and h

$$D(g,h) \ = \ \{(a,b) \in S^n \times S^n \,|\, g(a) = h(b) \in M\}$$

is finite. For each intersection point $(a,b) \in D(g,h)$ let

$$\gamma(a,b) \in \pi_1(M) \ = \ \pi_1(X)$$

be the homotopy class of the loop in M obtained by joining the path in $g(S^n) \subset M$ from the base point $* \in M$ to $g(a)$ to the path in $h(S^n) \subset M$ from $h(b)$ back to the base point. Choose an orientation for $\tau_*(M)$ and transport it to an orientation for $\tau_{g(a)}(M) = \tau_{h(b)}(M)$ by the path for g, and let

$$\epsilon(a,b) \ = \ [\tau_a(S^n) \oplus \tau_b(S^n) : \tau_{g(a)}(M)] \in \{\pm 1\}$$

be $+1$ (resp. -1) if the isomorphism

$$(dg \ dh) \ : \tau_a(S^n) \oplus \tau_b(S^n) \xrightarrow{\cong} \tau_{g(a)}(M)$$

is orientation-preserving (resp. reversing). The *geometric intersection* of $x, y \in K_n(M)$ is given by

$$\lambda(x,y) \ = \sum_{(a,b) \in D(g,h)} I(a,b) \in \Lambda$$

with

$$I(a,b) \ = \ \epsilon(a,b)\gamma(a,b) \in \Lambda \ .$$

It follows from

$$\begin{aligned} \epsilon(b,a) \ &= \ [\tau_b S^n \oplus \tau_a S^n : \tau_a S^n \oplus \tau_b S^n]\epsilon(a,b) \\ &= \ \det(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \mathbb{R}^n \oplus \mathbb{R}^n \to \mathbb{R}^n \oplus \mathbb{R}^n)\epsilon(a,b) \\ &= \ (-1)^n \epsilon(a,b) \in \{\pm 1\} \ , \\ \gamma(b,a) \ &= \ w(X)(\gamma(a,b))\gamma(a,b)^{-1} \in \pi_1(X) \ , \\ I(b,a) &= (-1)^n \overline{I(a,b)} \in \Lambda \end{aligned}$$

that

$$\lambda(y,x) \ = \ (-1)^n \overline{\lambda(x,y)} \in \Lambda$$

(which also holds from purely homological considerations).
(ii) The quadratic function $\mu : K_n(M) \to Q_{(-1)^n}(\Lambda)$ is defined by geometric self-intersection numbers, as follows. Represent $x \in K_n(M)$ by an immersion $g : S^n \looparrowright M$ as in (i), with transverse self-intersections. The double point set of g

$$\begin{aligned} D_2(g) \ &= \ D(g,g) \backslash \Delta(S^n) \\ &= \ \{(a,b) \in S^n \times S^n \,|\, a \neq b \in S^n, g(a) = g(b) \in M\} \end{aligned}$$

is finite, with a free $\mathbb{Z}_2$-action $(a,b) \mapsto (b,a)$. For each $(a,b) \in D_2(g)$ let $\gamma(a,b)$ be the loop in M obtained by transporting to the base point the image under g of a path in S^n from a to b. The *geometric self-intersection* of x is defined by

$$\mu(x) \; = \sum_{(a,b)\in D_2(g)/\mathbb{Z}_2} I(a,b) \in Q_{(-1)^n}(\Lambda) \; ,$$

with $I(a,b) = \epsilon(a,b)\gamma(a,b)$ as in (i). Note that $\mu(x)$ is independent of the choice of ordering of (a,b) since $I(b,a) \; = \; (-1)^n\overline{I(a,b)} \in \Lambda$.
(iii) The kernel $(-1)^n$-quadratic form $(K_n(M),\lambda,\mu)$ is such that $\mu(x)=0$ if (and for $n\geq 3$ only if) $x\in K_n(M)$ can be killed by surgery on $S^n\subset M^{2n}$, i.e. represented by an embedding $S^n\times D^n \hookrightarrow M$ with a nullhomotopy in X – the condition $\mu(x)=0$ allows the double points of a representative framed immersion $g: S^n \looparrowright M$ to be matched in pairs, which for $n\geq 3$ can be cancelled by the Whitney trick. The effect of the surgery is a bordant $(n-1)$-connected normal map

$$(f',b') \; : \; M'^{2n} \; = \; \mathrm{cl.}(M\backslash S^n\times D^n)\cup D^{n+1}\times S^{n-1} \to X$$

with kernel Λ-modules

$$K_i(M') \; = \; \begin{cases} \mathrm{coker}(x^*\lambda : K_n(M)\to\Lambda^*) & \text{if } i=n-1 \\ \dfrac{\ker(x^*\lambda : K_n(M)\to\Lambda^*)}{\mathrm{im}(x:\Lambda\to K_n(M))} & \text{if } i=n \\ \ker(x:\Lambda\to K_n(M)) & \text{if } i=n+1 \\ 0 & \text{otherwise .} \end{cases}$$

Thus (f',b') is n-connected if and only if x generates a direct summand $L=\langle x\rangle\subset K_n(M)$, in which case L is a sublagrangian of $(K_n(M),\lambda,\mu)$ in the terminology of §5, with

$$\begin{aligned} L\subseteq L^\perp \; &= \; \{y\in K_n(M)\,|\,\lambda(x,y)=0\} \; , \\ (K_n(M'),\lambda',\mu') \; &= \; (L^\perp/L,[\lambda],[\mu]) \; , \\ (K_n(M),\lambda,\mu) \; &\cong \; (K_n(M'),\lambda',\mu')\oplus H_{(-1)^n}(\Lambda) \; . \end{aligned}$$

(iv) The effect on (f,b) of a surgery on an X-nullhomotopic embedding $S^{n-1}\times D^{n+1}\hookrightarrow M$ is an n-connected bordant normal map

$$(f'',b'') \; : \; M''^{2n} \; = \; \mathrm{cl.}(M\backslash S^{n-1}\times D^{n+1})\cup D^n\times S^n \; = \; M\#(S^n\times S^n)\to X$$

with kernel Λ-modules

$$K_i(M'') \; = \; \begin{cases} K_n(M)\oplus\Lambda\oplus\Lambda^* & \text{if } i=n \\ 0 & \text{otherwise} \end{cases}$$

and kernel $(-1)^n$-quadratic form

$$(K_n(M''),\lambda'',\mu'') \; = \; (K_n(M),\lambda,\mu)\oplus H_{(-1)^n}(\Lambda) \; .$$

(v) The main result of even-dimensional surgery obstruction theory is that for $n\geq 3$ an n-connected $2n$-dimensional normal map $(f,b): M^{2n}\to X$ is normal bordant to a homotopy equivalence if and only if there exists an isomorphism of $(-1)^n$-quadratic forms over $\Lambda=\mathbb{Z}[\pi_1(X)]$ of the type

$$(K_n(M),\lambda,\mu)\oplus H_{(-1)^n}(\Lambda^k) \xrightarrow{\cong} H_{(-1)^n}(\Lambda^{k'})$$

for some $k, k' \geq 0$. $\square$

EXAMPLE 2.16 There is also a relative version of 2.15. An n-connected $2n$-dimensional normal map of pairs $(f,b) : (M^{2n}, \partial M) \to (X, \partial X)$ has a kernel $(-1)^n$-quadratic form $(K_n(M), \lambda, \mu)$ over $\mathbb{Z}[\pi_1(X)]$ is nonsingular if and only if $\partial f : \partial M \to \partial X$ is a homotopy equivalence (assuming $\pi_1(\partial X) \cong \pi_1(X)$). $\square$

REMARK 2.17 (Realization of even-dimensional surgery obstructions, Wall [29, 5.8])
(i) Let X^{2n-1} be a $(2n-1)$-dimensional manifold, and suppose given an embedding $e : S^{n-1} \times D^n \hookrightarrow X$, together with a null-homotopy δe of $e| : S^{n-1} \hookrightarrow X$ and a null-homotopy of the map $S^{n-1} \to O$ comparing the (stable) trivializations of $\nu_{e|} : S^{n-1} \to BO(n)$ given by e and δe. Then there is defined an n-connected $2n$-dimensional normal map

$$(f,b) \ : \ (M; \partial_- M, \partial_+ M) \to X \times ([0,1]; \{0\}, \{1\})$$

with

$$\begin{aligned} \partial_- f \ &= \ \text{id.} \ : \ \partial_- M \ = \ X \to X \ , \\ M^{2n} \ &= \ X \times [0,1] \cup_e D^n \times D^n \ , \\ \partial_+ M \ &= \ \text{cl.}(X \backslash e(S^{n-1} \times D^n)) \cup D^n \times S^{n-1} \ . \end{aligned}$$

The kernel $(-1)^n$-quadratic form (Λ, λ, μ) over Λ (2.16) is the (self-)-intersection of the framed immersion $S^{n-1} \times [0,1] \looparrowright X \times [0,1]$ defined by the track of a regular homotopy $e_0 \simeq e : S^{n-1} \times D^n \to X$ from a trivial unlinked embedding

$$e_0 \ : \ S^{n-1} \times D^n \hookrightarrow S^{2n-1} \ = \ S^{n-1} \times D^n \cup D^n \times S^{n-1} \hookrightarrow X \# S^{2n-1} \ = \ X \ .$$

Moreover, every form (Λ, λ, μ) arises in this way: starting with e_0 construct a regular homotopy $e_0 \simeq e$ to a (self-)linked embedding e such that the track has (self-)intersection (λ, μ).
(ii) Let (K, λ, μ) be a $(-1)^n$-quadratic form over $\mathbb{Z}[\pi]$, with π a finitely presented group and $K = \mathbb{Z}[\pi]^k$ f. g. free. Let $n \geq 3$, so that there exists a $(2n-1)$-dimensional manifold X^{2n-1} with $\pi_1(X) = \pi$. For any such $n \geq 3$, X there exists an n-connected $2n$-dimensional normal map

$$(f,b) \ : \ (M^{2n}; \partial_- M, \partial_+ M) \to X^{2n-1} \times ([0,1]; \{0\}, \{1\})$$

with kernel form (K, λ, μ) and

$$\begin{aligned} \partial_- f \ &= \ \text{id.} \ : \ \partial_- M \ = \ X \to X \ , \ K_n(M) \ = \ K \ , \\ K_{n-1}(\partial_+ M) \ &= \ \text{coker}(\lambda : K \to K^*) \ , \\ K_n(\partial_+ M) \ &= \ \ker(\lambda : K \to K^*) \ . \end{aligned}$$

The map $\partial_+ f : \partial_+ M \to X$ is a homotopy equivalence if and only if the form (K, λ, μ) is nonsingular. Given (K, λ, μ), X the construction of (f,b)

proceeds as in (i). □

EXAMPLE 2.18 For $\pi_1(X) = \{1\}$ the realization of even-dimensional surgery obstructions (2.17) is essentially the same as the Milnor [11], [12] construction of $(n-1)$-connected $2n$-dimensional manifolds by *plumbing* together n-plane bundles over S^n. Let G be a finite connected graph without loops (= edges joining a vertex to itself), with vertices $v_1, v_2, \dots, v_k$. Suppose given an oriented n-plane bundle over S^n at each vertex

$$\omega_1, \omega_2, \dots, \omega_k \in \pi_n(BSO(n)) \;=\; \pi_{n-1}(SO(n)) \;,$$

regarded as a weight. Let $(\mathbb{Z}^k, \lambda)$ be the $(-1)^n$-symmetric form over $\mathbb{Z}$ defined by the $(-1)^n$-symmetrized adjacency matrix of G and the Euler numbers $\chi(\omega_i) \in \mathbb{Z}$, with

$$\lambda_{ij} \;=\; \begin{cases} \text{no. of edges in } G \text{ joining } v_i \text{ to } v_j & \text{if } i < j \\ (-1)^n(\text{no. of edges in } G \text{ joining } v_i \text{ to } v_j) & \text{if } i > j \\ \chi(\omega_i) & \text{if } i = j \;, \end{cases}$$

$$\lambda \;:\; \mathbb{Z}^k \times \mathbb{Z}^k \to \mathbb{Z} \;;\; ((x_1, x_2, \dots, x_k), (y_1, y_2, \dots, y_k)) \mapsto \sum_{i=1}^{k} \sum_{j=1}^{k} \lambda_{ij} x_i y_j \;.$$

The graph G and the Euler numbers $\chi(\omega_i)$ determine and are determined by the form $(\mathbb{Z}^k, \lambda)$.

(i) See Browder [1, Chapter V] for a detailed account of the plumbing construction which uses G to glue together the (D^n, S^{n-1})-bundles

$$(D^n, S^{n-1}) \to (E(\omega_i), S(\omega_i)) \to S^n \quad (i = 1, 2, \dots, k)$$

to obtain a connected $2n$-dimensional manifold with boundary

$$(P, \partial P) \;=\; (P(G, \omega), \partial P(G, \omega))$$

such that P is an identification space

$$P \;=\; (\coprod_{i=1}^{k} E(\omega_i))/\sim$$

with 1-skeleton homotopy equivalent to G, fundamental group

$$\pi_1(P) \;=\; \pi_1(G) \;=\; *_g \mathbb{Z}$$

the free group on $g = 1 - \chi(G)$ generators, homology

$$H_r(P) \;=\; \begin{cases} \mathbb{Z} & \text{if } r = 0 \\ \mathbb{Z}^g & \text{if } r = 1 \\ \mathbb{Z}^k & \text{if } r = n \\ 0 & \text{otherwise} \;, \end{cases}$$

and intersection form $(H_n(P), \lambda)$. Killing $\pi_1(P)$ by surgeries removing g embeddings $S^1 \times D^{2n-1} \subset P$ representing the generators, there is obtained an $(n-1)$-connected $2n$-dimensional manifold with boundary

$$(M, \partial M) = (M(G,\omega), \partial M(G,\omega))$$

such that

$$\begin{aligned} H_r(M) &= \begin{cases} \mathbb{Z} & \text{if } r = 0 \\ \mathbb{Z}^k & \text{if } r = n \\ 0 & \text{otherwise ,} \end{cases} \\ \lambda &: H_n(M) \times H_n(M) \to \mathbb{Z} ; \\ & ((x_1, x_2, \dots, x_k), (y_1, y_2, \dots, y_k)) \mapsto \sum_{i=1}^{k} \sum_{j=1}^{k} \lambda_{ij} x_i y_j , \\ \tau_M &\simeq \bigvee_{i=1}^{k} (\omega_i \oplus \epsilon^n) : M \simeq \bigvee_{i=1}^{k} S^n \to BSO(2n) , \\ H_r(\partial M) &= \begin{cases} \mathbb{Z} & \text{if } r = 0, 2n-1 \\ \mathrm{coker}(\lambda : \mathbb{Z}^k \to \mathbb{Z}^k) & \text{if } r = n-1 \\ \ker(\lambda : \mathbb{Z}^k \to \mathbb{Z}^k) & \text{if } r = n \\ 0 & \text{otherwise .} \end{cases} \end{aligned}$$

If G is a tree then $g = 0$, $\pi_1(P(G,\omega)) = \{1\}$, and

$$(M(G,\omega), \partial M(G,\omega)) = (P(G,\omega), \partial P(G,\omega)) .$$

(ii) By Wall [27] for $n \geq 3$ an integral $(-1)^n$-symmetric matrix $(\lambda_{ij})_{1 \leq i,j \leq k}$ and elements $\omega_1, \omega_2, \dots, \omega_k \in \pi_n(BSO(n))$ with

$$\lambda_{ii} = \chi(\omega_i) \in \mathbb{Z} \quad (i = 1, 2, \dots, k)$$

determine an embedding

$$x = \bigcup_k x_i : \bigcup_k S^{n-1} \times D^n \hookrightarrow S^{2n-1}$$

such that:

(a) for $1 \leq i < j \leq k$

$$\text{linking number}(x_i(S^{n-1} \times 0) \cap x_j(S^{n-1} \times 0) \hookrightarrow S^{2n-1}) = \lambda_{ij} \in \mathbb{Z} ,$$

(b) for $1 \leq i \leq k$ $x_i : S^{n-1} \times D^n \hookrightarrow S^{2n-1}$ is isotopic to the embedding

$$\begin{aligned} e_{\omega_i} : S^{n-1} \times D^n \hookrightarrow S^{2n-1} &= S^{n-1} \times D^n \cup D^n \times S^{n-1} ; \\ (s,t) &\mapsto (s, \omega_i(s)(t)) . \end{aligned}$$

Using x to attach k n-handles to D^{2n} there is obtained an oriented $(n-1)$-connected $2n$-dimensional manifold

$$M(G,\omega) = D^{2n} \cup_x \bigcup_k n\text{-handles } D^n \times D^n$$

with boundary an oriented $(n-2)$-connected $(2n-1)$-dimensional manifold

$$\partial M(G,\omega) \;=\; \mathrm{cl.}(S^{2n-1}\backslash x(\bigcup_k S^{n-1}\times D^n))\cup\bigcup_k D^n\times S^{n-1}\ .$$

Moreover, every oriented $(n-1)$-connected $2n$-dimensional manifold with non-empty $(n-2)$-connected boundary is of the form $(M(G,\omega),\partial M(G,\omega))$, with (λ_{ij},ω_i) the complete set of diffeomorphism invariants.

(iii) Stably trivialized n-plane bundles over S^n are classified by $Q_{(-1)^n}(\mathbb{Z})$, with an isomorphism

$$Q_{(-1)^n}(\mathbb{Z}) \xrightarrow{\cong} \pi_{n+1}(BSO,BSO(n))\ ;\ 1\mapsto(\delta\tau_{S^n},\tau_{S^n})$$

with

$$\delta\tau_{S^n}\ :\ \tau_{S^n}\oplus\epsilon\ \cong\ \epsilon^{n+1}$$

the stable trivialization given by the standard embedding $S^n\subset S^{n+1}$. The map

$$Q_{(-1)^n}(\mathbb{Z}) \;=\; \pi_{n+1}(BSO,BSO(n))\to\pi_n(BSO(n))\ ;\ 1\mapsto\tau_{S^n}$$

is an injection for $n\neq 1,3,7$. With G as above, suppose now that the vertices $v_1,v_2,\dots,v_k$ are weighted by elements

$$\mu_1,\mu_2,\dots,\mu_k\in\pi_{n+1}(BSO,BSO(n)) \;=\; Q_{(-1)^n}(\mathbb{Z})\ .$$

Define

$$\begin{aligned}\omega_i \;&=\; [\mu_i]\in\mathrm{im}(\pi_{n+1}(BSO,BSO(n))\to\pi_n(BSO(n)))\\ &=\; \ker(\pi_n(BSO(n))\to\pi_n(BSO))\end{aligned}$$

and let $(\mathbb{Z}^k,\lambda,\mu)$ be the $(-1)^n$-quadratic form over $\mathbb{Z}$ with λ as before and

$$\mu\ :\ \mathbb{Z}^k\to Q_{(-1)^n}\mathbb{Z}\ ;\ (x_1,x_2,\dots,x_k)\mapsto\sum_{1\le i<j\le k}\lambda_{ij}x_ix_j+\sum_{i=1}^k\mu_i(x_i)^2\ ,$$

such that

$$\lambda_{ii} \;=\; \chi(\omega_i) \;=\; (1+(-1)^n)\mu_i\in\mathbb{Z}\ .$$

The $(n-1)$-connected $2n$-dimensional manifold

$$M(G,\mu_1,\mu_2,\dots,\mu_n) \;=\; M(G,\omega_1,\omega_2,\dots,\omega_n)$$

is stably parallelizable, with an n-connected normal map

$$(M(G,\mu_1,\mu_2,\dots,\mu_n),\partial M(G,\mu_1,\mu_2,\dots,\mu_n))\to(D^{2n},S^{2n-1})$$

with kernel form $(\mathbb{Z}^k,\lambda,\mu)$.

(iv) For $n\ge 3$ the realization of a $(-1)^n$-quadratic form $(\mathbb{Z}^k,\lambda,\mu)$ over $\mathbb{Z}$ (2.17) is an n-connected $2n$-dimensional normal map

$$\begin{aligned}(f,b)\ :\ &(M^{2n};S^{2n-1},\partial_+M)\\ &=\ (\mathrm{cl.}(M(G,\mu_1,\dots,\mu_k)\backslash D^{2n});S^{2n-1},\partial M(G,\mu_1,\dots,\mu_k))\\ &\qquad\to S^{2n-1}\times([0,1];\{0\},\{1\})\ .\end{aligned}$$

with kernel form $(\mathbb{Z}^k, \lambda, \mu)$. If $(\mathbb{Z}^k, \lambda, \mu)$ is nonsingular then

$$\partial_+ f \ : \ \Sigma^{2n-1} \ = \ \partial M(G, \mu_1, \dots, \mu_k) \to S^{2n-1}$$

is a homotopy equivalence, and Σ^{2n-1} is a homotopy sphere with a potentially exotic differentiable structure (Milnor [10], Kervaire and Milnor [7]) – see 2.20, 3.6 and 3.7 below. □

EXAMPLE 2.19 (i) Consider the special case $k = 1$ of 2.18 (i). Here $G = \{v_1\}$ is the graph with one vertex, and

$$\omega \in \pi_n(BSO(n)) \ = \ \pi_{n-1}(SO(n))$$

classifies an n-plane bundle over S^n. The plumbed $(n-1)$-connected $2n$-dimensional manifold with boundary is the (D^n, S^{n-1})-bundle over S^n

$$(M(G, \omega), \partial M(G, \omega)) \ = \ (E(\omega), S(\omega))$$

with

$$\begin{aligned} E(\omega) \ &= \ S^{n-1} \times D^n \cup_{(x,y)\sim(x,\omega(x)(y))} S^{n-1} \times D^n \\ &= \ D^{2n} \cup_{e_\omega} D^n \times D^n \end{aligned}$$

obtained from D^{2n} by attaching an n-handle along the embedding

$$\begin{aligned} e_\omega \ : \ S^{n-1} \times D^n \hookrightarrow S^{2n-1} \ &= \ S^{n-1} \times D^n \cup D^n \times S^{n-1} \ ; \\ (x, y) \mapsto (x, &\omega(x)(y)) \ . \end{aligned}$$

(ii) Consider the special case $k = 1$ of 2.18 (iii), the realization of a $(-1)^n$-quadratic form $(\mathbb{Z}, \lambda, \mu)$ over $\mathbb{Z}$, with $G = \{v_1\}$ as in (i). An element

$$\begin{aligned} \mu \ = \ (\delta\omega, \omega) \in \pi_{n+1}(BSO, BSO(n)) \ &= \ Q_{(-1)^n}(\mathbb{Z}) \\ &= \ \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 (\operatorname{mod} 2) \\ \mathbb{Z}_2 & \text{if } n \equiv 1 (\operatorname{mod} 2) \end{cases} \end{aligned}$$

classifies an n-plane bundle $\omega : S^n \to BSO(n)$ with a stable trivialization

$$\delta\omega \ : \ \omega \oplus \epsilon^\infty \ \cong \ \epsilon^{n+\infty} \ ,$$

and

$$(M(G, \mu), \partial M(G, \mu)) \ = \ (E(\omega), S(\omega)) \ .$$

For $n \neq 1, 3, 7$ $\delta\omega$ is determined by ω. For even n $\mu \in Q_{+1}(\mathbb{Z}) = \mathbb{Z}$ and

$$\omega \ = \ \mu^* \tau_{S^n} \ : \ S^n \xrightarrow{\ \mu\ } S^n \xrightarrow{\ \tau_{S^n}\ } BSO(n)$$

is the unique stably trivial n-plane bundle over S^n with Euler number

$$\chi(\omega) \ = \ 2\mu \in \mathbb{Z} \ .$$

For odd $n \neq 1, 3, 7$ $\mu \in Q_{-1}(\mathbb{Z}) = \mathbb{Z}_2$ and

$$\omega \ = \ \begin{cases} \tau_{S^n} & \text{if } \mu = 1 \\ \epsilon^n & \text{if } \mu = 0 \ . \end{cases}$$

For $n = 1, 3, 7$

$$\omega \;=\; \tau_{S^n} \;=\; \epsilon^n \;:\; S^n \to BSO(n)$$

and $\delta\omega$ is the (stable) trivialization of ω with mod 2 Hopf invariant μ. The plumbed $(n-1)$-connected $2n$-dimensional manifold

$$(M(G,\mu), \partial M(G,\mu)) \;=\; (M(G,\omega), \partial M(G,\omega)) \;=\; (E(\omega), S(\omega))$$

(as in (i)) is stably parallelizable. The trace of the surgery on the normal map

$$(f_-, b_-) \;=\; \mathrm{id.} \;:\; \partial_- M_\omega \;=\; S^{2n-1} \to S^{2n-1}$$

killing $e_\omega : S^{n-1} \times D^n \hookrightarrow S^{2n-1}$ is an n-connected $2n$-dimensional normal map

$$(f_\omega, b_{\delta\omega}) \;:\; (M^{2n}_\omega; \partial_- M_\omega, \partial_+ M_\omega) \to S^{2n-1} \times ([0,1]; \{0\}, \{1\})$$

with

$$\begin{aligned} M_\omega \;&=\; \mathrm{cl.}(M(G,\mu) \backslash D^{2n}) \;=\; \mathrm{cl.}(E(\omega) \backslash D^{2n}) \\ &=\; S^{2n-1} \times [0,1] \cup_{e_\omega} D^n \times D^n \;, \\ \partial_+ M_\omega \;&=\; \mathrm{cl.}(S^{2n-1} \backslash e_\omega(S^{n-1} \times D^n)) \cup D^n \times S^{n-1} \\ &=\; D^n \times S^{n-1} \cup_\omega D^n \times S^{n-1} \;=\; S(\omega) \;, \\ K_n(M_\omega) \;&=\; \mathbb{Z} \end{aligned}$$

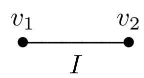
and kernel form $(\mathbb{Z}, \lambda, \mu)$. If $\mu = 0 \in Q_{(-1)^n}(\mathbb{Z})$ then

$$\omega \;=\; \epsilon^n \;:\; S^n \to BSO(n) \;\;,\;\; \partial_+ M_\omega \;=\; S(\epsilon^n) \;=\; S^{n-1} \times S^n \;,$$

If $\mu = 1 \in Q_{(-1)^n}(\mathbb{Z})$ then

$$\omega \;=\; \tau_{S^n} \;:\; S^n \to BSO(n) \;\;,\;\; \partial_+ M_\omega \;=\; S(\tau_{S^n}) \;=\; O(n+1)/O(n-1) \;.$$

(iii) Consider the special case $k = 2$ of 2.18 (i), with $G = I$ the graph with 1 edge and 2 vertices



For any weights $\omega_1, \omega_2 \in \pi_n(BSO(n))$ there is obtained an $(n-1)$-connected $2n$-dimensional manifold

$$M(I, \omega_1, \omega_2) \;=\; D^{2n} \cup_{e_{\omega_1} \cup e_{\omega_2}} (D^n \times D^n \cup D^n \times D^n)$$

by plumbing as in Milnor [11], [12], with intersection form the $(-1)^n$-symmetric form $(\mathbb{Z} \oplus \mathbb{Z}, \lambda)$ over $\mathbb{Z}$ defined by

$$\begin{aligned} &\lambda \;:\; \mathbb{Z} \oplus \mathbb{Z} \times \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \;; \\ &\quad ((x_1, x_2), (y_1, y_2)) \mapsto \chi(\omega_1) x_1 y_1 + \chi(\omega_2) x_2 y_2 + x_1 y_2 + (-1)^n x_2 y_1 \;. \end{aligned}$$

(iv) Consider the special case $k = 2$ of 2.18 (iii), with $G = I$ as in (iii). For $\mu_1, \mu_2 \in Q_{(-1)^n}(\mathbb{Z})$ and

$$\omega_i \;=\; [\mu_i] \in \mathrm{im}(Q_{(-1)^n}(\mathbb{Z}) \to \pi_n(BSO(n)))$$

the $(-1)^n$-quadratic form $(\mathbb{Z}\oplus\mathbb{Z},\lambda,\mu)$ over $\mathbb{Z}$ defined by

$$\mu\ :\ \mathbb{Z}\oplus\mathbb{Z}\to Q_{(-1)^n}(\mathbb{Z})\ ;\ (x_1,x_2)\mapsto \mu_1(x_1)^2+\mu_2(x_2)^2+x_1x_2$$

is the kernel form of an n-connected $2n$-dimensional normal map

$$(f,b)\ :\ M(I,\mu_1,\mu_2)\ =\ M(I,\omega_1,\omega_2)\to D^{2n}\ .$$

If $\mu_1=\mu_2=0$ then $(\mathbb{Z}\oplus\mathbb{Z},\lambda,\mu)=H_{(-1)^n}(\mathbb{Z})$ is hyperbolic $(-1)^n$-quadratic form over $\mathbb{Z}$, with

$$\begin{aligned}\lambda\ &:\ \mathbb{Z}\oplus\mathbb{Z}\times\mathbb{Z}\oplus\mathbb{Z}\to\mathbb{Z}\ ;\ ((x_1,x_2),(y_1,y_2))\mapsto x_1y_2+(-1)^nx_2y_1\ ,\\ \mu\ &:\ \mathbb{Z}\oplus\mathbb{Z}\to Q_{(-1)^n}(\mathbb{Z})\ =\ \mathbb{Z}/\{1+(-1)^{n-1}\}\ ;\ (x_1,x_2)\mapsto x_1x_2\ ,\end{aligned}$$

and the plumbed manifold is a punctured torus

$$(M(I,0,0)^{2n},\partial M(I,0,0))\ =\ (\mathrm{cl.}(S^n\times S^n\backslash D^{2n}),S^{2n-1})\ .$$

The hyperbolic form is the kernel of the n-connected $2n$-dimensional normal map

$$\begin{aligned}(f,b)\ :\ (M;\partial_-M,\partial_+M)\ &=\ (\mathrm{cl.}(M(I,0,0)\backslash D^{2n});S^{2n-1},S^{2n-1})\\ &\to S^{2n-1}\times([0,1];\{0\},\{1\})\end{aligned}$$

defined by the trace of surgeries on the linked spheres

$$S^{n-1}\cup S^{n-1}\hookrightarrow S^{2n-1}\ =\ S^{n-1}\times D^n\cup D^n\times S^{n-1}$$

with no self-linking. These are the attaching maps for the cores of the n-handles in the decomposition

$$M(I,0,0)\ =\ D^{2n}\cup D^n\times D^n\cup D^n\times D^n\ ,$$

using the standard framings of $S^{n-1}\subset S^{2n-1}$. If n is odd, say $n=2k+1$, and $\mu_0=\mu_1=1\in Q_{-1}(\mathbb{Z})$ the form in 2.18 (i) is just the Arf (-1)-quadratic form over $\mathbb{Z}$ $(\mathbb{Z}\oplus\mathbb{Z},\lambda,\mu')$ with

$$\mu'\ :\ \mathbb{Z}\oplus\mathbb{Z}\to Q_{(-1)}(\mathbb{Z})\ =\ \mathbb{Z}_2\ ;\ (x,y)\longmapsto x^2+xy+y^2\ .$$

The plumbed manifold

$$M(I,1,1)^{4k+2}\ =\ D^{4k+2}\cup D^{2k+1}\times D^{2k+1}\cup D^{2k+1}\times D^{2k+1}$$

has the same attaching maps for the cores of the $(2k+1)$-handles as $M(I,0,0)$, but now using the framings of $S^{2k}\subset S^{4k+1}$ classified by

$$\tau_{S^{2k+1}}\in\pi_{2k+1}(BSO(2k+1))\ =\ \pi_{2k}(SO(2k+1))$$

(which is zero if and only if $k=0,1,3$). The Arf form is the kernel of the $(2k+1)$-connected $(4k+2)$-dimensional normal map

$$\begin{aligned}(f',b')\ :\ (M';\partial_-M',\partial_+M')\ &=\ (\mathrm{cl.}(M(I,1,1)\backslash D^{4k+2});S^{4k+1},\Sigma^{4k+1})\\ &\to S^{4k+1}\times([0,1];\{0\},\{1\})\end{aligned}$$

defined by the trace of surgeries on the linked spheres

$$S^{2k}\cup S^{2k}\hookrightarrow S^{4k+1}\ =\ S^{2k}\times D^{2k+1}\cup D^{2k+1}\times S^{2k}\ ,$$

with self-linking given by the non-standard framing. (See 3.7 below for a brief account of the exotic sphere Σ^{4k+1}). □

EXAMPLE 2.20 The sphere bundles $S(\omega)$ of certain oriented 4-plane bundles ω over S^4 (the special case $n=4$ of 2.19 (i)) give explicit exotic 7-spheres. An oriented 4-plane bundle $\omega : S^4 \to BSO(4)$ is determined by the Euler number and first Pontrjagin class

$$\chi(\omega), p_1(\omega) \in H^4(S^4) \;=\; \mathbb{Z} \;,$$

which must be such that

$$2\chi(\omega) \equiv p_1(\omega)(\bmod 4) \;,$$

with an isomorphism

$$\pi_4(BSO(4)) \xrightarrow{\cong} \mathbb{Z}\oplus\mathbb{Z} \;;\; \omega \longmapsto ((2\chi(\omega)+p_1(\omega))/4, (2\chi(\omega)-p_1(\omega))/4) \;.$$

If $\chi(\omega)=1$ then $S(\omega)$ is a homotopy 7-sphere, and

$$p_1(\omega) \;=\; 2\ell \in H^4(S^4) \;=\; \mathbb{Z}$$

for some odd integer ℓ. The 7-dimensional differentiable manifold $\Sigma^7_\ell = S(\omega)$ is homeomorphic to S^7 (by Smale's generalized Poincaré conjecture, or by a direct Morse-theoretic argument). If Σ^7_ℓ is diffeomorphic to S^7 then

$$M^8 \;=\; E(\omega)\cup_{\Sigma^7_\ell} D^8$$

is a closed 8-dimensional differentiable manifold with

$$p_1(M) \;=\; p_1(\omega) \;=\; 2\ell \in H^4(M) \;=\; \mathbb{Z} \;\;,\;\; \sigma(M) \;=\; 1 \in \mathbb{Z} \;.$$

By the Hirzebruch signature theorem

$$\begin{aligned} \sigma(M) \;&=\; \langle \mathcal{L}(M), [M] \rangle \\ &=\; (7p_2(M) - p_1(M)^2)/45 \\ &=\; (7p_2(M) - 4\ell^2)/45 \;=\; 1 \in \mathbb{Z} \;. \end{aligned}$$

If $\ell \not\equiv \pm 1(\bmod 7)$ then

$$p_2(M) \;=\; (45+4\ell^2)/7 \notin H^8(M) \;=\; \mathbb{Z}$$

so that there is no such diffeomorphism, and Σ^7_ℓ is an exotic 7-sphere (Milnor [10], Milnor and Stasheff [14, p.247]). □

DEFINITION 2.21 An *isomorphism* of ϵ-symmetric forms

$$f \;:\; (K,\lambda) \xrightarrow{\cong} (K',\lambda')$$

is a Λ-module isomorphism $f : K \cong K'$ such that

$$\lambda'(f(x), f(y)) \;=\; \lambda(x,y) \in \Lambda \;.$$

An *isomorphism* of ϵ-quadratic forms $f : (K,\lambda,\mu) \cong (K',\lambda',\mu')$ is an isomorphism of the underlying ϵ-symmetric forms $f : (K,\lambda) \cong (K',\lambda')$ such that

$$\mu'(f(x)) \;=\; \mu(x) \in Q_\epsilon(\Lambda) \;. \qquad \square$$

Proposition 2.22 *If there exists a central element $s \in \Lambda$ such that*

$$s + \overline{s} = 1 \in \Lambda$$

there is an identification of categories

$$\{\epsilon\text{-}quadratic\ forms\ over\ \Lambda\} \;=\; \{\epsilon\text{-}symmetric\ forms\ over\ \Lambda\} \;.$$

Proof: The ϵ-symmetrization map $1+T_\epsilon : Q_\epsilon(K) \to Q^\epsilon(K)$ is an isomorphism for any Λ-module K, with inverse

$$Q^\epsilon(K) \to Q_\epsilon(K) \;;\; \lambda \longmapsto ((x,y) \longmapsto s\lambda(x,y)) \;.$$

For any ϵ-quadratic form (K,λ,μ) the ϵ-quadratic function μ is determined by the ϵ-symmetric pairing λ, with

$$\mu(x) \;=\; s\lambda(x,x) \in Q_\epsilon(\Lambda) \;. \qquad \square$$

Example 2.23 If $2 \in \Lambda$ is invertible then 2.22 applies with $s = 1/2 \in \Lambda$. $\square$

For any ϵ-symmetric form (K,λ) and $x \in K$

$$\lambda(x,x) \in Q^\epsilon(\Lambda) \;.$$

Definition 2.24 An ϵ-symmetric form (K,λ) is *even* if for all $x \in K$

$$\lambda(x,x) \in \mathrm{im}(1+T_\epsilon : Q_\epsilon(\Lambda) \to Q^\epsilon(\Lambda)) \;. \qquad \square$$

Proposition 2.25 *Let $\epsilon = 1$ or -1. If the ϵ-symmetrization map*

$$1+T_\epsilon \;:\; Q_\epsilon(\Lambda) \to Q^\epsilon(\Lambda)$$

is an injection there is an identification of categories

$$\{\epsilon\text{-}quadratic\ forms\ over\ \Lambda\} \;=\; \{even\ \epsilon\text{-}symmetric\ forms\ over\ \Lambda\} \;.$$

Proof: Given an even ϵ-symmetric form (K,λ) over Λ there is a unique function $\mu : K \to Q_\epsilon(\Lambda)$ such that for all $x \in K$

$$(1+T_\epsilon)(\mu(x)) \;=\; \lambda(x,x) \in Q^\epsilon(\Lambda) \;,$$

which then automatically satisfies the conditions of 2.13 for (K,λ,μ) to be an ϵ-quadratic form. $\square$

Example 2.26 The symmetrization map

$$1+T \;=\; 2 \;:\; Q_{+1}(\mathbb{Z}) \;=\; \mathbb{Z} \to Q^{+1}(\mathbb{Z}) \;=\; \mathbb{Z}$$

is an injection, so that quadratic forms over $\mathbb{Z}$ coincide with the even symmetric forms. $\square$

EXAMPLE 2.27 The (-1)-symmetrization map

$$1+T_- \ :\ Q_{-1}(\mathbb{Z})\ =\ \mathbb{Z}_2 \to Q^{-1}(\mathbb{Z})\ =\ 0$$

is not an injection, so that (-1)-quadratic forms over $\mathbb{Z}$ have a richer structure than even (-1)-symmetric forms. The hyperbolic (-1)-symmetric form $(K,\lambda)=H^{-1}(\mathbb{Z})$ over $\mathbb{Z}$

$$K\ =\ \mathbb{Z}\oplus\mathbb{Z}\ \ ,\ \ \lambda\ :\ K\times K\to\mathbb{Z}\ ;\ ((a,b),(c,d))\longmapsto ad-bc$$

admits two distinct (-1)-quadratic refinements (K,λ,μ), (K,λ,μ'), with

$$\mu\ :\ K\to Q_{-1}(\mathbb{Z})\ =\ \mathbb{Z}/2\ ;\ (x,y)\longmapsto xy\ ,$$
$$\mu'\ :\ K\to Q_{-1}(\mathbb{Z})\ =\ \mathbb{Z}/2\ ;\ (x,y)\longmapsto x^2+xy+y^2\ .$$

See §3 below for the definition of the Arf invariant, which distinguishes the hyperbolic (-1)-quadratic form $(K,\lambda,\mu)=H_{-1}(\mathbb{Z})$ from the Arf form (K,λ,μ') (which already appeared in 2.19 (iv)). □

§3. The even-dimensional L-groups

The even-dimensional surgery obstruction groups $L_{2n}(\Lambda)$ will now be defined, using the following preliminary result.

LEMMA 3.1 *For any nonsingular ϵ-quadratic form (K,λ,μ) there is defined an isomorphism*

$$(K,\lambda,\mu)\oplus(K,-\lambda,-\mu)\ \cong\ H_\epsilon(K)\ ,$$

with $H_\epsilon(K)$ the hyperbolic ϵ-quadratic form (2.14).

PROOF: Let L be a f. g. projective Λ-module such that $K\oplus L$ is f. g. free, with basis elements $\{x_1,x_2,\dots,x_k\}$ say. Let

$$\lambda_{ij}\ =\ (\lambda\oplus 0)(x_j,x_i)\in\Lambda\ \ (1\le i<j\le k)$$

and choose representatives $\mu_i\in\Lambda$ of $\mu(x_i)\in Q_\epsilon(\Lambda)$ $(1\le i\le k)$. Define the Λ-module morphism

$$\begin{aligned}&\psi_{K\oplus L}\ :\ K\oplus L\to(K\oplus L)^*\ ;\\ &\quad\sum_{i=1}^k a_ix_i\longmapsto(\sum_{j=1}^k b_jx_j\longmapsto\sum_{i=1}^k b_i\mu_i\overline{a}_i+\sum_{1\le i<j\le k}b_j\lambda_{ij}\overline{a}_i)\ .\end{aligned}$$

The Λ-module morphism defined by

$$\psi\ :\ K\xrightarrow{\text{inclusion}}K\oplus L\xrightarrow{\psi_{K\oplus L}}(K\oplus L)^*\ =\ K^*\oplus L^*\xrightarrow{\text{projection}}K^*$$

is such that

$$\begin{aligned}\lambda\ &=\ \psi+\epsilon\psi^*\ :\ K\to K^*,\\ \mu(x)\ &=\ \psi(x,x)\in Q_\epsilon(\Lambda)\ (x\in K)\ .\end{aligned}$$

As (K,λ,μ) is nonsingular $\psi+\epsilon\psi^*:K\to K^*$ is an isomorphism. The Λ-module morphism defined by

$$\widetilde{\psi} \;=\; (\psi+\epsilon\psi^*)^{-1}\psi(\psi+\epsilon\psi^*)^{-1} \;:\; K^*\to K$$

is such that

$$(\psi+\epsilon\psi^*)^{-1} \;=\; \widetilde{\psi}+\epsilon\widetilde{\psi}^* \;:\; K^*\to K\ .$$

Define an isomorphism of ϵ-quadratic forms

$$f \;:\; H_\epsilon(K) \xrightarrow{\cong} (K\oplus K,\lambda\oplus -\lambda,\mu\oplus -\mu)$$

by

$$f \;=\; \begin{pmatrix} 1 & -\epsilon\widetilde{\psi}^* \\ 1 & \widetilde{\psi} \end{pmatrix} \;:\; K\oplus K^*\to K\oplus K\ . \qquad \square$$

Definition 3.2 The *$2n$-dimensional L-group* $L_{2n}(\Lambda)$ is the group of equivalence classes of nonsingular $(-1)^n$-quadratic forms (K,λ,μ) on stably f. g. free Λ-modules, subject to the equivalence relation

$(K,\lambda,\mu)\sim(K',\lambda',\mu')$

if there exists an isomorphism of $(-1)^n$-quadratic forms

$(K,\lambda,\mu)\oplus H_{(-1)^n}(\Lambda^k) \xrightarrow{\cong} (K',\lambda',\mu')\oplus H_{(-1)^n}(\Lambda^{k'})$

for some f. g. free Λ-modules $\Lambda^k,\Lambda^{k'}$.

Addition and inverses in $L_{2n}(\Lambda)$ are given by

$$\begin{aligned}(K_1,\lambda_1,\mu_1)+(K_2,\lambda_2,\mu_2) \;&=\; (K_1\oplus K_2,\lambda_1\oplus\lambda_2,\mu_1\oplus\mu_2)\ ,\\ -(K,\lambda,\mu) \;&=\; (K,-\lambda,-\mu)\in L_{2n}(\Lambda)\ . \end{aligned} \qquad \square$$

The groups $L_{2n}(\Lambda)$ only depend on the residue $n(\mathrm{mod}\,2)$, so that only two L-groups have actually been defined, $L_0(\Lambda)$ and $L_2(\Lambda)$. Note that 3.2 uses Lemma 3.1 to justify $(K,\lambda,\mu)\oplus(K,-\lambda,-\mu)\sim 0$.

Remark 3.3 The surgery obstruction of an n-connected $2n$-dimensional normal map $(f,b):M^{2n}\to X$ is an element

$$\sigma_*(f,b) \;=\; (K_n(M),\lambda,\mu)\in L_{2n}(\mathbb{Z}[\pi_1(X)])$$

such that $\sigma_*(f,b)=0$ if (and for $n\geq 3$ only if) (f,b) is normal bordant to a homotopy equivalence. $\square$

Example 3.4 Let $M=M^2_g$ be the orientable 2-manifold (= surface) of genus g, with degree 1 map $f:M\to S^2$. A choice of framing of the stable normal bundle of an embedding $M\hookrightarrow\mathbb{R}^3$ determines a 1-connected

2-dimensional normal map $(f,b) : M \to S^2$. For a standard choice of framing (i.e. one which extends to a 3-manifold N with $\partial N = M$) the kernel form and the surgery obstruction are given by

$$\sigma_*(f,b) \;=\; H_{-1}(\mathbb{Z}^g) \;=\; 0 \;\in L_2(\mathbb{Z})$$

and (f,b) is normal bordant to a homotopy equivalence, i.e. M is framed null-cobordant. □

EXAMPLE 3.5 The even-dimensional L-groups of the ring $\Lambda = \mathbb{R}$ of real numbers with the identity involution are given by

$$L_{2n}(\mathbb{R}) \;=\; \begin{cases} \mathbb{Z} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd .} \end{cases}$$

Since $1/2 \in \mathbb{R}$ there is no difference between symmetric and quadratic forms over $\mathbb{R}$. □

The *signature* (alias index) of a nonsingular symmetric form (K,λ) over $\mathbb{R}$ is defined by

$$\begin{aligned}\sigma(K,\lambda) \;=\; &\text{no. of positive eigenvalues of } \lambda \\ &\qquad - \text{no. of negative eigenvalues of } \lambda \in \mathbb{Z} \;.\end{aligned}$$

Here, the symmetric form $\lambda \in Q^{+1}(K)$ is identified with the symmetric $k\times k$ matrix $(\lambda(x_i,x_j)_{1\leq i,j\leq k}) \in M_{k,k}(\mathbb{R})$ determined by any choice of basis $x_1,x_2,\ldots,x_k$ for K. By Sylvester's law of inertia the rank k and the signature $\sigma(K,\lambda)$ define a complete set of invariants for the isomorphism classification of nonsingular symmetric forms (K,λ) over $\mathbb{R}$, meaning that two forms are isomorphic if and only if they have the same rank and signature. A nonsingular quadratic form (K,ψ) over $\mathbb{R}$ is isomorphic to a hyperbolic form if and only if it has signature 0. Two such forms (K,λ), (K',λ') are related by an isomorphism

$$(K,\lambda)\oplus H_+(\mathbb{R}^m) \xrightarrow{\cong} (K',\lambda')\oplus H_+(\mathbb{R}^{m'})$$

if and only if they have the same signature

$$\sigma(K,\lambda) \;=\; \sigma(K',\lambda') \in \mathbb{Z} \;.$$

Moreover, every integer is the signature of a form, since $1 \in \mathbb{Z}$ is the signature of the nonsingular symmetric form $(\mathbb{R},1)$ with

$$1 \;:\; \mathbb{R} \to \mathbb{R}^* \;;\; x \longmapsto (y \longmapsto xy)$$

and for any nonsingular symmetric forms $(K,\lambda),(K',\lambda')$ over $\mathbb{R}$

$$\begin{aligned}\sigma((K,\lambda)\oplus(K',\lambda')) \;&=\; \sigma(K,\lambda)+\sigma(K',\lambda') \;, \\ \sigma(K,-\lambda) \;&=\; -\sigma(K,\lambda) \in \mathbb{Z} \;.\end{aligned}$$

The isomorphism of 3.5 in the case $n \equiv 0 (\bmod 2)$ is defined by

$$L_0(\mathbb{R}) \xrightarrow{\cong} \mathbb{Z} \ ; \ (K,\lambda) \longmapsto \sigma(K,\lambda) \ .$$

$L_2(\mathbb{R}) = 0$ because every nonsingular (-1)-symmetric (alias symplectic) form over $\mathbb{R}$ admits is isomorphic to a hyperbolic form.

It is not possible to obtain a complete isomorphism classification of nonsingular symmetric and quadratic forms over $\mathbb{Z}$ – see Chapter II of Milnor and Husemoller [13] for the state of the art in 1973. Fortunately, it is much easier to decide if two forms become isomorphic after adding hyperbolics then whether they are actually isomorphic. Define the signature of a nonsingular symmetric form (K,λ) over $\mathbb{Z}$ to be the signature of the induced nonsingular symmetric form over $\mathbb{R}$

$$\sigma(K,\lambda) \ = \ \sigma(\mathbb{R}\otimes K, 1\otimes\lambda) \in \mathbb{Z} \ .$$

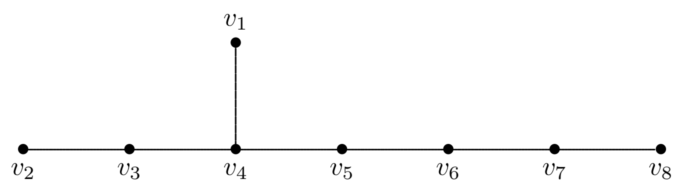
It is a non-trivial theorem that two nonsingular even symmetric forms (K,λ), (K',λ') are related by an isomorphism

$$(K,\lambda)\oplus H_+(\mathbb{Z}^m) \xrightarrow{\cong} (K',\lambda')\oplus H_+(\mathbb{Z}^{m'})$$

if and only if they have the same signature

$$\sigma(K,\lambda) \ = \ \sigma(K',\lambda') \in \mathbb{Z} \ .$$

Moreover, not every integer arises as the signature of an even symmetric form, only those divisible by 8. The Dynkin diagram of the exceptional Lie group E_8 is a tree



Weighing each vertex by $1 \in Q_{+1}(\mathbb{Z}) = \mathbb{Z}$ gives (by the method recalled in 2.18) a nonsingular quadratic form $(\mathbb{Z}^8, \lambda_{E_8}, \mu_{E_8})$ with signature

$$\sigma(\mathbb{Z}^8, \lambda_{E_8}) \ = \ 8 \in \mathbb{Z} \ ,$$

where

$$\lambda_{E_8} = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} : \mathbb{Z}^8 \to (\mathbb{Z}^8)^*$$

and μ_{E_8} is determined by λ_{E_8}.

EXAMPLE 3.6 (i) The signature divided by 8 defines an isomorphism

$$\sigma \ : \ L_{4k}(\mathbb{Z}) \to \mathbb{Z} \ ; \ (K, \lambda, \mu) \mapsto \sigma(K, \lambda)/8$$

so that $(\mathbb{Z}^8, \lambda_{E_8}, \mu_{E_8}) \in L_{4k}(\mathbb{Z})$ represents a generator.
(ii) See Kervaire and Milnor [7] and Levine [9] for the surgery classification of high-dimensional exotic spheres, including the expression of the h-cobordism group Θ^n of n-dimensional exotic spheres for $n \geq 5$ as

$$\Theta^n \ = \ \pi_n(TOP/O) \ = \ \pi_n(PL/O)$$

and the exact sequence

$$\ldots \to \pi_{n+1}(G/O) \to L_{n+1}(\mathbb{Z}) \to \Theta^n \to \pi_n(G/O) \to \ldots \ .$$

(iii) In the original case $n = 7$ (Milnor [10]) there is defined an isomorphism

$$\Theta^7 \xrightarrow{\cong} \mathbb{Z}_{28} \ ; \ \Sigma^7 \mapsto \sigma(W)/8$$

for any framed 8-dimensional manifold W with $\partial W = \Sigma^7$. The realization (2.17) of $(\mathbb{Z}^8, \lambda_{E_8}, \mu_{E_8})$ as the kernel form of a 4-connected 8-dimensional normal bordism

$$\begin{aligned}(f, b) \ : (M^8, S^7, \partial_+ M) \ &= \ (\mathrm{cl.}(M(E_8, 1, \ldots, 1)\backslash D^8); S^7, \partial M(E_8, 1, \ldots, 1)) \\ &\to S^7 \times ([0, 1]; \{0\}, \{1\})\end{aligned}$$

gives the exotic sphere

$$\Sigma^7 \ = \ \partial_+ M \ = \ \partial M(E_8, 1, \ldots, 1)$$

generating Θ^7: the framed 8-dimensional manifold $W = M(E_8, 1, \ldots, 1)$ obtained by the E_8-plumbing of 8 copies of τ_{S^4} (2.18) has $\sigma(W) = 8$. The 7-dimensional homotopy sphere Σ^7_ℓ defined for any odd integer ℓ in 2.20 is the boundary of a framed 8-dimensional manifold W_ℓ with $\sigma(W_\ell) = 8(\ell^2 - 1)$.

□

For any nonsingular (-1)-quadratic form (K, λ, μ) over $\mathbb{Z}$ there exists a symplectic basis $x_1, \dots, x_{2m}$ for K, such that

$$\lambda(x_i, x_j) \;=\; \begin{cases} 1 & \text{if } i - j = m \\ -1 & \text{if } j - i = m \\ 0 & \text{otherwise .} \end{cases}$$

The *Arf invariant* of (K, λ, μ) is defined using any such basis to be

$$c(K, \lambda, \mu) \;=\; \sum_{i=1}^{m} \mu(x_i)\mu(x_{i+m}) \in \mathbb{Z}_2 \ .$$

Example 3.7 (i) The Arf invariant defines an isomorphism

$$c \;:\; L_{4k+2}(\mathbb{Z}) \to \mathbb{Z}_2 \ ; \ (K, \lambda, \mu) \mapsto c(K, \lambda, \mu)$$

The nonsingular (-1)-quadratic form $(\mathbb{Z} \oplus \mathbb{Z}, \lambda, \mu)$ over $\mathbb{Z}$ defined by

$$\begin{aligned} \lambda((x, y), (x', y')) \;&=\; x'y - xy' \in \mathbb{Z} \ , \\ \mu(x, y) \;&=\; x^2 + xy + y^2 \in Q_{-1}(\mathbb{Z}) \;=\; \mathbb{Z}_2 \end{aligned}$$

has Arf invariant $c(\mathbb{Z} \oplus \mathbb{Z}, \lambda, \mu) = 1$, and so generates $L_{4k+2}(\mathbb{Z})$.
(ii) The realization (2.19 (iv)) of the Arf form $(\mathbb{Z} \oplus \mathbb{Z}, \lambda, \mu)$ as the kernel form of a 5-connected 10-dimensional normal bordism

$$\begin{aligned} (f, b) \;:\; (M^{10}, S^9, \partial_+ M) \;&=\; (\mathrm{cl.}(M(I,1,1)\backslash D^{10}); S^9, \partial M(I,1,1)) \\ &\to S^9 \times ([0,1]; \{0\}, \{1\}) \end{aligned}$$

is obtained by plumbing together 2 copies of τ_{S^5} (2.18) where I is the tree with 1 edge and 2 vertices

I

and $\Sigma^9 = \partial_+ M = \partial M(I,1,1)$ is the exotic 9-sphere generating $\Theta^9 = \mathbb{Z}_2$. Coning off the boundary components gives the closed 10-dimensional PL manifold $cS^9 \cup M \cup c\Sigma^9$ without differentiable structure of Kervaire [5]. □

§4. Split forms

A "split form" on a Λ-module K is an element

$$\psi \in S(K) \;=\; \mathrm{Hom}_\Lambda(K, K^*) \ ,$$

which can be regarded as a sesquilinear pairing

$$\psi \;:\; K \times K \to \Lambda \ ; \ (x, y) \mapsto \psi(x, y) \ .$$

Split forms are more convenient to deal with than ϵ-quadratic forms in describing the algebraic effects of even-dimensional surgery (in §5 below), and are closer to the geometric applications such as knot theory.

The main result of §4 is that the ϵ-quadratic structures (λ, μ) on a f.g. projective Λ-module K correspond to the elements of the ϵ-quadratic group of 2.6

$$Q_\epsilon(K) \;=\; \operatorname{coker}(1-T_\epsilon : S(K) \to S(K)) \ .$$

The pair of functions (λ,μ) used to define an ϵ-quadratic form (K,λ,μ) can thus be replaced by an equivalence class of Λ-module morphisms $\psi : K \to K^*$ such that

$$\begin{aligned}\lambda(x,y) \;&=\; \psi(x,y)+\epsilon\overline{\psi(y,x)} \in \Lambda \ ,\\ \mu(x) \;&=\; \psi(x,x) \in Q_\epsilon(\Lambda)\end{aligned}$$

i.e. by an equivalence class of split forms.

Definition 4.1 (i) A *split form* (K,ψ) over Λ is a f. g. projective Λ-module K together with an element $\psi \in S(K)$.
(ii) A *morphism* (resp. *isomorphism*) of split forms over Λ

$$f \;:\; (K,\psi) \to (K',\psi')$$

is a Λ-module morphism (resp. isomorphism) $f : K \to K'$ such that

$$f^*\psi' f \;=\; \psi \;:\; K \to K^* \ .$$

(iii) An *ϵ-quadratic morphism* (resp. *isomorphism*) of split forms over Λ

$$(f,\chi) \;:\; (K,\psi) \to (K',\psi')$$

is a Λ-module morphism (resp. isomorphism) $f : K \to K'$ together with an element $\chi \in Q_{-\epsilon}(K)$ such that

$$f^*\psi' f - \psi \;=\; \chi - \epsilon\chi^* \;:\; K \to K^* \ .$$

(iv) A split form (K,ψ) is *ϵ-nonsingular* if $\psi+\epsilon\psi^* : K \to K^*$ is a Λ-module isomorphism. □

Proposition 4.2 (i) *A split form (K,ψ) determines an ϵ-quadratic form (K,λ,μ) by*

$$\begin{aligned}&\lambda \;=\; \psi+\epsilon\psi^* \;:\; K \to K^* \ ; \ x \longmapsto (y \longmapsto \psi(x,y)+\epsilon\overline{\psi(y,x)}) \ ,\\ &\mu \;:\; K \to Q_\epsilon(\Lambda) \ ; \ x \longmapsto \psi(x,x) \ .\end{aligned}$$

(ii) *Every ϵ-quadratic form (K,λ,μ) is determined by a split form (K,ψ), which is unique up to*

$$\psi \;\sim\; \psi' \ \textit{if} \ \psi'-\psi \;=\; \chi-\epsilon\chi^* \ \textit{for some} \ \chi : K \to K^* \ .$$

(iii) *The isomorphism classes of (nonsingular) ϵ-quadratic forms (K,λ,μ) over Λ are in one-one correspondence with the ϵ-quadratic isomorphism classes of (ϵ-nonsingular) split forms (K,ψ) over Λ.*
Proof: (i) By construction.
(ii) There is no loss of generality in taking K to be f.g. free, $K=\Lambda^k$. An ϵ-quadratic form (Λ^k,λ,μ) over Λ is determined by a $k\times k$-matrix

$\lambda = \{\lambda_{ij} \in \Lambda \,|\, 1 \leq i, j \leq k\}$ such that

$$\overline{\lambda_{ij}} ~=~ \epsilon\lambda_{ji} \in \Lambda$$

and a collection of elements $\mu = \{\mu_i \in Q_\epsilon(\Lambda) \,|\, 1 \leq i \leq k\}$ such that

$$\mu_i + \epsilon\overline{\mu_i} ~=~ \lambda_{ii} \in Q^\epsilon(\Lambda) ~.$$

Choosing any representatives $\mu_i \in \Lambda$ of $\mu_i \in Q_\epsilon(\Lambda^k)$ there is defined a split form (Λ^k, ψ) with $\psi = \{\psi_{ij} \in \Lambda \,|\, 1 \leq i, j \leq k\}$ the $k \times k$ matrix defined by

$$\psi_{ij} ~=~ \begin{cases} \lambda_{ij} & \text{if } i < j \\ \mu_i & \text{if } i = j \\ 0 & \text{otherwise} ~. \end{cases}$$

(iii) An ϵ-quadratic (iso)morphism $(f, \chi) : (K, \psi) \to (K', \psi')$ of split forms determines an (iso)morphism $f : (K, \lambda, \mu) \to (K', \lambda', \mu')$ of ϵ-quadratic forms. Conversely, an ϵ-quadratic form (K, λ, μ) determines an ϵ-quadratic isomorphism class of split forms (K, ψ) as in 3.1, and every (iso)morphism of ϵ-quadratic forms lifts to an ϵ-quadratic (iso)morphism of split forms. □

Thus $Q_\epsilon(K)$ is both the group of isomorphism classes of ϵ-quadratic forms and the group of ϵ-quadratic isomorphism classes of split forms on a f. g. projective Λ-module K.

The following algebraic result will be used in 4.6 below to obtain a homological split form ψ on the kernel $\mathbb{Z}[\pi_1(X)]$-module $K_n(M)$ of an n-connected $2n$-dimensional normal map $(f, b) : M \to X$ with some extra structure, which determines the kernel $(-1)^n$-quadratic form $(K_n(M), \lambda, \mu)$ as in 4.2 (i).

Lemma 4.3 *Let (K, λ, μ) be an ϵ-quadratic form over Λ.*
(i) *If $s : K \to K$ is an endomorphism such that*

$$\begin{pmatrix} s \\ 1 - s \end{pmatrix} ~:~ (K, 0, 0) \to (K, \lambda, \mu) \oplus (K, -\lambda, -\mu)$$

defines a morphism of ϵ-quadratic forms then $(K, \lambda s)$ is a split form which determines the ϵ-quadratic form (K, λ, μ).
(ii) *If (K, λ, μ) is nonsingular and (K, ψ) is a split form which determines (K, λ, μ) then*

$$s ~=~ \lambda^{-1}\psi ~:~ K \to K$$

is an endomorphism such that

$$\begin{pmatrix} s \\ 1 - s \end{pmatrix} ~:~ (K, 0, 0) \to (K, \lambda, \mu) \oplus (K, -\lambda, -\mu)$$

defines a morphism of ϵ-quadratic forms.
Proof : (i) By 4.2 there exist a split form (K, ψ) which determines

(K,λ,μ) and an ϵ-quadratic morphism of split forms

$$(\begin{pmatrix} s \\ 1-s \end{pmatrix},\chi)\ :\ (K,0)\to(K,\psi)\oplus(K,-\psi)\ .$$

It follows from

$$\lambda\ =\ \psi+\epsilon\psi^*\ :\ K\to K^*\ ,$$

$$\begin{pmatrix} s \\ 1-s \end{pmatrix}^*\begin{pmatrix} \psi & 0 \\ 0 & -\psi \end{pmatrix}\begin{pmatrix} s \\ 1-s \end{pmatrix}\ =\ \chi-\epsilon\chi^*\ :\ K\to K^*$$

that

$$\lambda s-\psi\ =\ \chi'-\epsilon\chi'^*\ :\ K\to K^*$$

with

$$\chi'\ =\ \chi-s^*\psi\ :\ K\to K^*\ .$$

(ii) From the definitions. □

In the terminology of §5 the morphism of 4.3 (ii)

$$\begin{pmatrix} s \\ 1-s \end{pmatrix}\ :\ (K,0,0)\to(K,\lambda,\mu)\oplus(K,-\lambda,-\mu)$$

is the inclusion of a lagrangian

$$\begin{aligned} L\ &=\ \mathrm{im}(\begin{pmatrix} s \\ 1-s \end{pmatrix}:K\to K\oplus K) \\ &=\ \ker((\,(-1)^{n-1}\psi^*\quad \psi\,):K\oplus K\to K^*)\ . \end{aligned}$$

Example 4.4 A $(2n-1)$-*knot* is an embedding of a homotopy $(2n-1)$-sphere in a standard $(2n+1)$-sphere

$$\ell\ :\ \Sigma^{2n-1}\hookrightarrow S^{2n+1}\ .$$

For $n=1$ this is just a classical knot $\ell:\Sigma^1=S^1\hookrightarrow S^3$; for $n\geq 3$ Σ^{2n-1} is homeomorphic to S^{2n-1}, by the generalized Poincaré conjecture, but may have an exotic differentiable structure. Split forms (K,ψ) first appeared as the Seifert forms over $\mathbb{Z}$ of $(2n-1)$-knots, originally for $n=1$. See Ranicki [21, 7.8], [24] for a surgery treatment of high-dimensional knot theory. In particular, a Seifert form is an integral refinement of an even-dimensional surgery kernel form, as follows.

(i) A $(2n-1)$-knot $\ell:\Sigma^{2n-1}\hookrightarrow S^{2n+1}$ is *simple* if

$$\pi_r(S^{2n+1}\backslash\ell(\Sigma^{2n-1}))\ =\ \pi_r(S^1)\quad(1\leq r\leq n-1)\ .$$

(Every 1-knot is simple). A simple $(2n-1)$-knot ℓ has a simple Seifert surface, that is an $(n-1)$-connected framed codimension 1 submanifold $M^{2n}\subset S^{2n+1}$ with boundary

$$\partial M\ =\ \ell(\Sigma^{2n-1})\subset S^{2n+1}\ .$$

The kernel of the n-connected normal map

$$(f,b) \ = \ \text{inclusion} \ : \ (M,\partial M) \to (X,\partial X) \ = \ (D^{2n+2},\ell(\Sigma^{2n-1}))$$

is a nonsingular $(-1)^n$-quadratic form $(H_n(M),\lambda,\mu)$ over $\mathbb{Z}$. The *Seifert form* of ℓ with respect to M is the refinement of $(H_n(M),\lambda,\mu)$ to a $(-1)^n$-nonsingular split form $(H_n(M),\psi)$ over $\mathbb{Z}$ which is defined using Alexander duality and the universal coefficient theorem

$$\psi \ = \ i_* \ : \ H_n(M) \to H_n(S^{2n+1}\backslash M) \ \cong \ H^n(M) \ \cong \ H_n(M)^*$$

with $i : M \to S^{2n+1}\backslash M$ the map pushing M off itself along a normal direction in S^{2n+1}. If $i' : M \to S^{2n+1}\backslash M$ pushes M off itself in the opposite direction

$$i'_* \ = \ (-1)^{n+1}\psi^* \ : \ H_n(M) \to H_n(S^{2n+1}\backslash M) \ \cong \ H^n(M) \ \cong \ H_n(M)^*$$

with

$$\begin{aligned} i_* - i'_* \ &= \ \psi + (-1)^n\psi^* \ = \ \lambda \\ &= \ ([M]\cap -)^{-1} \ : \ H_n(M) \to H^n(M) \ \cong \ H_n(M)^* \end{aligned}$$

the Poincaré duality isomorphism. If $x_1, x_2, \dots, x_k \in H_n(M)$ is a basis then $(\psi(x_j, x_{j'}))$ is a *Seifert matrix* for the $(2n-1)$-knot ℓ. For any embeddings $x, y : S^n \hookrightarrow M$

$$\begin{aligned} \psi(x,y) \ &= \ \text{linking number}(ix(S^n)\cup y(S^n)\subset S^{2n+1}) \\ &= \ \text{degree}(y^*ix : S^n \to S^n) \in \mathbb{Z} \end{aligned}$$

with

$$y^*ix \ : \ S^n \xrightarrow{\ x\ } M \xrightarrow{\ i\ } S^{2n+1}\backslash M \xrightarrow{\ y^*\ } S^{2n+1}\backslash y(S^n) \simeq S^n \ .$$

For $n \geq 3$ every element $x \in H_n(M)$ is represented by an embedding $e : S^n \hookrightarrow M$, using the Whitney embedding theorem, and $\pi_1(M) = \{1\}$. Moreover, for any embedding $x : S^n \hookrightarrow M$ the framed embedding $M \hookrightarrow S^{2n+1}$ determines a stable trivialization of the normal bundle $\nu_x : S^n \to BSO(n)$

$$\delta\nu_x \ : \ \nu_x \oplus \epsilon \ \cong \ \epsilon^{n+1}$$

such that

$$\psi(x,x) \ = \ (\delta\nu_x,\nu_x) \in \pi_{n+1}(BSO(n+1),BSO(n)) \ = \ \mathbb{Z} \ .$$

Every element $x \in H_n(M)$ is represented by an embedding

$$e_1 \times e_2 \ : \ S^n \hookrightarrow M \times \mathbb{R}$$

with $e_1 : S^n \looparrowright M$ a framed immersion such that the composite

$$S^n \xrightarrow{\ e_1\times e_2\ } M\times\mathbb{R} \hookrightarrow S^{2n+1}$$

is isotopic to the standard framed embedding $S^n \hookrightarrow S^{2n+1}$. Then

$$\psi(x,x) \;=\; \sum_{(a,b)\in D_2(e_1),e_2(a)<e_2(b)} I(a,b) \in \mathbb{Z}$$

is an integral lift of the geometric self-intersection (2.15 (ii))

$$\mu(x) \;=\; \sum_{(a,b)\in D_2(e_1)/\mathbb{Z}_2} I(a,b) \in Q_{(-1)^n}(\mathbb{Z}) \ ,$$

with

$$D_2(e_1) \;=\; \{(a,b)\in S^n\times S^n \,|\, a\neq b\in S^n, e_1(a)=e_1(b)\in M\}$$

the double point set. For even n $\psi(x,x) = \mu(x) \in Q_{+1}(\mathbb{Z}) = \mathbb{Z}$, while for odd n $\psi(x,x) \in \mathbb{Z}$ is a lift of $\mu(x) \in Q_{-1}(\mathbb{Z}) = \mathbb{Z}_2$. The Seifert form $(H_n(M),\psi)$ is such that $\psi(x,x) = 0$ if (and for $n \geq 3$ only if) $x \in H_n(M)$ can be killed by an ambient surgery on $M^{2n} \subset S^{2n+1}$, i.e. represented by a framed embedding of pairs

$$x \;:\; (D^{n+1}\times D^n, S^n\times D^n) \hookrightarrow (S^{2n+1}\times[0,1], M\times\{0\})$$

so that the effect of the surgery on M is another Seifert surface for the $(2n-1)$-knot ℓ

$$M' \;=\; \mathrm{cl.}(M\times x(S^n\times D^n)) \cup D^{n+1}\times S^{n-1} \subset S^{2n+1} \ .$$

If $x \in H_n(M)$ generates a direct summand $L = \langle x\rangle \subset H_n(M)$ then M' is also $(n-1)$-connected, with Seifert form

$$(H_n(M'),\psi') \;=\; (L^{\perp}/L,[\psi]) \ ,$$

where

$$L^{\perp} \;=\; \{y\in H_n(M)\,|\,(\psi+(-1)^n\psi^*)(x)(y)=0 \text{ for } x\in L\} \subseteq H_n(M) \ .$$

(ii) Every $(-1)^n$-nonsingular split form (K,ψ) over $\mathbb{Z}$ is realized as the Seifert form of a simple $(2n-1)$-knot $\ell : \Sigma^{2n-1} \hookrightarrow S^{2n+1}$ (Kervaire [6]). From the algebraic surgery point of view the realization proceeds as follows. By 2.17 the nonsingular $(-1)^n$-quadratic form (K,λ,μ) determined by (K,ψ) (4.2 (i)) is the kernel form of an n-connected $2n$-dimensional normal map

$$(f,b) \;:\; (M^{2n},\Sigma^{2n-1}) \to (D^{2n},S^{2n-1})$$

with $f| : \Sigma^{2n-1} \to S^{2n-1}$ a homotopy equivalence. The double of (f,b) defines an n-connected $2n$-dimensional normal map

$$(g,c) \;=\; (f,b)\cup -(f,b) \;:\; N^{2n} \;=\; M\cup_{\Sigma^{2n-1}} -M \to D^{2n}\cup_{S^{2n-1}} -D^{2n} \;=\; S^{2n}$$

with kernel form $(K\oplus K,\lambda\oplus -\lambda,\mu\oplus -\mu)$. The direct summand

$$L \;=\; \ker(\begin{pmatrix}(-1)^{n-1}\psi^* & \psi\end{pmatrix} : K\oplus K \to K^*) \subset K\oplus K$$

is such that for any $(x,y) \in L$

$$\mu(x)-\mu(y) \;=\; (1+(-1)^{n-1})\psi(x,x) \;=\; 0 \in Q_{(-1)^n}(\mathbb{Z}) \ .$$

Let $k = \mathrm{rank}_{\mathbb{Z}}(K)$. The trace of the k surgeries on (g,c) killing a basis $(x_j, y_j) \in K \oplus K$ $(j = 1,2,\dots,k)$ for L is an n-connected $(2n+1)$-dimensional normal map

$$(W^{2n+1}; N, S^{2n}) \to S^{2n} \times ([0,1];\{0\},\{1\})$$

such that

$$\ell \;:\; \Sigma^{2n-1} \hookrightarrow (\Sigma^{2n-1} \times D^2) \cup (W \cup D^{2n+1}) \cup (M \times [0,1]) \;\cong\; S^{2n+1}$$

is a simple $(2n-1)$-knot with Seifert surface M and Seifert form (K,ψ). Note that M itself is entirely determined by the $(-1)^n$-quadratic form (K,λ,μ), with $\mathrm{cl.}(M\backslash D^{2n})$ the trace of k surgeries on S^{2n-1} removing

$$\bigcup_k S^{n-1} \times D^n \hookrightarrow S^{2n-1}$$

with (self-)linking numbers (λ,μ). The embedding $M \hookrightarrow S^{2n+1}$ is determined by the choice of split structure ψ for (λ,μ).

(iii) In particular, (ii) gives a knot version of the plumbing construction (2.18): let G be a finite graph with vertices $v_1, v_2, \dots, v_k$, weighted by $\mu_1, \mu_2, \dots, \mu_k \in Q_{(-1)^n}(\mathbb{Z})$, so that there are defined a $(-1)^n$-quadratic form $(\mathbb{Z}^k, \lambda, \mu)$ and a plumbed stably parallelizable $(n-1)$-connected $2n$-dimensional manifold with boundary

$$M^{2n} \;=\; M(G, \mu_1, \mu_2, \dots, \mu_k)\ ,$$

killing $H_1(G)$ by surgery if G is not a tree. A choice of split form ψ for (λ,μ) determines a compression of a framed embedding $M \hookrightarrow S^{2n+j}$ (j large) to a framed embedding $M \hookrightarrow S^{2n+1}$, so that $\partial M \hookrightarrow S^{2n+1}$ is a codimension 2 framed embedding. The form $(\mathbb{Z}^k,\lambda,\mu)$ is nonsingular if and only if $\Sigma^{2n-1} = \partial M$ is a homotopy $(2n-1)$-sphere, in which case $\Sigma^{2n-1} \hookrightarrow S^{2n+1}$ is a simple $(2n-1)$-knot with simple Seifert surface M.

(iv) Given a simple $(2n-1)$-knot $\ell : \Sigma^{2n-1} \hookrightarrow S^{2n+1}$ and a simple Seifert surface $M^{2n} \hookrightarrow S^{2n+1}$ there is defined an n-connected $2n$-dimensional normal map

$$(f,b) \;=\; \text{inclusion} \;:\; (M,\partial M) \to (X,\partial X) \;=\; (D^{2n+2}, \ell(\Sigma^{2n-1}))$$

as in (i). The knot complement is a $(2n+1)$-dimensional manifold with boundary

$$(W,\partial W) \;=\; (\mathrm{cl.}(S^{2n+1}\backslash(\ell(\Sigma^{2n-1}) \times D^2)), \ell(\Sigma^{2n-1}) \times S^1)$$

with a $\mathbb{Z}$-homology equivalence $p : (W,\partial W) \to S^1$ such that

$$\begin{aligned} p| \;&=\; \text{projection} \;:\; \partial W \;=\; \Sigma^{2n-1} \times S^1 \to S^1\ ,\\ p^{-1}(\mathrm{pt.}) \;&=\; M \subset W\ . \end{aligned}$$

Cutting W along $M \subset W$ there is obtained a cobordism $(N; M, M')$ with M' a copy of M, and N a deformation retract of $S^{2n+1}\backslash M$, such that (f,b)

extends to an n-connected normal map

$$(g,c)\ :\ (N;M,M')\to X\times([0,1];\{0\},\{1\})$$

with $(g,c)|=(f',b'):M'\to X$ a copy of (f,b). The n-connected $(2n+1)$-dimensional normal map

$$\begin{aligned}(h,d)\ &=\ (g,c)/((f,b)=(f',b'))\ :\\ (W,\partial W)\ &=\ (N;M,M')/(M=M')\to(X,\partial X)\times S^1\end{aligned}$$

is a $\mathbb{Z}$-homology equivalence which is the identity on ∂W, and such that

$$(f,b)\ =\ (h,d)|\ :\ (M,\partial M)\ =\ h^{-1}((X,\partial X)\times\{\text{pt.}\})\to(X,\partial X)\ .\quad\square$$

EXAMPLE 4.5 (i) Split forms over group rings arise in the following geometric situation, generalizing 4.4 (iv).
Let X be a $2n$-dimensional Poincaré complex, and let $(h,d):W\to X\times S^1$ be an n-connected $(2n+1)$-dimensional normal map which is a $\mathbb{Z}[\pi_1(X)]$-homology equivalence. Cut (h,d) along $X\times\{\text{pt.}\}\subset X\times S^1$ to obtain an n-connected $2n$-dimensional normal map

$$(f,b)\ =\ (h,d)|\ :\ M\ =\ h^{-1}(\{\text{pt.}\})\to X$$

and an n-connected normal bordism

$$(g,c)\ :\ (N;M,M')\to X\times([0,1];\{0\},\{1\})$$

with N a deformation retract of $W\backslash M$, such that $(g,c)|=(f,b):M\to X$, and such that $(g,c)|=(f',b'):M'\to X$ is a copy of (f,b). The inclusions $i:M\hookrightarrow N$, $i':M'\hookrightarrow N$ induce $\mathbb{Z}[\pi_1(X)]$-module morphisms

$$i_*\ :\ K_n(M)\to K_n(N)\quad,\quad i'_*\ :\ K_n(M')\ =\ K_n(M)\to K_n(N)$$

which fit into an exact sequence

$$K_{n+1}(W)=0\longrightarrow K_n(M)\xrightarrow{i_*-i'_*}K_n(N)\longrightarrow K_n(W)=0\ ,$$

so that $i_*-i'_*:K_n(M)\to K_n(N)$ is an isomorphism. Let $(K_n(M),\lambda,\mu)$ be the kernel $(-1)^n$-quadratic form of (f,b). The endomorphism

$$s\ =\ (i_*-i'_*)^{-1}i_*\ :\ K_n(M)\to K_n(M)$$

is such that

$$\begin{pmatrix}s\\1-s\end{pmatrix}\ :\ (K_n(M),0,0)\to(K_n(M),\lambda,\mu)\oplus(K_n(M),-\lambda,-\mu)$$

defines a morphism of $(-1)^n$-quadratic forms, so that by 4.3 the split form $(K_n(M),\psi)$ with

$$\psi\ =\ \lambda s\ :\ K_n(M)\to K_n(M)^*$$

determines $(K_n(M),\lambda,\mu)$. Every element $x\in K_n(M)$ can be represented by a framed immersion $x:S^n\looparrowright M$ with a null-homotopy $fx\simeq *:S^n\to$

X. Use the null-homotopy and the normal $\mathbb{Z}[\pi_1(X)]$-homology equivalence $(h,d) : W \to X \times S^1$ to extend x to a framed immersion $\delta x : D^{n+1} \looparrowright W$. If $x_1, x_2, \dots, x_k \in K_n(M)$ is a basis for the kernel f.g. free $\mathbb{Z}[\pi_1(X)]$-module then

$$s(x_j) \;=\; \sum_{j'=1}^{k} s_{jj'} x_{j'} \in K_n(M)$$

with

$$\begin{aligned} s_{jj'} &= \text{ linking number}(ix_j(S^n) \cup x_{j'}(S^n) \subset W) \\ &= \text{ intersection number}(ix_j(S^n) \cap \delta x_{j'}(D^{n+1}) \subset W) \in \mathbb{Z}[\pi_1(X)] \ . \end{aligned}$$

The split form $(K_n(M), \psi)$ is thus a (non-simply connected) Seifert form.
(ii) Suppose given an n-connected $2n$-dimensional normal map $(f,b) : (M, \partial M) \to (X, \partial X)$, with kernel $(-1)^n$-quadratic form $(K_n(M), \lambda, \mu)$ over $\mathbb{Z}[\pi_1(X)]$. A choice of split form ψ for (λ, μ) can be realized by an $(n+1)$-connected $(2n+2)$-dimensional normal map

$$(g,c) \;:\; (L, \partial L) \to (X \times D^2, X \times S^1 \cup \partial X \times D^2)$$

which is a $\mathbb{Z}[\pi_1(X)]$-homology equivalence with

$$\begin{aligned} (f,b) \;&=\; (g,c)| \;:\; (M, \partial M) \;=\; g^{-1}((X, \partial X) \times \{0\}) \to (X, \partial X) \ , \\ H_{*+1}(\widetilde{L}, \widetilde{M}) \;&=\; K_*(M) \ (= \ 0 \text{ for } * \neq n) \end{aligned}$$

as follows. The inclusion $\partial M \hookrightarrow \partial L$ is a codimension 2 embedding with Seifert surface $M \hookrightarrow \partial L$ and Seifert form $(K_n(M), \psi)$ as in the relative version of (i), with

$$(h,d) \;=\; (g,c)| \;:\; W \;=\; \partial L \to X \times S^1 \cup \partial X \times D^2 \ .$$

The choice of split form ψ for (λ, μ) determines a sequence of surgeries on the n-connected $(2n+1)$-dimensional normal map

$$(f,b) \times 1_{[0,1]} \;:\; M \times ([0,1]; \{0\}, \{1\}) \to X \times ([0,1]; \{0\}, \{1\})$$

killing the (stably) f. g. free $\mathbb{Z}[\pi_1(X)]$-module

$$K_n(M \times [0,1]) \;=\; K_n(M) \ ,$$

obtaining a rel ∂ normal bordant map

$$(f_N, b_N) \;:\; (N; M, M') \to X \times ([0,1]; \{0\}, \{1\})$$

with $K_i(N) = 0$ for $i \neq n$. The $\mathbb{Z}[\pi_1(X)]$-module morphisms induced by the inclusions $i : M \hookrightarrow N$, $i' : M' \hookrightarrow N$

$$i_* \;:\; K_n(M) \to K_n(N) \quad , \quad i'_* \;:\; K_n(M) \;=\; K_n(M') \to K_n(N)$$

are such that $i_* - i'_* : K_n(M) \to K_n(N)$ is a $\mathbb{Z}[\pi_1(X)]$-module isomorphism, with

$$\psi \;:\; K_n(M) \xrightarrow{(i_* - i'_*)^{-1} i_*} K_n(M) \xrightarrow{\text{adjoint}(\lambda)} K_n(M)^* \ .$$

Thus it is possible to identify

$$i_* \;=\; \psi \;:\; K_n(M) \to K_n(N) \;\cong\; K_n(M)^* \;,$$
$$i'_* \;=\; (-1)^{n+1}\psi^* \;:\; K_n(M) \;=\; K_n(M') \to K_n(N) \;\cong\; K_n(M)^*$$

with

$$i_* - i'_* \;=\; \psi + (-1)^n\psi^* \;=\; \text{adjoint}(\lambda) \;:\; K_n(M) \xrightarrow{\cong} K_n(M)^* \;.$$

The $(2n+1)$-dimensional manifold with boundary defined by

$$(V, \partial V) \;=\; (N/(M = M'), \partial M \times S^1)$$

is equipped with a normal map

$$(f_V, b_V) \;:\; (V, \partial V) \to (X \times S^1, \partial X \times S^1)$$

which is an n-connected $\mathbb{Z}[\pi_1(X)]$-homology equivalence, with $K_j(V) = 0$ for $j \neq n+1$ and

$$K_{n+1}(V) \;=\; \text{coker}(z\psi + (-1)^n\psi^* : K_n(M)[z, z^{-1}] \to K_n(M)^*[z, z^{-1}])$$

identifying

$$\mathbb{Z}[\pi_1(X \times S^1)] \;=\; \mathbb{Z}[\pi_1(X)][z, z^{-1}] \;\;(\overline{z} = z^{-1}) \;.$$

The trace of the surgeries on $(f, b) \times 1_{[0,1]}$ gives an extension of (f_V, b_V) to an $(n+1)$-connected $(2n+2)$-dimensional normal bordism

$$(f_U, b_U) \;:\; (U; V, M \times S^1) \to X \times S^1 \times ([0,1]; \{0\}, \{1\})$$

with $K_i(U) = 0$ for $i \neq n+1$ and (singular) kernel $(-1)^{n+1}$-quadratic form over $\mathbb{Z}[\pi_1(X)][z, z^{-1}]$

$$\begin{aligned}&(K_{n+1}(U), \lambda_U, \mu_U)\\ &\;=\; (K_n(M)[z, z^{-1}], (1-z)\psi + (-1)^{n+1}(1-z^{-1})\psi^*, (1-z)\psi) \;.\end{aligned}$$

The $(2n+2)$-dimensional manifold with boundary defined by

$$(W, \partial W) \;=\; (M \times D^2 \cup U, \partial M \times D^2 \cup V)$$

is such that (f, b) extends to an $(n+1)$-connected normal map

$$(g, c) \;=\; (f, b) \times 1_{D^2} \cup (f_U, b_U) \;:\; (W, \partial W) \to (X \times D^2, \partial X \times D^2 \cup X \times S^1)$$

which is a $\mathbb{Z}[\pi_1(X)]$-homology equivalence, with $H_{n+1}(\widetilde{W}, \widetilde{M}) = K_n(M)$. See Example 27.9 of Ranicki [24] for further details (noting that the split form ψ here corresponds to the asymmetric form λ there).

(iii) Given a simple knot $\ell : \Sigma^{2n-1} \hookrightarrow S^{2n+1}$ and a simple Seifert surface $M^{2n} \subset S^{2n+1}$ there is defined an n-connected normal map

$$(f, b) \;=\; \text{inclusion} \;:\; (M^{2n}, \partial M) \to (X, \partial X) \;=\; (D^{2n+2}, \ell(\Sigma^{2n-1}))$$

with a Seifert form ψ on $K_n(M) = H_n(M)$, as in 4.4. For $n \geq 2$ the surgery construction of (i) applied to (f, b), ψ recovers the knot

$$\ell \;:\; \Sigma^{2n-1} \;=\; \partial M \hookrightarrow \partial W \;=\; S^{2n+1}$$

with $M^{2n} \subset W = D^{2n+2}$ the Seifert surface pushed into the interior of D^{2n+2}. The knot complement

$$(V^{2n+1}, \partial V) = (\mathrm{cl.}\big(S^{2n+1}\backslash(\ell(\Sigma^{2n-1}) \times D^2)\big), \ell(\Sigma^{2n-1}) \times S^1)$$

is such that there is defined an n-connected $(2n+1)$-dimensional normal map

$$(f_V, b_V) : (V, \partial V) \to (X, \partial X) \times S^1$$

which is a homology equivalence, with

$$(f_V, b_V)| = (f, b) : (M, \partial M) = (f_V)^{-1}((X, \partial X) \times \{*\}) \to (X, \partial X) .$$

Cutting (f_V, b_V) along (f, b) results in a normal map as in (i)

$$(f_N, b_N) : (N^{2n+1}; M^{2n}, M'^{2n}) \to X \times ([0,1]; \{0\}, \{1\}) . \qquad \square$$

§5. Surgery on forms

§5 develops algebraic surgery on forms. The effect of a geometric surgery on an n-connected $2n$-dimensional normal map is an algebraic surgery on the kernel $(-1)^n$-quadratic form. Moreover, geometric surgery is possible if and only if algebraic surgery is possible.

Given an ϵ-quadratic form (K, λ, μ) over Λ it is possible to kill an element $x \in K$ by algebraic surgery if and only if $\mu(x) = 0 \in Q_\epsilon(\Lambda)$ and x generates a direct summand $\langle x \rangle = \Lambda x \subset K$. The effect of the surgery is the ϵ-quadratic form (K', λ', μ') defined on the subquotient $K' = \langle x \rangle^\perp / \langle x \rangle$ of K, with $\langle x \rangle^\perp = \{y \in K \,|\, \lambda(x, y) = 0 \in \Lambda\}$.

Definition 5.1 (i) Given an ϵ-symmetric form (K, λ) and a submodule $L \subseteq K$ define the *orthogonal* submodule

$$\begin{aligned} L^\perp &= \{x \in K \,|\, \lambda(x, y) = 0 \in \Lambda \text{ for all } y \in L\} \\ &= \ker(i^*\lambda : K \to L^*) \end{aligned}$$

with $i : L \to K$ the inclusion. If (K, λ) is nonsingular and L is a direct summand of K then so is $L^\perp$.
(ii) A *sublagrangian* of a nonsingular ϵ-quadratic form (K, λ, μ) over Λ is a direct summand $L \subseteq K$ such that

$$\mu(L) = \{0\} \subseteq Q_\epsilon(\Lambda) ,$$

and

$$\lambda(L)(L) = \{0\} \ , \ L \subseteq L^\perp .$$

(iii) A *lagrangian* of (K, λ, μ) is a sublagrangian L such that $L^\perp = L$. $\square$

The main result of §5 is that the inclusion of a sublagrangian is a morphism of ϵ-quadratic forms

$$i \;:\; (L,0,0) \to (K,\lambda,\mu)$$

which extends to an isomorphism

$$f \;:\; H_\epsilon(L) \oplus (L^\perp/L,[\lambda],[\mu]) \xrightarrow{\cong} (K,\lambda,\mu)$$

with $H_\epsilon(L)$ the hyperbolic ϵ-quadratic form (2.14).

Example 5.2 Let $(f,b) : M^{2n} \to X$ be an n-connected $2n$-dimensional normal map with kernel $(-1)^n$-quadratic form $(K_n(M),\lambda,\mu)$ over $\mathbb{Z}[\pi_1(X)]$, and $n \geq 3$. An element $x \in K_n(M)$ generates a sublagrangian $L = \langle x\rangle \subset K_n(M)$ if and only if it can be killed by surgery on $S^n \times D^n \hookrightarrow M$ with trace an n-connected normal bordism

$$((g,c);(f,b),(f',b')) \;:\; (W^{2n+1};M^{2n},M'^{2n}) \to X \times ([0,1];\{0\},\{1\})$$

such that $K_{n+1}(W,M') = 0$. The kernel form of the effect of such a surgery

$$(f',b') \;:\; M' \;=\; \mathrm{cl.}(M\backslash S^n \times D^n) \cup D^{n+1} \times S^{n-1} \to X$$

is given by

$$(K_n(M'),\lambda',\mu') \;=\; (L^\perp/L,[\lambda],[\mu]) \;.$$

There exists an n-connected normal bordism (g,c) of (f,b) to a homotopy equivalence (f',b') with $K_{n+1}(W,M') = 0$ if and only if $(K_n(M),\lambda,\mu)$ admits a lagrangian. □

Remark 5.3 There are other terminologies. In the classical theory of quadratic forms over fields a lagrangian is a "maximal isotropic subspace". Wall called hyperbolic forms "kernels" and the lagrangians "subkernels". Novikov called hyperbolic forms "hamiltonian", and introduced the name "lagrangian", because of the analogy with the hamiltonian formulation of physics. □

Example 5.4 An n-connected $(2n+1)$-dimensional normal bordism

$$((g,c);(f,b),(f',b')) \;:\; (W^{2n+1};M^{2n},M'^{2n}) \to X \times ([0,1];\{0\},\{1\})$$

with $K_{n+1}(W,M') = 0$ determines a sublagrangian

$$L \;=\; \mathrm{im}(K_{n+1}(W,M) \to K_n(M)) \subset K_n(M)$$

of the kernel $(-1)^n$-quadratic form (K,λ,μ) of (f,b), with $K = K_n(M)$. The sublagrangian L is a lagrangian if and only if (f',b') is a homotopy equivalence. W has a handle decomposition on M of the type

$$W \;=\; M \times I \cup \bigcup_k (n+1)\text{-handles } D^{n+1} \times D^n \;,$$

and $L \cong K_{n+1}(W,M) \cong \mathbb{Z}[\pi_1(X)]^k$ is a f. g. free $\mathbb{Z}[\pi_1(X)]$-module with rank the number k of $(n+1)$-handles. The exact sequences of stably f. g. free $\mathbb{Z}[\pi_1(X)]$-modules

$$0 \to K_{n+1}(W,M) \to K_n(M) \to K_n(W) \to 0 \ ,$$
$$0 \to K_n(M') \to K_n(W) \to K_n(W,M') \to 0$$

are isomorphic to

$$0 \to \ L \xrightarrow{\ i\ } K \to K/L \to 0 \ ,$$
$$0 \to L^{\perp}/L \to K/L \xrightarrow{[i^*\lambda]} L^* \to 0 \ . \qquad \square$$

Definition 5.5 (i) A *sublagrangian* of an ϵ-nonsingular split form (K,ψ) is an ϵ-quadratic morphism of split forms

$$(i,\theta) \ : \ (L,0) \to (K,\psi)$$

with $i : L \to K$ a split injection.
(ii) A *lagrangian* of (K,ψ) is a sublagrangian such that the sequence

$$0 \to L \xrightarrow{\ i\ } K \xrightarrow{i^*(\psi+\epsilon\psi^*)} L^* \to 0$$

is exact. $\square$

An ϵ-nonsingular split form (K,ψ) admits a (sub)lagrangian if and only if the associated ϵ-quadratic form (K,λ,μ) admits a (sub)lagrangian. (Sub)lagrangians in split ϵ-quadratic forms are thus (sub)lagrangians in ϵ-quadratic forms with the $(-\epsilon)$-quadratic structure θ, which (following Novikov) is sometimes called the "hessian" form.

Definition 5.6 The ϵ-nonsingular *hyperbolic* split form $H_\epsilon(L)$ is given for any f. g. projective Λ-module L by

$$H_\epsilon(L) \ = \ (L \oplus L^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ : \ L \oplus L^* \to (L\oplus L^*)^* \ = \ L^* \oplus L) \ ,$$

with lagrangian $(i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, 0) : (L,0) \to H_\epsilon(L)$. $\square$

Theorem 5.7 *An ϵ-nonsingular split form (K,ψ) admits a lagrangian if and only if it is ϵ-quadratic isomorphic isomorphic to the hyperbolic form $H_\epsilon(L)$. Moreover, the inclusion $(i,\theta) : (L,0) \to (K,\psi)$ of a lagrangian extends to an ϵ-quadratic isomorphism of split forms $(f,\chi) : H_\epsilon(L) \cong (K,\psi)$.*
Proof: An isomorphism of forms sends lagrangians to lagrangians, so any form isomorphic to a hyperbolic has at least one lagrangian. Conversely suppose that (K,ψ) has a lagrangian $(i,\theta) : (L,0) \to (K,\psi)$. An

extension of (i,θ) to an ϵ-quadratic isomorphism $(f,\chi) : H_\epsilon(L) \cong (K,\psi)$ determines a lagrangian $f(L^*) \subset K$ complementary to L. Construct an isomorphism f by choosing a complementary lagrangian to L in (K,ψ). Let $i \in \mathrm{Hom}_\Lambda(L,K)$ be the inclusion, and choose a splitting $j' \in \mathrm{Hom}_\Lambda(L^*,K)$ of $i^*(\psi+\epsilon\psi^*) \in \mathrm{Hom}_\Lambda(K,L^*)$, so that

$$i^*(\psi+\epsilon\psi^*)j' \;=\; 1 \in \mathrm{Hom}_\Lambda(L^*,L^*)\ .$$

In general, $j' : L^* \to K$ is not the inclusion of a lagrangian, with $j'^*\psi j' \neq 0 \in Q_\epsilon(L^*)$. Given any $k \in \mathrm{Hom}_\Lambda(L^*,L)$ there is defined another splitting

$$j \;=\; j'+ik \;:\; L^* \to K$$

such that

$$\begin{aligned} j^*\psi j \;&=\; j'^*\psi j' + k^*i^*\psi ik + k^*i^*\psi j' + j'^*\psi ik \\ &=\; j'^*\psi j' + k \in Q_\epsilon(L^*)\ . \end{aligned}$$

Choosing a representative $\psi \in \mathrm{Hom}_\Lambda(K,K^*)$ of $\psi \in Q_\epsilon(K)$ and setting

$$k \;=\; -j'^*\psi j' \;:\; L^* \to L^*$$

there is obtained a splitting $j : L^* \to K$ which is the inclusion of a lagrangian

$$(j,\nu) \;:\; (L^*,0) \to (K,\psi)\ .$$

The isomorphism of ϵ-quadratic forms

$$(i\ j) \;=\; \begin{pmatrix} \theta & 0 \\ j^*\psi i & \psi \end{pmatrix} \;:\; H_\epsilon(L) \xrightarrow{\cong} (K,\psi)$$

is an ϵ-quadratic isomorphism of split forms. □

Remark 5.8 Theorem 5.7 is a generalization of Witt's theorem on the extension to isomorphism of an isometry of quadratic forms over fields. The procedure for modifying the choice of complement to be a lagrangian is a generalization of the Gram-Schmidt method of constructing orthonormal bases in an inner product space. Ignoring the split structure 5.7 shows that a nonsingular ϵ-quadratic form admits a lagrangian (in the sense of 5.1 (iii)) if and only if it is isomorphic to a hyperbolic form. □

Corollary 5.9 *For any ϵ-nonsingular split form (K,ψ) the diagonal inclusion*

$$\Delta \;:\; K \to K\oplus K\ ;\ x \longmapsto (x,x)$$

extends to an ϵ-quadratic isomorphism of split forms

$$H_\epsilon(K) \xrightarrow{\cong} (K,\psi)\oplus(K,-\psi)\ .$$

Proof: Apply 5.7 to the inclusion of the lagrangian

$$(\Delta,0) \;:\; (K,0) \to (K\oplus K,\psi\oplus-\psi)\ .$$

(This result has already been used in 3.1). □

PROPOSITION 5.10 *The inclusion $(i,\theta):(L,0)\to(K,\psi)$ of a sublagrangian in an ϵ-nonsingular split form (K,ψ) extends to an isomorphism of forms*

$$(f,\chi)\ :\ H_\epsilon(L)\oplus(L^\perp/L,[\psi])\xrightarrow{\cong}(K,\psi)\ .$$

PROOF: For any direct complement L_1 to $L^\perp$ in K there is defined a Λ-module isomorphism

$$e\ :\ L_1\xrightarrow{\cong}L^*\ ;\ x\longmapsto(y\longmapsto(1+T_\epsilon)\psi(x,y))\ .$$

Define a Λ-module morphism

$$j\ :\ L^*\xrightarrow{e^{-1}}L_1\xrightarrow{\text{inclusion}}K\ .$$

The ϵ-nonsingular split form defined by

$$(H,\phi)\ =\ (L\oplus L^*,\begin{pmatrix}0&1\\0&j^*\psi j\end{pmatrix})$$

has lagrangian L, so that it is isomorphic to the hyperbolic form $H_\epsilon(L)$ by 5.7. Also, there is defined an ϵ-quadratic morphism of split forms

$$(g\ =\ (i\ j)\ ,\begin{pmatrix}\theta&i^*\psi j\\0&0\end{pmatrix})\ :\ (H,\phi)\to(K,\psi)$$

with $g:H\to K$ an injection split by

$$h\ =\ ((1+T_\epsilon)\phi)^{-1}g^*(1+T_\epsilon)\psi\ :\ K\to H\ .$$

The direct summand of K defined by

$$\begin{aligned}H^\perp\ &=\ \{x\in K\,|\,(1+T_\epsilon)\psi(x,gy)\ =\ 0\text{ for all }y\in H\}\\&=\ \ker(g^*(1+T_\epsilon)\psi:K\to H^*)\ =\ \ker(h:K\to H)\end{aligned}$$

is such that

$$K\ =\ g(H)\oplus H^\perp\ .$$

It follows from the factorization

$$i^*(1+T_\epsilon)\psi\ :\ K\xrightarrow{h}H\ =\ L\oplus L^*\xrightarrow{\text{projection}}L^*$$

that

$$L^\perp\ =\ \ker(i^*(1+T_\epsilon)\psi:K\to L^*)\ =\ L\oplus H^\perp\ .$$

The restriction of $\psi\in S(K)$ to $H^\perp$ defines an ϵ-nonsingular split form $(H^\perp,\phi^\perp)$. The injection g and the inclusion $g^\perp:H^\perp\to K$ are the components of a Λ-module isomorphism

$$f\ =\ (g\ g^\perp)\ :\ H\oplus H^\perp\to K$$

which defines an ϵ-quadratic isomorphism of split forms

$$(f,\chi)\ :\ (H,\phi)\oplus(H^\perp,\phi^\perp)\xrightarrow{\cong}(K,\psi)$$

with

$$(H^{\perp}, \phi^{\perp}) \cong (L^{\perp}/L, [\psi]) . \qquad \square$$

EXAMPLE 5.11 An n-connected $(2n+1)$-dimensional normal bordism

$$((g,c);(f,b),(f',b')) : (W^{2n+1}; M^{2n}, M'^{2n}) \to X \times ([0,1];\{0\},\{1\})$$

is such that W has a handle decomposition on M of the type

$$W = M \times I \cup \bigcup_k n\text{-handles } D^n \times D^{n+1} \cup \bigcup_{k'} (n+1)\text{-handles } D^{n+1} \times D^n .$$

Let

$$(W; M, M') = (W'; M, M'') \cup_{M''} (W''; M'', M')$$

with

$$\begin{aligned} W' &= M \times [0,1] \cup \bigcup_k n\text{-handles } D^n \times D^{n+1} , \\ M'' &= \mathrm{cl.}(\partial W' \backslash M) , \\ W'' &= M'' \times [0,1] \cup \bigcup_{k'} (n+1)\text{-handles } D^{n+1} \times D^n . \end{aligned}$$

The restriction of (g,c) to M'' is an n-connected $2n$-dimensional normal map

$$(f'', b'') : M'' \cong M\#(\#_k S^n \times S^n) \cong M'\#(\#_{k'} S^n \times S^n) \to X$$

with kernel $(-1)^n$-quadratic form

$$\begin{aligned} (K_n(M''), \lambda'', \mu'') &\cong (K_n(M), \lambda, \mu) \oplus H_{(-1)^n}(\mathbb{Z}[\pi_1(X)]^k) \\ &\cong (K_n(M'), \lambda', \mu') \oplus H_{(-1)^n}(\mathbb{Z}[\pi_1(X)]^{k'}) . \end{aligned}$$

Thus $(K_n(M''), \lambda'', \mu'')$ has sublagrangians

$$\begin{aligned} L &= \mathrm{im}(K_{n+1}(W', M'') \to K_n(M'')) \cong \mathbb{Z}[\pi_1(X)]^k , \\ L' &= \mathrm{im}(K_{n+1}(W'', M'') \to K_n(M'')) \cong \mathbb{Z}[\pi_1(X)]^{k'} \end{aligned}$$

such that

$$\begin{aligned} (L^{\perp}/L, [\lambda''], [\mu'']) &\cong (K_n(M), \lambda, \mu) , \\ (L'^{\perp}/L', [\lambda'']', [\mu'']') &\cong (K_n(M'), \lambda', \mu') . \end{aligned}$$

Note that L is a lagrangian if and only if $(f,b) : M \to X$ is a homotopy equivalence. Similarly for L' and $(f',b') : M' \to X$. $\square$

§6. Short odd complexes

A "$(2n+1)$-complex" is the algebraic structure best suited to describing the surgery obstruction of an n-connected $(2n+1)$-dimensional normal map. In essence it is a 1-dimensional chain complex with $(-1)^n$-quadratic Poincaré duality.

As before, let Λ be a ring with involution.

DEFINITION 6.1 A $(2n+1)$-*complex over* Λ (C,ψ) is a f. g. free Λ-module chain complex of the type

$$C \ : \ \dots \to 0 \to \ C_{n+1} \ \xrightarrow{d} \ C_n \to 0 \to \dots$$

together with two Λ-module morphisms

$$\psi_0 \ : \ C^n \ = \ (C_n)^* \to C_{n+1} \ , \ \psi_1 \ : \ C^n \to C_n$$

such that

$$d\psi_0 + \psi_1 + (-1)^{n+1}\psi_1^* \ = \ 0 \ : \ C^n \to C_n \ ,$$

and such that the chain map

$$(1+T)\psi_0 \ : \ C^{2n+1-*} \to C$$

defined by

$$\begin{aligned} d_{C^{2n+1-*}} \ &= \ (-1)^{n+1}d^* \ : \\ &\quad (C^{2n+1-*})_{n+1} \ = \ C^n \to (C^{2n+1-*})_n \ = \ C^{n+1} \ , \\ (1+T)\psi_0 \ &= \ \begin{cases} \psi_0 \ : \ (C^{2n+1-*})_{n+1} \ = \ C^n \to C_{n+1} \\ \psi_0^* \ : \ (C^{2n+1-*})_n \ = \ C^{n+1} \to C_n \ , \end{cases} \\ (C^{2n+1-*})_r \ &= \ C^{2n+1-r} \ = \ 0 \text{ for } r \neq n, n+1 \end{aligned}$$

is a chain equivalence

$$\begin{array}{ccccccccccc} C^{2n+1-*} : \dots & \longrightarrow & 0 & \longrightarrow & C^n & \xrightarrow{(-1)^{n+1}d^*} & C^{n+1} & \longrightarrow & 0 & \longrightarrow & \cdots \\ \Big\downarrow (1+T)\psi_0 & & \Big\downarrow & & \Big\downarrow \psi_0 & & \Big\downarrow \psi_0^* & & \Big\downarrow & & \\ C \ : \dots & \longrightarrow & 0 & \longrightarrow & C_{n+1} & \xrightarrow{d} & C_n & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

□

REMARK 6.2 A $(2n+1)$-complex is essentially the inclusion of a lagrangian in a hyperbolic split $(-1)^n$-quadratic form

$$(\begin{pmatrix} \psi_0 \\ d^* \end{pmatrix}, -\psi_1) \ : \ (C^n, 0) \to H_{(-1)^n}(C_{n+1}) \ .$$

The chain map $(1+T)\psi_0 : C^{2n+1-*} \to C$ is a chain equivalence if and only if the algebraic mapping cone

$$0 \to C^n \xrightarrow{\begin{pmatrix} \psi_0 \\ d^* \end{pmatrix}} C_{n+1} \oplus C^{n+1} \xrightarrow{(d \quad (-1)^n \psi_0^*)} C_n \to 0$$

is contractible, which is just the lagrangian condition. The triple

$$(\text{ form ; lagrangian , lagrangian }) \;=\; (H_{(-1)^n}(C_{n+1}); C_{n+1}, \mathrm{im}\begin{pmatrix} \psi_0 \\ d^* \end{pmatrix})$$

is an example of a "$(-1)^n$-quadratic formation". Formations will be studied in greater detail in §9 below. □

Example 6.3 Define a *presentation* of an n-connected $(2n+1)$-dimensional normal map $(f,b) : M^{2n+1} \to X$ to be a normal bordism

$$((g,c);(f,b),(f',b')) \;:\; (W^{2n+2}; M^{2n+1}, M'^{2n+1}) \to X \times ([0,1];\{0\},\{1\})$$

such that $W \to X \times [0,1]$ is n-connected, with

$$K_r(W) \;=\; 0 \text{ for } r \neq n+1 \;.$$

Then $K_{n+1}(W)$ a f. g. free $\mathbb{Z}[\pi_1(X)]$-module and W has a handle decomposition on M of the type

$$W \;=\; M \times I \cup \bigcup_k (n+1)\text{-handles } D^{n+1} \times D^{n+1} \;,$$

and $K_{n+1}(W,M) \cong \mathbb{Z}[\pi_1(X)]^k$ is a f. g. free $\mathbb{Z}[\pi_1(X)]$-module with rank the number k of $(n+1)$-handles. Thus $(W;M,M')$ is the trace of surgeries on k disjoint embeddings $S^n \times D^{n+1} \hookrightarrow M^{2n+1}$ with null-homotopy in X representing a set of $\mathbb{Z}[\pi_1(X)]$-module generators of $K_n(M)$. For every n-connected $(2n+1)$-dimensional normal map $(f,b) : M^{2n+1} \to X$ the kernel $\mathbb{Z}[\pi_1(X)]$-module $K_n(M)$ is f. g., so that there exists a presentation $(g,c) : (W;M,M') \to X \times ([0,1];\{0\},\{1\})$. Poincaré duality and the universal coefficient theorem give natural identifications of f. g. free $\mathbb{Z}[\pi_1(X)]$-modules

$$K_{n+1}(W) \;=\; K^{n+1}(W,\partial W) \;=\; K_{n+1}(W,\partial W)^* \;\; (\partial W = M \cup M') \;,$$

$$K_{n+1}(W,M) \;=\; K^{n+1}(W,M') \;=\; K_{n+1}(W,M')^* \;.$$

The presentation determines a $(2n+1)$-complex (C,ψ) such that

$$H_*(C) \;=\; K_*(M) \;,$$

with

$$d \;=\; (\text{inclusion})_* \;:\; C_{n+1} \;=\; K_{n+1}(W,M') \;\to C_n \;=\; K_{n+1}(W,\partial W) \;,$$

$$\psi_0 \;=\; (\text{inclusion})_* \;:\; C^n \;=\; K_{n+1}(W) \to \; C_{n+1} \;=\; K_{n+1}(W,M') \;.$$

The hessian $(C^n, -\psi_1 \in Q_{(-1)^{n+1}}(C^n))$ is the geometric self-intersection $(-1)^{n+1}$-quadratic form on the kernel $C^n = K_{n+1}(W)$ of the normal map

$W^{2n+2} \to X \times [0,1]$, such that

$$\begin{aligned} & -(\psi_1 + (-1)^{n+1}\psi_1^*) \;=\; d\psi_0 \;=\; \text{inclusion}_* \;: \\ & \qquad C^n \;=\; K_{n+1}(W) \to C_n \;=\; K_{n+1}(W, \partial W) \;=\; K_{n+1}(W)^* \;. \end{aligned}$$

The chain equivalence $(1+T)\psi_0 : C^{2n+1-*} \to C$ induces the Poincaré duality isomorphisms

$$[M] \cap - \;:\; H^{2n+1-*}(C) \;=\; K^{2n+1-*}(M) \xrightarrow{\cong} H_*(C) \;=\; K_*(M) \;. \qquad \square$$

Remark 6.4 The $(2n+1)$-complex (C, ψ) of 6.3 can also be obtained by working inside M, assuming that X has a single $(2n+1)$-cell

$$X \;=\; X_0 \cup D^{2n+1}$$

(as is possible by the Poincaré disc theorem of Wall [28]) so that there is defined a degree 1 map

$$\text{collapse} \;:\; X \to X/X_0 \;=\; S^{2n+1} \;.$$

Let $U \subset M^{2n+1}$ be the disjoint union of the k embeddings $S^n \times D^{n+1} \hookrightarrow M$ with null-homotopies in X, so that (f, b) has a Heegaard splitting as a union of normal maps

$$\begin{aligned} & (f, b) \;=\; (e, a) \cup (f_0, b_0) \;: \\ & M \;=\; (U, \partial U) \cup (M_0, \partial M_0) \to X \;=\; (D^{2n+1}, S^{2n}) \cup (X_0, \partial X_0) \end{aligned}$$

with the inclusion (6.2) of the lagrangian

$$\begin{pmatrix} \psi_0 \\ d^* \end{pmatrix} \;:\; C^n \to C_{n+1} \oplus C^{n+1}$$

in the hyperbolic $(-1)^n$-quadratic form $H_{(-1)^n}(C_{n+1})$ given by

$$\text{inclusion}_* \;:\; K_{n+1}(M_0, \partial U) \to K_n(\partial U) \;=\; K_{n+1}(U, \partial U) \oplus K_n(U) \;.$$

Wall obtained the surgery obstruction of (f, b) using an extension (cf. 5.7) of this inclusion to an automorphism

$$\alpha \;:\; H_{(-1)^n}(C_{n+1}) \xrightarrow{\cong} H_{(-1)^n}(C_{n+1}) \;,$$

which will be discussed further in §10 below. The presentation of (f, b) used to obtain (C, ψ) in 6.3 is the trace of the k surgeries on $U \subset M$

$$\begin{aligned} & (g, c) \;=\; (e_1, a_1) \cup (f_0, b_0) \times \text{id} \;: \\ & (W; M, M') \;=\; (V; U, U') \cup M_0 \times ([0,1]; \{0\}, \{1\}) \to X \times ([0,1]; \{0\}, \{1\}) \end{aligned}$$

with

$$(V; U, U') \;=\; \bigcup_k (D^{n+1} \times D^{n+1}; S^n \times D^{n+1}, D^{n+1} \times S^n) \;. \qquad \square$$

EXAMPLE 6.5 There is also a relative version of 6.3. A presentation of an n-connected normal map $(f,b) : M^{2n} \to X$ from a $(2n+1)$-dimensional manifold with boundary $(M, \partial M)$ to a geometric Poincaré pair $(X, \partial X)$ with $\partial f = f| : \partial M \to \partial X$ a homotopy equivalence is a normal map of triads

$$\begin{aligned}(W^{2n+2}; M^{2n+1}, M'^{2n+1}; \partial M \times [0,1])\\ \to (X \times [0,1]; X \times \{0\}, X \times \{1\}; \partial X \times [0,1])\end{aligned}$$

such that $W \to X \times [0,1]$ is n-connected. Again, the presentation determines a $(2n+1)$-complex (C, ψ) over $\mathbb{Z}[\pi_1(X)]$ with

$$C_n = K_{n+1}(W, \partial W) \ , \ C_{n+1} = K_{n+1}(W, M') \ , \ H_*(C) = K_*(M) \ . \ \square$$

REMARK 6.6 (Realization of odd-dimensional surgery obstructions, Wall [29, 6.5]) The theorem of [29] realizing automorphisms of hyperbolic forms as odd-dimensional surgery obstructions has the following interpretation in terms of complexes. Let (C, ψ) be a $(2n+1)$-complex over $\mathbb{Z}[\pi]$, with π a finitely presented group. Let $n \geq 2$, so that there exists a $2n$-dimensional manifold X^{2n} with $\pi_1(X) = \pi$. For any such $n \geq 2$, X there exists an n-connected $(2n+1)$-dimensional normal map

$$(f,b) \ : \ (M^{2n+1}; \partial_- M, \partial_+ M) \to X^{2n} \times ([0,1]; \{0\}, \{1\})$$

with $\partial_- M = X \to X$ the identity and $\partial_+ M \to X$ a homotopy equivalence, and with a presentation with respect to which (f,b) has kernel $(2n+1)$-complex (C, ψ). Such a normal map is constructed from the identity $X \to X$ in two stages. First, choose a basis $\{b_1, b_2, \ldots, b_k\}$ for C_{n+1}, and perform surgeries on k disjoint trivial embeddings $S^{n-1} \times D^{n+1} \hookrightarrow X^{2n}$ with trace

$$\begin{aligned}(U; X, \partial_+ U) \ = \ (X \times [0,1] \cup \bigcup_k D^n \times D^{n+1}; X \times \{0\}, X \#\#_k S^n \times S^n)\\ \to X \times ([0,1/2]; \{0\}, \{1/2\}) \ .\end{aligned}$$

The n-connected $2n$-dimensional normal map $\partial_+ U \to X \times \{1/2\}$ has kernel $(-1)^n$-quadratic form

$$(K_n(\partial_+ U), \lambda, \mu) \ = \ H_{(-1)^n}(\mathbb{Z}[\pi]^k) \ = \ H_{(-1)^n}(C_{n+1}) \ .$$

Second, choose a basis $\{c_1, c_2, \ldots, c_k\}$ for C^n and realize the inclusion of the lagrangian in $H_{(-1)^n}(C_{n+1})$ by surgeries on k disjoint embeddings $S^n \times D^n \hookrightarrow \partial_+ U$ with trace

$$(M_0; \partial_+ U, \partial_+ M) \to X \times ([0,1]; \{0\}, \{1\})$$

such that

$$\begin{aligned}\begin{pmatrix} \psi_0 \\ d^* \end{pmatrix} \ = \ \partial \ : \ C^n \ = \ K_{n+1}(M_0, \partial_+ U)\\ \to C_{n+1} \oplus C^{n+1} \ = \ K_{n+1}(U, \partial_+ U) \oplus K_n(U) \ = \ K_n(\partial_+ U) \ .\end{aligned}$$

The required $(2n+1)$-dimensional normal map realizing (C,ψ) is the union

$$(M;\partial_-M,\partial_+M) \;=\; (U;X,\partial_+U)\cup(M_0;\partial_+U,\partial_+M) \to X\times([0,1];\{0\},\{1\}) \;.$$

The corresponding presentation is the trace of surgeries on k disjoint embeddings $S^n\times D^{n+1}\hookrightarrow U\subset M^{2n+1}$. This is the terminology (and result) of Wall [29, Chapter 6]. □

The choice of presentation (6.3) for an n-connected $(2n+1)$-dimensional normal map $(f,b): M^{2n+1}\to X$ does not change the "homotopy type" of the associated $(2n+1)$-complex (C,ψ), in the following sense.

DEFINITION 6.7 (i) A *map* of $(2n+1)$-complexes over Λ

$$f \;:\; (C,\psi)\to(C',\psi')$$

is a chain map $f: C\to C'$ such that there exist Λ-module morphisms

$$\chi_0 \;:\; C'^{n+1}\to C'_{n+1} \;\;,\;\; \chi_1 \;:\; C'^{n}\to C'_{n}$$

with

$$\begin{aligned} f\psi_0f^*-\psi'_0 \;&=\; (\chi_0+(-1)^{n+1}\chi_0^*)d'^* \;:\; C'^n\to C'_{n+1} \;, \\ f\psi_1f^*-\psi'_1 \;&=\; -d'\chi_0d'^*+\chi_1+(-1)^n\chi_1^* \;:\; C'^n\to C'_n \;. \end{aligned}$$

(ii) A *homotopy equivalence* of $(2n+1)$-complexes is a map with $f: C\to C'$ a chain equivalence.
(iii) An *isomorphism* of $(2n+1)$-complexes is a map with $f: C\to C'$ an isomorphism of chain complexes. □

PROPOSITION 6.8 *Homotopy equivalence is an equivalence relation on* $(2n+1)$*-complexes.*

PROOF: For $m\geq 0$ let $E(m)$ be the contractible f. g. free Λ-module chain complex defined by

$$\begin{aligned} d_{E(m)} \;&=\; 1 \;:\; E(m)_{n+1} \;=\; \Lambda^m\to E(m)_n \;=\; \Lambda^m \;, \\ E(m)_r \;&=\; 0 \text{ for } r\neq n,n+1 \;. \end{aligned}$$

A map $f:(C,\psi)\to(C',\psi')$ is a homotopy equivalence if and only if for some $m,m'\geq 0$ there exists an isomorphism

$$f' \;:\; (C,\psi)\oplus(E(m),0) \xrightarrow{\cong} (C',\psi')\oplus(E(m'),0)$$

such that the underlying chain map f' is chain homotopic to

$$f\oplus 0 \;:\; C\oplus E(m)\to C'\oplus E(m') \;.$$

Isomorphism is an equivalence relation on $(2n+1)$-complexes, and hence so is homotopy equivalence. □

EXAMPLE 6.9 The $(2n+1)$-complexes (C,ψ), (C',ψ') associated by 6.3 to any two presentations

$$(W;M,\widehat{M})\to X\times([0,1];\{0\},\{1\})\ ,\ (W';M,\widehat{M}')\to X\times([0,1];\{0\},\{1\})$$

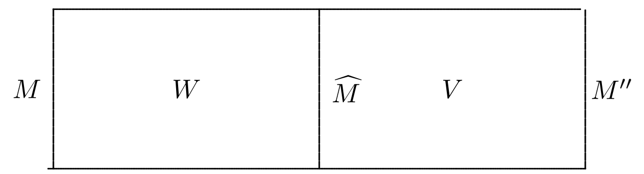
of an n-connected normal map $M^{2n+1}\to X$ are homotopy equivalent. Without loss of generality it may be assumed that W and W' are the traces of surgeries on disjoint embeddings

$$g^i\ :\ S^n\times D^{n+1}\hookrightarrow M\ ,\ g'^j\ :\ S^n\times D^{n+1}\hookrightarrow M\ ,$$

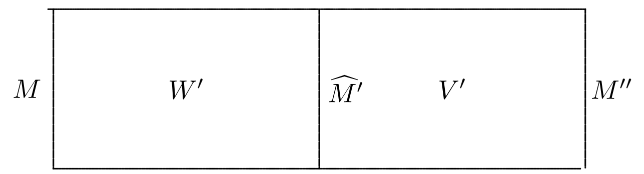
corresponding to two sets of $\mathbb{Z}[\pi_1(X)]$-module generators of $K_n(M)$. Define a presentation of $M\to X$

$$(W'';M,M'')\ =\ (W;M,\widehat{M})\cup(V;\widehat{M},M'')\ =\ (W';M,\widehat{M}')\cup(V';\widehat{M}',M'')$$

with $(V;\widehat{M},M'')$ the presentation of $\widehat{M}\to X$ defined by the trace of the surgeries on the copies $\widehat{g}'^j : S^n\times D^{n+1}\hookrightarrow\widehat{M}$ of $g'^j : S^n\times D^{n+1}\hookrightarrow M$, and $(V';\widehat{M}',M'')$ the presentation of $\widehat{M}'\to X$ defined by the trace of the surgeries on the copies $\widehat{g}^i : S^n\times D^{n+1}\hookrightarrow\widehat{M}'$ of $g^i : S^n\times D^{n+1}\hookrightarrow M$.



$$W''=W\cup_{\widehat{M}}V=W'\cup_{\widehat{M}'}V'$$



The projections $C''\to C$, $C''\to C'$ define homotopy equivalences of $(2n+1)$-complexes

$$(C'',\psi'')\to(C,\psi)\quad,\quad(C'',\psi'')\to(C',\psi')\ .\qquad\square$$

Definition 6.10 A $(2n+1)$-complex (C,ψ) over Λ is *contractible* if it is homotopy equivalent to the zero complex $(0,0)$, or equivalently if $d : C_{n+1}\to C_n$ is a Λ-module isomorphism. $\square$

Example 6.11 A $(2n+1)$-complex (C,ψ) associated to an n-connected $(2n+1)$-dimensional normal map $(f,b) : M^{2n+1}\to X$ is contractible if (and for $n\geq 2$ only if) f is a homotopy equivalence, by the theorem of J.H.C. Whitehead. The $(2n+1)$-complexes (C,ψ) associated to the various presentations of a homotopy equivalence $(f,b) : M^{2n+1}\to X$ are contractible, by 6.9. The zero complex $(0,0)$ is associated to the presentation

$$(f,b)\times \mathrm{id.}\ :\ M\times([0,1];\{0\},\{1\})\to X\times([0,1];\{0\},\{1\})\ . \qquad \square$$

§7. Complex cobordism

The cobordism of $(2n+1)$-complexes is the equivalence relation which corresponds to the normal bordism of n-connected $(2n+1)$-dimensional normal maps. The $(2n+1)$-dimensional surgery obstruction group $L_{2n+1}(\Lambda)$ will be defined in §8 below to be the cobordism group of $(2n+1)$-complexes over Λ.

Definition 7.1 A *cobordism* of $(2n+1)$-complexes (C,ψ), (C',ψ')

$$((j\ j'):C\oplus C'\to D,(\delta\psi,\psi\oplus-\psi'))$$

is a f. g. free Λ-module chain complex of the type

$$D\ :\ \ldots\to 0\to D_{n+1}\to 0\to\ldots$$

together with Λ-module morphisms

$$\begin{aligned} j\ :\ C_{n+1}\to D_{n+1}\quad ,\quad j'\ :\ C'_{n+1}\to D_{n+1}\ ,\\ \delta\psi_0\ :\ D^{n+1}\ =\ (D_{n+1})^*\to D_{n+1}\end{aligned}$$

such that the *duality* Λ-module chain map

$$(1+T)(\delta\psi_0,\psi_0\oplus-\psi'_0)\ :\ \mathcal{C}(j')^{2n+2-*}\to\mathcal{C}(j)$$

defined by

$$\begin{aligned}(1+T)(\delta\psi_0,\psi_0\oplus-\psi'_0)\ &=\ \begin{pmatrix}\delta\psi_0+(-1)^{n+1}\delta\psi_0^* & j'\psi'_0\\ \psi_0^*j^* & 0\end{pmatrix}\\ :\ \mathcal{C}(j')^{2n+1}\ &=D^{n+1}\oplus C'^n\to\mathcal{C}(j)_{n+1}\ =\ D_{n+1}\oplus C_n\end{aligned}$$

is a chain equivalence, with $\mathcal{C}(j)$, $\mathcal{C}(j')$ the algebraic mapping cones of the chain maps $j:C\to D$, $j':C'\to D$. $\square$

The duality chain map $\mathcal{C}(j')^{2n+2-*}\to\mathcal{C}(j)$ is given by

$$\begin{tikzcd}[column sep=small] \mathcal{C}(j')^{2n+2-*}:\ldots \arrow[r] \arrow[d, "(1+T)\delta\psi_0"'] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & D^{n+1}\oplus C'^n \arrow[r] \arrow[d] & C'^{n+1} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \mathcal{C}(j):\ldots \arrow[r] & 0 \arrow[r] & C_{n+1} \arrow[r] & D_{n+1}\oplus C_n \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

The condition for it to be a chain equivalence is just that the Λ-module morphism

$$\begin{pmatrix} d & 0 & \psi_0^* j^* \\ 0 & d'^* & j'^* \\ (-1)^{n+1} j & j'\psi_0' & \delta\psi_0 + (-1)^{n+1}\delta\psi_0^* \end{pmatrix}$$
$$: \ C_{n+1} \oplus C'^n \oplus D^{n+1} \to C_n \oplus C'^{n+1} \oplus D_{n+1}$$

be an isomorphism.

EXAMPLE 7.2 Suppose given two n-connected $(2n+1)$-dimensional normal maps $M^{2n+1} \to X$, $M'^{2n+1} \to X$ with presentations (6.3)

$$(W^{2n+2}; M^{2n+1}, \widehat{M}^{2n+1}) \to X \times ([0,1]; \{0\}, \{1\}) \ ,$$
$$(W'^{2n+2}; M'^{2n+1}, \widehat{M}'^{2n+1}) \to X \times ([0,1]; \{0\}, \{1\})$$

and corresponding $(2n+1)$-complexes (C, ψ), (C', ψ'). An n-connected normal bordism

$$(V^{2n+2}; M^{2n+1}, M'^{2n+1}) \to X \times ([0,1]; \{0\}, \{1\})$$

determines a cobordism $((j\ j') : C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi'))$ (again, up to some choices) from (C, ψ) to (C', ψ'). Define an n-connected normal bordism

$$(V'; \widehat{M}, \widehat{M}') \ = \ (W; \widehat{M}, M) \cup (V; M, M') \cup (W'; M', \widehat{M}')$$
$$\to X \times ([0,1]; \{0\}, \{1\}) \ .$$

The exact sequence of stably f. g. free $\mathbb{Z}[\pi_1(X)]$-modules

$$0 \to K_{n+1}(V) \ \to K_{n+1}(V', \partial V')$$
$$\to K_{n+1}(W, \partial W) \oplus K_{n+1}(W', \partial W') \to 0$$

splits. Choosing any splitting $K_{n+1}(V', \partial V') \to K_{n+1}(V)$ define j, j' by

$$(j\ j') \ : \ C_{n+1} \oplus C'_{n+1} \ = \ K_{n+1}(W, \widehat{M}) \oplus K_{n+1}(W', \widehat{M}')$$
$$\xrightarrow{\text{incl}_* \oplus \text{incl}_*} K_{n+1}(V', \partial V') \ \to K_{n+1}(V) \ = \ D_{n+1} \ .$$

Geometric intersection numbers provide a $(-1)^{n+1}$-quadratic form $(D^{n+1}, \delta\psi_0)$ over $\mathbb{Z}[\pi_1(X)]$ such that the duality chain map $\mathcal{C}(j')^{2n+2-*} \to \mathcal{C}(j)$ is a chain equivalence inducing the Poincaré duality isomorphisms

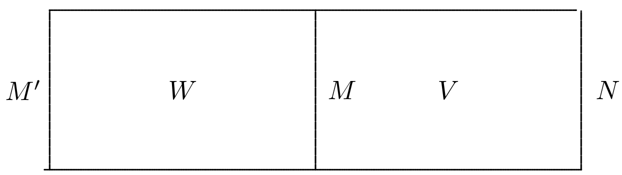
$$[V] \cap - \ : \ H^{2n+2-*}(j') \ = \ K^{2n+2-*}(V, M') \xrightarrow{\cong} H_*(j) \ = \ K_*(V, M) \ . \ \square$$

DEFINITION 7.3 A *null-cobordism* of a $(2n+1)$-complex (C, ψ) is a cobordism $(j : C \to D, (\delta\psi, \psi))$ to $(0, 0)$. $\square$

EXAMPLE 7.4 Let $(W^{2n+2}; M'^{2n+1}, M^{2n+1}) \to X \times ([0,1]; \{0\}, \{1\})$ be a presentation of an n-connected $(2n+1)$-dimensional normal map $M \to X$, with $(2n+1)$-complex (C,ψ). For $n \geq 2$ there is a one-one correspondence between n-connected normal bordisms of $M \to X$

$$(V^{2n+2}; M^{2n+1}, N^{2n+1}) \to X \times ([0,1]; \{0\}, \{1\})$$

to homotopy equivalences $N \to X$ and null-cobordisms $(j : C \to D, (\delta\psi, \psi))$. (Every normal bordism of n-connected $(2n+1)$-dimensional normal maps can be made n-connected by surgery below the middle dimension on the interior.)



$$V' = W \cup_M V$$

Any such $(V; M, N) \to X \times ([0,1]; \{0\}, \{1\})$ determines by 7.2 a null-cobordism $(j : C \to D, (\delta\psi, \psi))$ of (C, ψ). □

Cobordisms of $(2n+1)$-complexes arise in the following way:

Construction 7.5 An isomorphism of hyperbolic split $(-1)^n$-quadratic forms over Λ

$$\left(\begin{pmatrix} \gamma & \widetilde{\gamma} \\ \mu & \widetilde{\mu} \end{pmatrix}, \begin{pmatrix} \theta & 0 \\ \widetilde{\gamma}^*\mu & \widetilde{\theta} \end{pmatrix}\right) : H_{(-1)^n}(G) \xrightarrow{\cong} H_{(-1)^n}(F)$$

with F, G f. g. free determines a cobordism of $(2n+1)$-complexes

$$((j\ j') : C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi'))$$

by

$$\begin{aligned}
d &= \mu^* : C_{n+1} = F \to C_n = G^* , \\
\psi_0 &= \gamma : C^n = G \to C_{n+1} = F , \\
\psi_1 &= -\theta : C^n = G \to C_n = G^* , \\
j &= \widetilde{\mu}^* : C_{n+1} = F \to D_{n+1} = G , \\
d' &= \gamma^* : C'_{n+1} = F^* \to C_n = G^* , \\
\psi'_0 &= \mu : C'^n = G \to C'_{n+1} = F^* , \\
\psi'_1 &= -\theta : C'^n = G \to C'_n = G^* , \\
j' &= \widetilde{\gamma}^* : C'_{n+1} = F^* \to D_{n+1} = G , \\
\delta\psi_0 &= 0 : D^{n+1} = G^* \to D_{n+1} = G .
\end{aligned}$$

□

It can be shown that every cobordism of $(2n+1)$-complexes is homotopy equivalent to one constructed as in 7.5.

EXAMPLE 7.6 An n-connected $(2n+2)$-dimensional normal bordism

$$((g,c);(f,b),(f',b')) \ :\ (W^{2n+2};M^{2n+1},M'^{2n+1}) \to X \times ([0,1];\{0\},\{1\})$$

with $g : W \to X \times [0,1]$ n-connected can be regarded both as a presentation of (f,b) and as a presentation of (f',b'). The cobordism of $(2n+1)$-complexes $((j\ j') : C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi'))$ obtained in 7.2 with $W = V = W'$, $\widehat{M} = M'$, $\widehat{M}' = M'$ is the construction of 7.5 for an extension of the inclusion of the lagrangian (6.2)

$$(\begin{pmatrix} \gamma \\ \mu \end{pmatrix}, \theta) \ =\ (\begin{pmatrix} \psi_0 \\ d^* \end{pmatrix}, -\psi_1) \ :\ (C^n, 0) \to H_{(-1)^n}(C_{n+1})$$

to an isomorphism of hyperbolic split $(-1)^n$-quadratic forms

$$\left(\begin{pmatrix} \gamma & \widetilde{\gamma} \\ \mu & \widetilde{\mu} \end{pmatrix}, \begin{pmatrix} \theta & 0 \\ \widetilde{\gamma}^*\mu & \widetilde{\theta} \end{pmatrix}\right) \ :\ H_{(-1)^n}(C^n) \xrightarrow{\cong} H_{(-1)^n}(C_{n+1}) \ ,$$

with

$$\begin{aligned} j \ &=\ \widetilde{\mu}^* \ :\ C_{n+1} \ =\ K_{n+1}(W,M') \ \to D_{n+1} = K_{n+1}(W) \ , \\ j' \ &=\ \widetilde{\gamma}^* \ :\ C'_{n+1} \ =\ K_{n+1}(W,M) \to D_{n+1} = K_{n+1}(W) \ . \end{aligned}$$

□

REMARK 7.7 Fix a $(2n+1)$-dimensional geometric Poincaré complex X with reducible Spivak normal fibration, and choose a stable vector bundle $\nu_X : X \to BO$ in the Spivak normal class, e.g. a manifold with the stable normal bundle. Consider the set of n-connected normal maps $(f : M^{2n+1} \to X, b : \nu_M \to \nu_X)$. The relation defined on this set by

$$(M \to X) \sim (M' \to X) \text{ if there exists an } (n+1)\text{-connected normal bordism } (W;M,M') \to X \times ([0,1];\{0\},\{1\})$$

is an equivalence relation. Symmetry and transitivity are verified in the same way as for any geometric cobordism relation. For reflexivity form the cartesian product of an n-connected normal map $M^{2n+1} \to X$ with $([0,1];\{0\},\{1\})$, as usual. The product is an n-connected normal bordism

$$M \times ([0,1];\{0\},\{1\}) \to X \times ([0,1];\{0\},\{1\})$$

which can be made $(n+1)$-connected by surgery killing the n-dimensional kernel $K_n(M \times [0,1]) = K_n(M)$. The following verification that the cobordism of $(2n+1)$-complexes is an equivalence relation uses algebraic surgery in exactly the same way. □

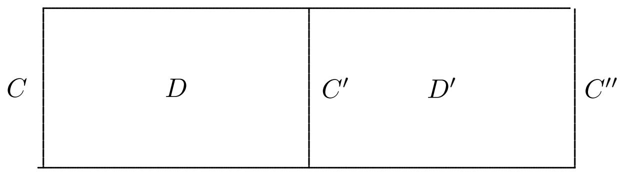
PROPOSITION 7.8 *Cobordism is an equivalence relation on $(2n+1)$-complexes (C,ψ) over Λ, such that $(C,\psi) \oplus (C,-\psi)$ is null-cobordant. Homotopy equivalent complexes are cobordant.*

Proof: Symmetry is easy: if $((j\ j') : C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi'))$ is a cobordism from (C,ψ) to (C',ψ') then

$$((j'\ j) : C' \oplus C \to D', (-\delta\psi, \psi' \oplus -\psi))$$

is a cobordism from (C',ψ') to (C,ψ). For transitivity, suppose given adjoining cobordisms of $(2n+1)$-complexes

$$\begin{aligned}&((j\ j') : C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi'))\ ,\\ &((\widetilde{j}'\ j'') : C' \oplus C'' \to D', (\delta\psi', \psi' \oplus -\psi''))\ .\end{aligned}$$



$D'' = D \cup_{C'} D'$

Define the *union* cobordism between (C,ψ) and (C'',ψ'')

$$((\widetilde{j}\ \widetilde{j}'') : C \oplus C'' \to D'', (\delta\psi'', \psi \oplus -\psi''))$$

by

$$\begin{aligned}D''_{n+1} &= \operatorname{coker}(i = \begin{pmatrix} j' \\ d' \\ \widetilde{j}' \end{pmatrix} : C'_{n+1} \to D_{n+1} \oplus C'_n \oplus D'_{n+1})\ ,\\ \widetilde{j} &= [j \oplus 0 \oplus 0]\ :\ C_{n+1} \to D''_{n+1},\\ \widetilde{j}'' &= [0 \oplus 0 \oplus j'']\ :\ C''_{n+1} \to D''_{n+1}\ ,\\ \delta\psi''_0 &= \begin{bmatrix} \delta\psi_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \delta\psi'_0 \end{bmatrix}\ :\ D''^{n+1} \to D''_{n+1}\ .\end{aligned}$$

The Λ-module morphism $i : C'_{n+1} \to D_{n+1} \oplus C'_n \oplus D'_{n+1}$ is a split injection since the dual Λ-module morphism i^* is a surjection, as follows from the Mayer-Vietoris exact sequence

$$H^{n+2}(D,C') \oplus H^{n+2}(D',C')\ =\ 0 \oplus 0 \to H^{n+2}(D'') \to H^{n+2}(C')\ =\ 0\ .$$

Given any $(2n+1)$-complex (C,ψ) let (C',ψ') be the $(2n+1)$-complex defined by

$$\begin{aligned}d' &= (-1)^n\psi_0^*\ :\ C'_{n+1}\ =\ C^{n+1} \to C'_n\ =\ C_n\ ,\\ \psi'_0 &= d^*\ :\ C'^n\ =\ C^n \to C'_{n+1}\ =\ C^{n+1}\ ,\\ \psi'_1 &= -\psi_1\ :\ C'^n\ =\ C^n \to C'_n\ =\ C_n\ .\end{aligned}$$

Apply 5.7 to extend the inclusion of the lagrangian in $H_{(-1)^n}(C_{n+1})$

$$\begin{pmatrix} \psi_0 \\ d^* \end{pmatrix} : C^n \to C_{n+1} \oplus C^{n+1}$$

to an isomorphism of $(-1)^n$-quadratic forms

$$\begin{pmatrix} \psi_0 & \widetilde{\psi}_0 \\ d^* & \widetilde{d}^* \end{pmatrix} : H_{(-1)^n}(C^n) \xrightarrow{\cong} H_{(-1)^n}(C_{n+1})$$

with $\widetilde{\psi}_0 \in \mathrm{Hom}_\Lambda(C_n, C_{n+1})$, $\widetilde{d} \in \mathrm{Hom}_\Lambda(C_{n+1}, C^n)$. Now apply 7.5 to construct from any such extension a cobordism

$$((j\ j') : C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi'))$$

with

$$\begin{aligned} j &= \widetilde{d} : C_{n+1} \to D_{n+1} = C^n , \\ j' &= \widetilde{\psi}_0^* : C'_{n+1} = C^{n+1} \to D_{n+1} = C^n , \\ d' &= \psi_0^* : C'_{n+1} = C^{n+1} \to C'_n = C_n , \\ \psi'_0 &= d^* : C'^n = C^n \to C'^{n+1} = C^{n+1} , \\ \delta\psi_0 &= 0 : D^{n+1} = C_n \to D_{n+1} = C^n . \end{aligned}$$

(This is the algebraic analogue of the construction of a presentation (6.3)

$$(W^{2n+2}; M^{2n+1}, M'^{2n+1}) \to X \times ([0,1]; \{0\}, \{1\})$$

of an n-connected $(2n+1)$-dimensional normal map $M^{2n+1} \to X$ by surgery on a finite set of $\mathbb{Z}[\pi_1(X)]$-module generators of $K_n(M)$). The union of the cobordisms

$$\begin{aligned} &((j\ j') : C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi')) , \\ &((j'\ j) : C' \oplus C \to D, (-\delta\psi, \psi' \oplus -\psi)) \end{aligned}$$

is a cobordism

$$((\widetilde{j}\ \widetilde{j}') : C \oplus C \to D', (\delta\psi', \psi \oplus -\psi))$$

with a Λ-module isomorphism

$$\begin{bmatrix} 1 & 0 & -1 \\ \psi_0 & (-1)^n\widetilde{\psi}_0 & 0 \end{bmatrix} :$$

$$D'_{n+1} = \mathrm{coker}(\begin{pmatrix} \widetilde{\psi}_0^* \\ \psi_0^* \\ \widetilde{\psi}_0^* \end{pmatrix} : C^{n+1} \to C^n \oplus C_n \oplus C^n) \xrightarrow{\cong} C^n \oplus C_{n+1} .$$

This verifies that cobordism is reflexive, and also that $(C, \psi) \oplus (C, -\psi)$ is null-cobordant.

Suppose given a homotopy equivalence of $(2n+1)$-complexes

$$f : (C, \psi) \to (C', \psi') ,$$

with $\chi_0 : C'^{n+1} \to C'_{n+1}$ as in 6.6. By reflexivity there exists a cobordism $((j''\ j') : C' \oplus C' \to D, (\delta\psi', \psi' \oplus -\psi'))$ from (C', ψ') to itself. Define a cobordism $((j\ j') : C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi'))$ from (C, ψ) to (C', ψ') by

$$\begin{aligned} j \ &= \ j''f \ : \ C_{n+1} \xrightarrow{f} C'_{n+1} \xrightarrow{j''} D_{n+1} \ , \\ \delta\psi_0 \ &= \ \delta\psi'_0 + j''\chi_0 j''^* \ : \ D^{n+1} \to D_{n+1} \ . \end{aligned}$$

□

DEFINITION 7.9 (i) A *weak map* of $(2n+1)$-complexes over Λ

$$f \ : \ (C, \psi) \to (C', \psi')$$

is a chain map $f : C \to C'$ such that there exist Λ-module morphisms

$$\chi_0 \ : \ C'^{n+1} \to C'_{n+1} \ \ , \ \ \chi_1 \ : \ C'^n \to C'^n$$

with

$$f\psi_0 f^* - \psi'_0 \ = \ (\chi_0 + (-1)^{n+1}\chi_0^*)d'^* \ : \ C'^n \to C'_{n+1} \ .$$

(ii) A *weak equivalence* of $(2n+1)$-complexes is a weak map with $f : C \to C'$ a chain equivalence.
(iii) A *weak isomorphism* of $(2n+1)$-complexes is a weak map with $f : C \to C'$ an isomorphism of chain complexes. □

PROPOSITION 7.10 *Weakly equivalent $(2n+1)$-complexes are cobordant.*
PROOF: The proof in 7.8 that homotopy equivalent $(2n+1)$-complexes are cobordant works just as well for weakly equivalent ones. □

Given a $(2n+1)$-complex (C, ψ) let

$$(\begin{pmatrix} \psi_0 \\ d^* \end{pmatrix}, -\psi_1) \ : \ (C^n, 0) \to H_{(-1)^n}(C_{n+1})$$

be the inclusion of a lagrangian in a hyperbolic split $(-1)^n$-quadratic form given by 6.2. The result of 7.10 is that the cobordism class of (C, ψ) is independent of the hessian $(-1)^{n+1}$-quadratic form $(C^n, -\psi_1)$.

§8. The odd-dimensional L-groups

The odd-dimensional surgery obstruction groups $L_{2n+1}(\Lambda)$ of a ring with involution Λ will now be defined to be the cobordism groups of $(2n+1)$-complexes over Λ.

DEFINITION 8.1 Let $L_{2n+1}(\Lambda)$ be the abelian group of cobordism classes of $(2n+1)$-complexes over Λ, with addition and inverses by

$$\begin{aligned} (C, \psi) + (C', \psi') \ &= \ (C \oplus C', \psi \oplus \psi') \ , \\ -\,(C, \psi) \ &= \ (C, -\psi) \in L_{2n+1}(\Lambda) \ . \end{aligned}$$

□

The groups $L_{2n+1}(\Lambda)$ only depend on the residue $n(\bmod 2)$, so that only two L-groups have actually been defined, $L_1(\Lambda)$ and $L_3(\Lambda)$. Note that 8.1 uses 7.8 to justify $(C,\psi)\oplus(C,-\psi) = 0 \in L_{2n+1}(\Lambda)$.

EXAMPLE 8.2 The odd-dimensional L-groups of $\Lambda = \mathbb{Z}$ are trivial

$$L_{2n+1}(\mathbb{Z}) \;=\; 0 \;. \qquad \square$$

8.2 was implicit in the work of Kervaire and Milnor [7] on the surgery classification of even-dimensional exotic spheres.

EXAMPLE 8.3 The surgery obstruction of an n-connected $(2n+1)$-dimensional normal map $(f,b) : M^{2n+1} \to X$ is the cobordism class

$$\sigma_*(f,b) \;=\; (C,\psi) \in L_{2n+1}(\mathbb{Z}[\pi_1(X)])$$

of the $(2n+1)$-complex (C,ψ) associated in 6.3 to any choice of presentation

$$(W;M,M') \to X \times ([0,1];\{0\},\{1\}) \;.$$

The surgery obstruction vanishes $\sigma_*(f,b) = 0$ if (and for $n \geq 2$ only if) (f,b) is normal bordant to a homotopy equivalence. $\square$

DEFINITION 8.4 A *surgery* $(j : C \to D, (\delta\psi,\psi))$ on a $(2n+1)$-complex (C,ψ) is a Λ-module chain map $j : C \to D$ with $D_r = 0$ for $r \neq n+1$ and D_{n+1} a f. g. free Λ-module, together with a Λ-module morphism

$$\delta\psi_0 \;:\; D^{n+1} \;=\; (D_{n+1})^* \to D_{n+1} \;,$$

such that the Λ-module morphism

$$(\, d \quad \psi_0^* j^* \,) \;:\; C_{n+1} \oplus D^{n+1} \to C_n$$

is onto. The *effect* of the surgery is the $(2n+1)$-complex (C',ψ') defined by

$$\begin{aligned} d' \;&=\; \begin{pmatrix} d & \psi_0^* j^* \\ (-1)^{n+1} j & \delta\psi_0 + (-1)^{n+1}\delta\psi_0^* \end{pmatrix} \\ &\qquad : \; C'_{n+1} \;=\; C_{n+1} \oplus D^{n+1} \to C'_n \;=\; C_n \oplus D_{n+1} \;, \\ \psi'_0 \;&=\; \begin{pmatrix} \psi_0 & 0 \\ 0 & 1 \end{pmatrix} \;:\; C'^n \;=\; C^n \oplus D^{n+1} \to C'_{n+1} \;=\; C_{n+1} \oplus D^{n+1} \;, \\ \psi'_1 \;&=\; \begin{pmatrix} \psi_1 & -\psi_0^* j^* \\ 0 & -\delta\psi_0 \end{pmatrix} \;:\; C'^n = C^n \oplus D^{n+1} \to C'_n \;=\; C_n \oplus D_{n+1} \;. \end{aligned}$$

The *trace* of the surgery is the cobordism of $(2n+1)$-complexes $((j' \; j'') : C \oplus C' \to D', (0, \psi \oplus -\psi'))$, with

$$\begin{aligned} j'' \;=\; (j' \;\, k) \; &: C'_{n+1} \;=\; C_{n+1} \oplus D^{n+1} \\ &\to D'_{n+1} \;=\; \ker((\, d \quad \psi_0 j^* \,) : C_{n+1} \oplus D^{n+1} \to C_n) \end{aligned}$$

a splitting of the split injection $(d\ \psi_0^* j^*) : C_{n+1} \oplus D^{n+1} \to C_n$. $\square$

EXAMPLE 8.5 Let

$$((e,a);(f,b),(f',b')) \ : \ (V^{2n+2}; M^{2n+1}, M'^{2n+1}) \to X \times ([0,1];\{0\},\{1\})$$

be the trace of a sequence of k surgeries on an n-connected $(2n+1)$-dimensional normal map $(f,b) : M \to X$ killing elements $x_1, x_2, \dots, x_k \in K_n(M)$, with e n-connected and f' n-connected. V has a handle decomposition on M of the type

$$V \ = \ M \times I \cup \bigcup_k (n+1)\text{-handles } D^{n+1} \times D^{n+1} ,$$

and also a handle decomposition on M' of the same type

$$V \ = \ M' \times I \cup \bigcup_k (n+1)\text{-handles } D^{n+1} \times D^{n+1} .$$

A presentation of (f,b)

$$((g,c);(\widehat{f},\widehat{b}),(f,b)) \ : \ (W^{2n+2}; \widehat{M}^{2n+1}, M^{2n+1}) \to X \times ([0,1];\{0\},\{1\})$$

with $(2n+1)$-complex (C,ψ) determines a presentation of (f',b')

$$((g',c');(\widehat{f},\widehat{b}),(f',b')) \ = \ ((g,c);(\widehat{f},\widehat{b}),(f,b)) \cup ((e,a);(f,b),(f',b')) \ :$$

$$(W';\widehat{M},M') \ = \ (W;\widehat{M},M) \cup (V;M,M') \to X \times ([0,1];\{0\},\{1\})$$

such that the $(2n+1)$-complex (C',ψ') is the effect of a surgery $(j : C \to D, (\delta\psi,\psi))$ on (C,ψ) with

$$D_{n+1} \ = \ K_{n+1}(V,M') \ = \ \mathbb{Z}[\pi_1(X)]^k ,$$

$$C'_{n+1} \ = \ K_{n+1}(W',\widehat{M}) \ = \ K_{n+1}(W,\widehat{M}) \oplus K_{n+1}(V,M) \ = \ C_{n+1} \oplus D^{n+1} ,$$

$$C'_n \ = \ K_{n+1}(W',\partial W') \ = \ K_{n+1}(W,\partial W) \oplus K_{n+1}(V,M') \ = \ C_n \oplus D_{n+1} .$$

Also, the geometric trace determines the algebraic trace, with

$$D'_{n+1} \ = \ K_{n+1}(V) \ .$$ $\square$

It can be shown that $(2n+1)$-complexes (C,ψ), (C',ψ') are cobordant if and only if (C',ψ') is homotopy equivalent to the effect of a surgery on (C,ψ). This result will only be needed for $(C',\psi') = (0,0)$, so it will only be proved in this special case:

PROPOSITION 8.6 *A $(2n+1)$-complex (C,ψ) represents 0 in $L_{2n+1}(\Lambda)$ if and only if there exists a surgery $(j : C \to D, (\delta\psi,\psi))$ with contractible effect.*

PROOF: The effect of a surgery is contractible if and only if it is a null-cobordism. $\square$

Given an n-connected $(2n+1)$-dimensional normal map $(f,b) : M^{2n+1} \longrightarrow X$ it is possible to kill every element $x \in K_n(M)$ by an embedding $S^n \times D^{n+1} \hookrightarrow M$ to obtain a bordant normal map

$$(f',b') \; : \; M'^{2n+1} \; = \; \mathrm{cl.}(M \backslash S^n \times D^{n+1}) \cup D^{n+1} \times S^n \to X \; .$$

There are many ways of carrying out the surgery, which are quantified by the surgeries on the kernel $(2n+1)$-complex (C,ψ). In general, $K_n(M')$ need not be smaller than $K_n(M)$.

Example 8.7 The kernel $(2n+1)$-complex (C,ψ) over $\mathbb{Z}$ of the identity normal map

$$(f,b) \; = \; \mathrm{id.} \; : \; M^{2n+1} \; = \; S^{2n+1} \to S^{2n+1}$$

is $(0,0)$. For any element

$$\mu \in \pi_{n+1}(SO, SO(n+1)) \; = \; Q_{(-1)^{n+1}}(\mathbb{Z})$$

let $\omega = \partial\mu \in \pi_n(SO(n+1))$, and define a null-homotopic embedding of S^n in M

$$e_\omega \; : \; S^n \times D^{n+1} \hookrightarrow M \; ; \; (x,y) \longmapsto (x, \omega(x)(y))/ \parallel (x,\omega(x)(y)) \parallel \; .$$

Use μ to kill $0 \in K_n(M)$ by surgery on (f,b), with effect a normal bordant n-connected $(2n+1)$-dimensional normal map

$$(f_\mu, b_\mu) \; : \; M_\mu^{2n+1} \; = \; \mathrm{cl.}(M \backslash e_\omega(S^n \times D^{n+1})) \cup D^{n+1} \times S^n \to S^{2n+1}$$

exactly as in 2.18, with the kernel complex (C',ψ') given by

$$d' \; = \; (1 + T_{(-1)^{n+1}})(\mu) \; : \; C'_{n+1} \; = \; \mathbb{Z} \to C'_n \; = \; \mathbb{Z} \; .$$

In particular, for $\mu = 0, 1$ this gives the $(2n+1)$-dimensional manifolds

$$\begin{aligned} M' \; &= \; M_0 \; = \; S^n \times S^{n+1} \; , \\ M'' \; &= \; M_1 \; = \; S(\tau_{S^{n+1}}), \text{ the tangent } S^n\text{-bundle of } S^{n+1} \\ &= \; O(n+2)/O(n) \\ &= \; V_{n+2,2}, \text{ the Stiefel manifold of orthonormal 2-frames in } \mathbb{R}^{n+2} \\ &(= \; SO(3) \; = \; \mathbb{RP}^3 \text{ for } n=1 \;) \; , \end{aligned}$$

corresponding to the algebraic surgeries on $(0,0)$

$$(0 : 0 \to D, (\delta\psi', 0)) \; , \; (0 : 0 \to D, (\delta\psi'', 0))$$

with

$$D_{n+1} \; = \; \mathbb{Z} \; , \; \delta\psi'_0 \; = \; 0 \; , \; \delta\psi''_0 \; = \; 1 \; . \qquad \square$$

§9. Formations

As before, let Λ be a ring with involution, and let $\epsilon = \pm 1$.

DEFINITION 9.1 An *ϵ-quadratic formation over* Λ $(Q, \phi; F, G)$ is a nonsingular ϵ-quadratic form (Q, ϕ) together with an ordered pair of lagrangians F,G. □

Formations with $\epsilon = (-1)^n$ are essentially the $(2n+1)$-complexes of §6 expressed in the language of forms and lagrangians of §4. In the general theory it is possible to consider formations $(Q, \phi; F, G)$ with Q, F, G f. g. projective, but in view of the more immediate topological applications only the f. g. free case is considered here. Strictly speaking, 9.1 defines a "nonsingular formation". In the general theory a formation $(Q, \phi; F, G)$ is a nonsingular form (Q, ϕ) together with a lagrangian F and a sublagrangian G. The automorphisms of hyperbolic forms in the original treatment due to Wall [29] of odd-dimensional surgery theory were replaced by formations by Novikov [16] and Ranicki [18].

In dealing with formations assume that the ground ring Λ is such that the rank of f. g. free Λ-modules is well-defined (e.g. $\Lambda = \mathbb{Z}[\pi]$). The rank of a f. g. free Λ-module K is such that

$$\mathrm{rank}_\Lambda(K) \;=\; k \in \mathbb{Z}^+$$

if and only if K is isomorphic to Λ^k. Also, since $\Lambda^k \cong (\Lambda^k)^*$

$$\mathrm{rank}_\Lambda(K) \;=\; \mathrm{rank}_\Lambda(K^*) \in \mathbb{Z}^+ \;.$$

DEFINITION 9.2 An *isomorphism* of ϵ-quadratic formations over Λ

$$f \;:\; (Q, \phi; F, G) \xrightarrow{\cong} (Q', \phi'; F', G')$$

is an isomorphism of forms $f : (Q, \phi) \cong (Q', \phi')$ such that

$$f(F) \;=\; F' \quad , \quad f(G) \;=\; G' \;. \qquad \square$$

PROPOSITION 9.3 (i) *Every ϵ-quadratic formation $(Q, \phi; F, G)$ is isomorphic to one of the type $(H_\epsilon(F); F, G)$.*
(ii) *Every ϵ-quadratic formation $(Q, \phi; F, G)$ is isomorphic to one of the type $(H_\epsilon(F); F, \alpha(F))$ for some automorphism $\alpha : H_\epsilon(F) \cong H_\epsilon(F)$.*
PROOF : (i) By Theorem 5.7 the inclusion of the lagrangian $F \to Q$ extends to an isomorphism of forms $f : H_\epsilon(F) \cong (Q, \phi)$, defining an isomorphism of formations

$$f \;:\; (H_\epsilon(F); F, f^{-1}(G)) \xrightarrow{\cong} (Q, \phi; F, G) \;.$$

(ii) As in (i) extend the inclusions of the lagrangians to isomorphisms of forms

$$f \ : \ H_\epsilon(F) \xrightarrow{\cong} (Q,\phi) \ , \ g \ : \ H_\epsilon(G) \xrightarrow{\cong} (Q,\phi) \ .$$

Then

$$\mathrm{rank}_\Lambda(F) \ = \ \mathrm{rank}_\Lambda(Q)/2 \ = \ \mathrm{rank}_\Lambda(G) \in \mathbb{Z}^+ \ ,$$

so that F is isomorphic to G. Choosing a Λ-module isomorphism $\beta : G \cong F$ there is defined an automorphism of $H_\epsilon(F)$

$$\alpha \ : \ H_\epsilon(F) \xrightarrow{f} (Q,\phi) \xrightarrow{g^{-1}} H_\epsilon(G) \xrightarrow{\begin{pmatrix} \beta & 0 \\ 0 & \beta^{*-1} \end{pmatrix}} H_\epsilon(F)$$

such that there is defined an isomorphism of formations

$$f \ : \ (H_\epsilon(F); F, \alpha(F)) \xrightarrow{\cong} (Q,\phi; F, G) \ . \qquad \square$$

PROPOSITION 9.4 *The weak isomorphism classes of* $(2n+1)$*-complexes* (C,ψ) *over* Λ *are in natural one-one correspondence with the isomorphism classes of* $(-1)^n$*-quadratic formations* $(Q,\phi;F,G)$ *over* Λ*, with*

$$H_n(C) \ = \ Q/(F+G) \quad , \quad H_{n+1}(C) \ = \ F \cap G \ .$$

Moreover, if the complex (C,ψ) *corresponds to the formation* $(Q,\phi;F,G)$ *then* $(C,-\psi)$ *corresponds to* $(Q,-\phi;F,G)$.

PROOF: Given a $(2n+1)$-complex (C,ψ) define a $(-1)^n$-quadratic formation

$$(Q,\phi;F,G) \ = \ (H_{(-1)^n}(C_{n+1}); C_{n+1}, \mathrm{im}(\begin{pmatrix} \psi_0 \\ d^* \end{pmatrix} : C^n \to C_{n+1} \oplus C^{n+1})) \ .$$

The formation associated in this way to the $(2n+1)$-complex $(C,-\psi)$ is isomorphic to $(Q,-\phi;F,G)$, by the isomorphism

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} : (Q,-\phi;F,G) \xrightarrow{\cong}$$
$$(H_{(-1)^n}(C_{n+1}); C_{n+1}, \mathrm{im}(\begin{pmatrix} -\psi_0 \\ d^* \end{pmatrix} : C^n \to C_{n+1} \oplus C^{n+1})) \ .$$

Conversely, suppose given an $(-1)^n$-quadratic formation $(Q,\phi;F,G)$. By 9.3 (i) this can be replaced by an isomorphic formation with $(Q,\phi) = H_{(-1)^n}(F)$. Let $\gamma \in \mathrm{Hom}_\Lambda(G,F)$, $\mu \in \mathrm{Hom}_\Lambda(G,F^*)$ be the components of the inclusion

$$i \ = \ \begin{pmatrix} \gamma \\ \mu \end{pmatrix} \ : \ G \to Q \ = \ F \oplus F^* \ .$$

Choose any $\theta \in \mathrm{Hom}_\Lambda(G,G^*)$ such that

$$\gamma^*\mu \ = \ \theta + (-1)^{n+1}\theta^* \in \mathrm{Hom}_\Lambda(G,G^*) \ .$$

Define a $(2n+1)$-complex (C,ψ) by

$$\begin{aligned} d &= \mu^* : C_{n+1} = F \to C_n = G^* , \\ \psi_0 &= \gamma : C^n = G \to C_{n+1} = F , \\ \psi_1 &= (-1)^n\theta : C^n = G \to C_n = G^* . \end{aligned}$$

The exact sequence

$$0 \to G \xrightarrow{i} Q \xrightarrow{i^*(\phi+(-1)^n\phi^*)} G^* \to 0$$

is the algebraic mapping cone

$$0 \to G \xrightarrow{\begin{pmatrix}\gamma \\ \mu\end{pmatrix}} F \oplus F^* \xrightarrow{(\mu^* \quad (-1)^n\gamma^*)} G^* \to 0$$

of the chain equivalence $(1+T)\psi_0 : C^{2n+1-*} \to C$. □

EXAMPLE 9.5 An n-connected $(2n+1)$-dimensional normal map $M^{2n+1} \to X$ together with a choice of presentation $(W;M,M') \to X\times([0,1];\{0\},\{1\})$ determine by 9.3 a $(2n+1)$-complex (C,ψ), and hence by 9.4 a $(-1)^n$-quadratic formation $(Q,\phi;F,G)$ over $\mathbb{Z}[\pi_1(X)]$ such that

$$\begin{aligned} Q/(F+G) &= H_n(C) = K_n(M) , \\ F\cap G &= H_{n+1}(C) = K_{n+1}(M) . \end{aligned}$$

□

The following equivalence relation on formations corresponds to the weak equivalence (7.9) of $(2n+1)$-complexes.

DEFINITION 9.6 (i) An ϵ-quadratic formation $(Q,\phi;F,G)$ is *trivial* if it is isomorphic to $(H_\epsilon(L);L,L^*)$ for some f. g. free Λ-module L.
(ii) A *stable isomorphism* of ϵ-quadratic formations

$$[f] : (Q,\phi;F,G) \overset{\cong}{\to} (Q',\phi';F',G')$$

is an isomorphism of ϵ-quadratic formations of the type

$$f : (Q,\phi;F,G)\oplus(\text{trivial}) \overset{\cong}{\to} (Q',\phi';F',G')\oplus(\text{trivial}') .$$

□

EXAMPLE 9.7 The $(-1)^n$-quadratic formations associated in 9.5 to all the presentations of an n-connected $(2n+1)$-dimensional normal map $M^{2n+1} \to X$ define a stable isomorphism class. □

PROPOSITION 9.8 *The weak equivalence classes of $(2n+1)$-complexes over Λ are in natural one-one correspondence with the stable isomorphism classes of $(-1)^n$-quadratic formations over Λ.*

PROOF: The $(2n+1)$-complex (C,ψ) associated (up to weak equivalence) to a $(-1)^n$-quadratic formation $(Q,\phi;F,G)$ in 9.4 is contractible if and only if the formation is trivial. $\square$

The following formations correspond to the null-cobordant complexes.

DEFINITION 9.9 The *boundary* of a $(-\epsilon)$-quadratic form (K,λ,μ) is the ϵ-quadratic formation

$$\partial(K,\lambda,\mu) \;=\; (H_\epsilon(K);K,\Gamma_{(K,\lambda)})$$

with $\Gamma_{(K,\lambda)}$ the *graph* lagrangian

$$\Gamma_{(K,\lambda)} \;=\; \{(x,\lambda(x))\in K\oplus K^* \,|\, x\in K\}\ . \qquad \square$$

Note that the form (K,λ,μ) may be singular, that is the Λ-module morphism $\lambda : K \to K^*$ need not be an isomorphism. The graphs $\Gamma_{(K,\lambda)}$ of $(-\epsilon)$-quadratic forms (K,λ,μ) are precisely the lagrangians of $H_\epsilon(K)$ which are direct complements of K^*.

PROPOSITION 9.10 *A $(-1)^n$-quadratic formation $(Q,\phi;F,G)$ is stably isomorphic to a boundary $\partial(K,\lambda,\mu)$ if and only if the corresponding $(2n+1)$-complex (C,ψ) is null-cobordant.*
PROOF: Given a $(-1)^{n+1}$-quadratic form (K,λ,μ) choose a split form $\theta : K\to K^*$ (4.2) and let (C,ψ) be the $(2n+1)$-complex associated by 9.4 to the boundary formation $\partial(K,\lambda,\mu)$, so that

$$\begin{aligned}
d \;=\; \lambda \;=\; \theta+(-1)^{n+1}\theta^* \;:\; C_{n+1} \;=\; K \to C_n \;=\; K^*\ ,\\
\psi_0 \;=\; 1 \;:\; C^n \;=\; K\to C_{n+1} \;=\; K\ ,\\
\psi_1 \;=\; -\theta \;:\; C^n \;=\; K\to C_n \;=\; K^*\ .
\end{aligned}$$

Then (C,ψ) is null-cobordant, with a null-cobordism $(j:C\to D,(\delta\psi,\psi))$ defined by

$$\begin{aligned}
j \;=\; 1 \;:\; C_{n+1} \;=\; K\to D_{n+1} \;=\; K\ ,\\
\delta\psi_0 \;=\; 0 \;:\; D^{n+1} \;=\; K^*\to D_{n+1} \;=\; K\ .
\end{aligned}$$

Conversely, suppose given a $(2n{+}1)$-complex (C,ψ) with a null-cobordism

$(j : C \to D, (\delta\psi, \psi))$ as in 8.1. The $(2n+1)$-complex (E, θ) defined by

$$d \;=\; \begin{pmatrix} \psi_1 + (-1)^{n+1}\psi_1^* & d & \psi_0^* j^* \\ (-1)^{n+1}d^* & 0 & -j^* \\ (-1)^{n+1}j\psi_0^* & (-1)^n j & \delta\psi_0 + (-1)^{n+1}\delta\psi_0^* \end{pmatrix} :$$

$$E_{n+1} \;=\; C^n \oplus C_{n+1} \oplus D^{n+1} \;\to\; E_n \;=\; C_n \oplus C^{n+1} \oplus D_{n+1}\;,$$

$$\theta_0 \;=\; 1 \;:\; E^n \;=\; C^n \oplus C_{n+1} \oplus D^{n+1} \;\to\; E_{n+1} \;=\; C^n \oplus C_{n+1} \oplus D^{n+1}\;,$$

$$\theta_1 \;=\; \begin{pmatrix} -\psi_1 & -d & -\psi_0^* j^* \\ 0 & 0 & j^* \\ 0 & 0 & -\delta\psi_0 \end{pmatrix} :$$

$$E^n \;=\; C^n \oplus C_{n+1} \oplus D^{n+1} \;\to\; E_n \;=\; C_n \oplus C^{n+1} \oplus D_{n+1}$$

corresponds to the boundary $(-1)^n$-quadratic formation $\partial(E^n, \lambda_1, \mu_1)$ of the $(-1)^{n+1}$-quadratic form (E^n, λ_1, μ_1) determined by the split form θ_1, and there is defined a homotopy equivalence $f : (E, \theta) \to (C, \psi)$ with

$$f_n \;=\; (1 \;\; \psi_0^* \;\; 0) \;:\; E_n \;=\; C_n \oplus C^{n+1} \oplus D_{n+1} \to C_n\;,$$

$$f_{n+1} \;=\; (0 \;\; 1 \;\; 0) \;:\; E_{n+1} \;=\; C^n \oplus C_{n+1} \oplus D^{n+1} \to C_{n+1}\;. \qquad \square$$

PROPOSITION 9.11 *The cobordism group $L_{2n+1}(\Lambda)$ of $(2n+1)$-complexes is naturally isomorphic to the abelian group of equivalence classes of $(-1)^n$-quadratic formations over Λ, subject to the equivalence relation*

$(Q, \phi; F, G) \sim (Q', \phi'; F', G')$ *if there exists a stable isomorphism*

$$[f] \;:\; (Q, \phi; F, G) \oplus (Q', -\phi'; F', G') \xrightarrow{\cong} \partial(K, \lambda, \mu)$$

for some $(-1)^{n+1}$-quadratic form (K, λ, μ) over Λ ,

with addition and inverses by

$$(Q, \phi; F, G) + (Q', \phi'; F', G') \;=\; (Q \oplus Q', \phi \oplus \phi'; F \oplus F', G \oplus G')\;,$$

$$-(Q, \phi; F, G) \;=\; (Q, -\phi; F, G) \in L_{2n+1}(\Lambda)\;.$$

PROOF: This is just the translation of the definition (8.1) of $L_{2n+1}(\Lambda)$ into the language of $(-1)^n$-quadratic formations, using 9.4, 9.8 and 9.10. $\square$

Use 9.11 as an identification of $L_{2n+1}(\Lambda)$ with the group of equivalence classes of $(-1)^n$-quadratic formations over Λ.

COROLLARY 9.12 *A $(-1)^n$-quadratic formation $(Q, \phi; F, G)$ over Λ is such that $(Q, \phi; F, G) = 0 \in L_{2n+1}(\Lambda)$ if and only if it is stably isomorphic to the boundary $\partial(K, \lambda, \mu)$ of a $(-1)^{n+1}$-quadratic form (K, λ, μ) on a f. g. free Λ-module K.*

PROOF: Immediate from 9.10. $\square$

Next, it is necessary to establish the relation

$$(Q,\phi;F,G)\oplus(Q,\phi;G,H) \ = \ (Q,\phi;F,H)\in L_{2n+1}(\Lambda)\ .$$

This is the key step in the identification in §10 below of $L_{2n+1}(\Lambda)$ with a stable unitary group.

LEMMA 9.13 (i) *An ϵ-quadratic formation $(Q,\phi;F,G)$ is trivial if and only if the lagrangians F and G are direct complements in Q.*
(ii) *An ϵ-quadratic formation $(Q,\phi;F,G)$ is isomorphic to a boundary if and only if (Q,ϕ) has a lagrangian H which is a direct complement of both the lagrangians F,G.*
PROOF: (i) If F and G are direct complements in Q express any representative $\phi\in\mathrm{Hom}_\Lambda(Q,Q^*)$ of $\phi\in Q_\epsilon(Q)$ as

$$\phi \ = \ \begin{pmatrix}\lambda-\epsilon\lambda^* & \gamma\\ \delta & \mu-\epsilon\mu^*\end{pmatrix} \ : \ Q \ = \ F\oplus G\to Q^* \ = \ F^*\oplus G^*\ .$$

Then $\gamma+\epsilon\delta^*\in\mathrm{Hom}_\Lambda(G,F^*)$ is an Λ-module isomorphism, and there is defined an isomorphism of ϵ-quadratic formations

$$\begin{pmatrix}1 & 0\\ 0 & (\gamma+\epsilon\delta^*)^{-1}\end{pmatrix} \ : \ (H_\epsilon(F);F,F^*)\xrightarrow{\cong}(Q,\phi;F,G)$$

so that $(Q,\phi;F,G)$ is trivial. The converse is obvious.
(ii) For the boundary $\partial(K,\lambda,\mu)$ of a $(-\epsilon)$-quadratic form (K,λ,μ) the lagrangian K^* of $H_\epsilon(K)$ is a direct complement of both the lagrangians $K,\Gamma_{(K,\lambda)}$. Conversely, suppose that $(Q,\phi;F,G)$ is such that there exists a lagrangian H in (Q,ϕ) which is a direct complement to both F and G. By the proof of (i) there exists an isomorphism of formations

$$f \ : \ (H_\epsilon(F);F,F^*)\xrightarrow{\cong}(Q,\phi;F,H)$$

which is the identity on F. Now $f^{-1}(G)$ is a lagrangian of $H_\epsilon(F)$ which is a direct complement of F^*, so that it is the graph $\Gamma_{(F,\lambda)}$ of a $(-\epsilon)$-quadratic form (F,λ,μ), and f defines an isomorphism of ϵ-quadratic formations

$$f \ : \ \partial(F,\lambda,\mu) \ = \ (H_\epsilon(F);F,\Gamma_{(F,\lambda)})\xrightarrow{\cong}(Q,\phi;F,G)\ . \qquad \square$$

PROPOSITION 9.14 *For any lagrangians F,G,H in a nonsingular $(-1)^n$-quadratic form (Q,ϕ) over Λ*

$$(Q,\phi;F,G)\oplus(Q,\phi;G,H) \ = \ (Q,\phi;F,H)\in L_{2n+1}(\Lambda)\ .$$

PROOF: Choose lagrangians F^*,G^*,H^* in (Q,ϕ) complementary to F,G,H respectively. The $(-1)^n$-quadratic formations $(Q_i,\phi_i;F_i,G_i)$ $(1\leq i\leq 4)$

defined by

$$\begin{aligned}
(Q_1,\phi_1;F_1,G_1) &= (Q,-\phi;G^*,G^*)\ ,\\
(Q_2,\phi_2;F_2,G_2) &= (Q\oplus Q,\phi\oplus -\phi;F\oplus F^*,H\oplus G^*)\\
&\qquad\qquad \oplus(Q\oplus Q,-\phi\oplus\phi;\Delta_Q,H^*\oplus G)\ ,\\
(Q_3,\phi_3;F_3,G_3) &= (Q\oplus Q,\phi\oplus -\phi,F\oplus F^*,G\oplus G^*)\ ,\\
(Q_4,\phi_4;F_4,G_4) &= (Q\oplus Q,\phi\oplus -\phi;G\oplus G^*,H\oplus G^*)\\
&\qquad\qquad \oplus(Q\oplus Q,-\phi\oplus\phi;\Delta_Q,H^*\oplus G)
\end{aligned}$$

are such that

$$\begin{aligned}
(Q,\phi;F,G)\ &\oplus(Q,\phi;G,H)\oplus(Q_1,\phi_1;F_1,G_1)\oplus(Q_2,\phi_2;F_2,G_2)\\
&= (Q,\phi;F,H)\oplus(Q_3,\phi_3;F_3,G_3)\oplus(Q_4,\phi_4;F_4,G_4)\ .
\end{aligned}$$

Each of $(Q_i,\phi_i;F_i,G_i)$ $(1\leq i\leq 4)$ is isomorphic to a boundary, since there exists a lagrangian H_i in (Q_i,ϕ_i) complementary to both F_i and G_i, so that 9.13 (ii) applies and $(Q_i,\phi_i;F_i,G_i)$ represents 0 in $L_{2n+1}(\Lambda)$. Explicitly, take

$$\begin{aligned}
H_1 &= G\subset Q_1\ =\ Q\ ,\\
H_2 &= \Delta_{Q\oplus Q}\subset Q_2\ =\ (Q\oplus Q)\oplus(Q\oplus Q)\ ,\\
H_3 &= \Delta_Q\subset Q_3\ =\ Q\oplus Q\ ,\\
H_4 &= \Delta_{Q\oplus Q}\subset Q_4\ =\ (Q\oplus Q)\oplus(Q\oplus Q)\ .
\end{aligned}$$

□

REMARK 9.15 It is also possible to express $L_{2n+1}(\Lambda)$ as the abelian group of equivalence classes of $(-1)^n$-quadratic formations over Λ subject to the equivalence relation generated by

(i) $(Q,\phi;F,G)\sim(Q',\phi';F',G')$ if $(Q,\phi;F,G)$ is stably isomorphic to $(Q',\phi';F',G')$,

(ii) $(Q,\phi;F,G)\oplus(Q,\phi;G,H)\sim(Q,\phi;F,H)$, with addition and inverses by

$$\begin{aligned}
(Q,\phi;F,G)+(Q',\phi';F',G') &= (Q\oplus Q',\phi\oplus\phi';F\oplus F',G\oplus G')\ ,\\
-(Q,\phi;F,G) &= (Q,\phi;G,F)\in L_{2n+1}(\Lambda)\ .
\end{aligned}$$

This is immediate from 9.13 and the observation that for any $(-1)^{n+1}$-quadratic form (K,λ,μ) on a f. g. free Λ-module K the lagrangian K^* in $H_{(-1)^n}(K)$ is a complement to both K and the graph $\Gamma_{(K,\lambda)}$, so that

$$\begin{aligned}
\partial(K,\lambda,\mu)\ &\sim\ (H_{(-1)^n}(K);K,\Gamma_{(K,\lambda)})\oplus(H_{(-1)^n}(K);\Gamma_{(K,\lambda)},K^*)\\
&\sim\ (H_{(-1)^n}(K);K,K^*)\ \sim\ 0\ .
\end{aligned}$$

□

§10. Automorphisms

The $(2n+1)$-dimensional L-group $L_{2n+1}(\Lambda)$ of a ring with involution Λ is identified with a quotient of the stable automorphism group of hyperbolic $(-1)^n$-quadratic forms over Λ, as in the original definition of Wall [29].

Given a Λ-module K let $\mathrm{Aut}_\Lambda(K)$ be the group of automorphisms $K \to K$, with the composition as group law.

EXAMPLE 10.1 The automorphism group of the f. g. free Λ-module Λ^k is the general linear group $GL_k(\Lambda)$ of invertible $k \times k$ matrices in Λ

$$\mathrm{Aut}_\Lambda(\Lambda^k) \;=\; GL_k(\Lambda)$$

with the multiplication of matrices as group law (cf. Remark 1.12). The general linear group is not abelian for $k \geq 2$, since

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} . \qquad \square$$

DEFINITION 10.2 For any ϵ-quadratic form (K, λ, μ) let $\mathrm{Aut}_\Lambda(K, \lambda, \mu)$ be the subgroup of $\mathrm{Aut}_\Lambda(K)$ consisting of the automorphisms $f : (K, \lambda, \mu) \to (K, \lambda, \mu)$. $\square$

DEFINITION 10.3 The (ϵ, k)-*unitary group* of Λ is defined for $\epsilon = \pm 1$, $k \geq 0$ to be the automorphism group

$$U_{\epsilon,k}(\Lambda) \;=\; \mathrm{Aut}_\Lambda(H_\epsilon(\Lambda^k))$$

of the ϵ-quadratic hyperbolic form $H_\epsilon(\Lambda^k)$. $\square$

PROPOSITION 10.4 $U_{\epsilon,k}(\Lambda)$ is the group of invertible $2k \times 2k$ matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_{2k}(\Lambda)$ such that

$$\alpha^*\delta + \epsilon\gamma^*\beta \;=\; 1 \in M_{k,k}(\Lambda) \;,\; \alpha^*\gamma \;=\; \beta^*\delta \;=\; 0 \in Q_\epsilon(\Lambda^k) \;.$$

PROOF : This is just the decoding of the condition

$$\begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \;=\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in Q_\epsilon(\Lambda^k \oplus (\Lambda^k)^*)$$

for $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ to define an automorphism of the hyperbolic (split) ϵ-quadratic form

$$H_\epsilon(\Lambda^k) \;=\; (\Lambda^k \oplus (\Lambda^k)^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) \;. \qquad \square$$

Use 10.4 to express the automorphisms of $H_\epsilon(\Lambda^k)$ as matrices.

EXAMPLE 10.5 $U_{\epsilon,1}(\Lambda)$ is the subgroup of $GL_2(\Lambda)$ consisting of the 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$d\bar{a}+\epsilon b\bar{c} \ = \ 1\in\Lambda \ , \ c\bar{a} \ = \ d\bar{b} \ = \ 0\in Q_{\epsilon}(\Lambda) \ . \qquad \square$$

DEFINITION 10.6 The *elementary (ϵ,k)-quadratic unitary group* of Λ is the normal subgroup

$$EU_{\epsilon,k}(\Lambda)\subseteq U_{\epsilon,k}(\Lambda)$$

of the full (ϵ,k)-quadratic unitary group generated by the elements of the following two types:

(i) $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*-1} \end{pmatrix}$ for any automorphism $\alpha\in GL_k(\Lambda)$,

(ii) $\begin{pmatrix} 1 & 0 \\ \theta-\epsilon\theta^* & 1 \end{pmatrix}$ for any split $(-\epsilon)$-quadratic form (Λ^k,θ). $\square$

LEMMA 10.7 *For any $(-\epsilon)$-quadratic form $(\Lambda^k,\theta\in Q_{-\epsilon}(\Lambda^k))$*

$$\begin{pmatrix} 1 & \theta-\epsilon\theta^* \\ 0 & 1 \end{pmatrix}\in EU_{\epsilon,k}(\Lambda) \ .$$

PROOF: This is immediate from the identity

$$\begin{pmatrix} 1 & \theta-\epsilon\theta^* \\ 0 & 1 \end{pmatrix} \ = \ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ \theta-\epsilon\theta^* & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \qquad \square$$

Use the identifications

$$\Lambda^{k+1} \ = \ \Lambda^k\oplus\Lambda \ , \ H_{\epsilon}(\Lambda^{k+1}) \ = \ H_{\epsilon}(\Lambda^k)\oplus H_{\epsilon}(\Lambda)$$

to define injections of groups

$$U_{\epsilon,k}(\Lambda)\to U_{\epsilon,k+1}(\Lambda) \ ; \ f \ \longmapsto f\oplus 1,$$

such that $EU_{\epsilon,k}(\Lambda)$ is sent into $EU_{\epsilon,k+1}(\Lambda)$.

DEFINITION 10.8 (i) The *stable ϵ-quadratic unitary group* of Λ is the union

$$U_{\epsilon}(\Lambda) \ = \ \bigcup_{k=1}^{\infty}U_{\epsilon,k}(\Lambda) \ .$$

(ii) The *elementary stable ϵ-quadratic unitary group* of Λ is the union

$$EU_{\epsilon}(\Lambda) \ = \ \bigcup_{k=1}^{\infty}EU_{\epsilon,k}(\Lambda) \ ,$$

a normal subgroup of $U_{\epsilon}(\Lambda)$.

(iii) The *ϵ-quadratic M-group* of Λ is the quotient

$$M_{\epsilon}(\Lambda) \ = \ U_{\epsilon}(\Lambda)/\{EU_{\epsilon}(\Lambda),\sigma_{\epsilon}\}$$

with $\sigma_\epsilon = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix} \in U_{\epsilon,1}(\Lambda) \subseteq U_\epsilon(\Lambda)$. □

The automorphism group $M_\epsilon(\Lambda)$ is the original definition due to Wall [29, Chap. 6] of the odd-dimensional L-group $L_{2n+1}(\Lambda)$, with $\epsilon = (-1)^n$. The original verification that $M_\epsilon(\Lambda)$ is abelian used a somewhat complicated matrix identity ([29, p.66]), corresponding to the formation identity 9.14. Formations will now be used to identify $M_{(-1)^n}(\Lambda)$ with the a priori abelian L-group $L_{2n+1}(\Lambda)$ defined in §8.

Given an automorphism of a hyperbolic $(-1)^n$-quadratic form

$$\alpha \ = \ \begin{pmatrix} \gamma & \widetilde{\gamma} \\ \mu & \widetilde{\mu} \end{pmatrix} \ : \ H_{(-1)^n}(\Lambda^k) \xrightarrow{\cong} H_{(-1)^n}(\Lambda^k)$$

define a $(2n+1)$-complex (C, ψ) by

$$d \ = \ \mu^* \ : \ C_{n+1} \ = \ \Lambda^k \to C_n \ = \ \Lambda^k \ ,$$

$$\psi_0 \ = \ \gamma \ : \ C^n \ = \ \Lambda^k \to C_{n+1} \ = \ \Lambda^k \ ,$$

corresponding to the $(-1)^n$-quadratic formation

$$\Phi_k(\alpha) \ = \ (H_{(-1)^n}(\Lambda^k); \Lambda^k, \mathrm{im}(\begin{pmatrix} \gamma \\ \mu \end{pmatrix} : \Lambda^k \to \Lambda^k \oplus (\Lambda^k)^*)) \ .$$

LEMMA 10.9 *The formations $\Phi_k(\alpha_1)$, $\Phi_k(\alpha_2)$ associated to two automorphisms*

$$\alpha_i \ = \ \begin{pmatrix} \gamma_i & \widetilde{\gamma}_i \\ \mu_i & \widetilde{\mu}_i \end{pmatrix} \ : \ H_{(-1)^n}(\Lambda^k) \xrightarrow{\cong} H_{(-1)^n}(\Lambda^k) \quad (i = 1, 2)$$

are isomorphic if and only if there exist $\beta_i \in GL_k(\Lambda)$, $\theta_i \in S(\Lambda^k)$ such that

$$\begin{aligned} &\begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_1^{*-1} \end{pmatrix} \begin{pmatrix} 1 & \theta_1 + (-1)^{n+1}\theta_1^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 & \widetilde{\gamma}_1 \\ \mu_1 & \widetilde{\mu}_1 \end{pmatrix} \\ &\qquad = \ \begin{pmatrix} \gamma_2 & \widetilde{\gamma}_2 \\ \mu_2 & \widetilde{\mu}_2 \end{pmatrix} \begin{pmatrix} \beta_2 & 0 \\ 0 & \beta_2^{*-1} \end{pmatrix} \begin{pmatrix} 1 & \theta_2 + (-1)^{n+1}\theta_2^* \\ 0 & 1 \end{pmatrix} \\ &\qquad\qquad : \ H_{(-1)^n}(\Lambda^k) \xrightarrow{\cong} H_{(-1)^n}(\Lambda^k) \ . \end{aligned}$$

PROOF: An automorphism α of the hyperbolic $(-1)^n$-quadratic form $H_{(-1)^n}(\Lambda^k)$ preserves the lagrangian $\Lambda^k \subset \Lambda^k \oplus (\Lambda^k)^*$ if and only if there exist $\beta \in GL_k(\Lambda)$, $\theta \in S(\Lambda^k)$ such that

$$\alpha \ = \ \begin{pmatrix} \beta & 0 \\ 0 & \beta^{*-1} \end{pmatrix} \begin{pmatrix} 1 & \theta + (-1)^{n+1}\theta^* \\ 0 & 1 \end{pmatrix} \ : \ H_{(-1)^n}(\Lambda^k) \xrightarrow{\cong} H_{(-1)^n}(\Lambda^k) \ .$$

□

PROPOSITION 10.10 *The function*

$$\Phi \ : \ M_{(-1)^n}(\Lambda) \to L_{2n+1}(\Lambda) \ ; \ \alpha \ \longmapsto \Phi_k(\alpha) \quad (\alpha \in U_{(-1)^n,k}(\Lambda))$$

is an isomorphism of groups.

PROOF: The function

$$\Phi_k \ : \ U_{(-1)^n,k}(\Lambda) \to L_{2n+1}(\Lambda) \ ; \ \alpha \longmapsto \Phi_k(\alpha)$$

is a group morphism by 9.14. Each of the generators (10.6) of the elementary subgroup $EU_{(-1)^n,k}(\Lambda)$ is sent to 0 with

(i) $\Phi_k \begin{pmatrix} \beta & 0 \\ 0 & \beta^{*-1} \end{pmatrix} = (H_{(-1)^n}(\Lambda^k); \Lambda^k, \Lambda^k) = \partial(\Lambda^k, 0, 0) = 0 \in L_{2n+1}(\Lambda),$

(ii) $\Phi_k \begin{pmatrix} 1 & 0 \\ \theta + (-1)^{n+1}\theta^* & 1 \end{pmatrix} = \partial(\Lambda^k, \theta + (-1)^{n+1}\theta^*, \theta) = 0 \in L_{2n+1}(\Lambda).$

Also, abbreviating $\sigma_{(-1)^n}$ to σ

$$\Phi_1(\sigma) \ = \ (H_{(-1)^n}(\Lambda); \Lambda, \Lambda^*) \ = \ 0 \ ,$$

$$\Phi_{k+1}(\alpha \oplus 1) \ = \ \Phi_k(\alpha) \oplus (H_{(-1)^n}(\Lambda); \Lambda, \Lambda) \ = \ \Phi_k(\alpha) \in L_{2n+1}(\Lambda) \ .$$

Thus the morphisms Φ_k $(k \geq 0)$ fit together to define a group morphism

$$\Phi \ : \ M_{(-1)^n}(\Lambda) \to L_{2n+1}(\Lambda) \ ; \ \alpha \longmapsto \Phi_k(\alpha) \text{ if } \alpha \in U_{(-1)^n,k}(\Lambda)$$

such that

$$\Phi(\alpha_1\alpha_2) \ = \ \Phi(\alpha_1 \oplus \alpha_2) \ = \ \Phi(\alpha_1) \oplus \Phi(\alpha_2) \in L_{2n+1}(\Lambda) \ .$$

Φ is onto by 9.3 (ii). It remains to prove that Φ is one-one.

For any $\alpha_i \in U_{(-1)^n,k_i}(\Lambda)$ $(i = 1, 2)$

$$\alpha_1 \oplus \alpha_2 \ = \ \alpha_2 \oplus \alpha_1 \in M_{(-1)^n}(\Lambda) \ ,$$

since

$$\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \ = \ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$: \ H_{(-1)^n}(\Lambda^{k_1+k_2}) \to H_{(-1)^n}(\Lambda^{k_1+k_2}) \ .$$

Now $\sigma = 1 \in M_{(-1)^n}(\Lambda)$ (by construction), so that for any $\alpha \in U_{(-1)^n,k}(\Lambda)$

$$\alpha \oplus \sigma \ = \ \sigma \oplus \alpha \ = \ (\sigma \oplus 1)(1 \oplus \alpha) \ = \ \alpha \in M_{(-1)^n}(\Lambda) \ .$$

It follows that for every $m \geq 1$

$$\sigma \oplus \sigma \oplus \ldots \oplus \sigma \ = \ 1 \in M_{(-1)^n}(\Lambda) \ (m\text{-fold sum}) \ .$$

If $\alpha \in U_{(-1)^n,k}(\Lambda)$ is such that $\Phi(\alpha) = 0 \in L_{2n+1}(\Lambda)$ then by 9.12 the $(-1)^n$-quadratic formation $\Phi_k(\alpha)$ is stably isomorphic to the boundary $\partial(\Lambda^{k'}, \lambda, \mu)$ of a $(-1)^{n+1}$-quadratic form $(\Lambda^{k'}, \lambda, \mu)$. Choosing a split form

$\theta \in S(\Lambda^{k'})$ for (λ, μ) this can be expressed as

$$\partial(\Lambda^{k'}, \lambda, \mu) \;=\; \Phi_{k'} \begin{pmatrix} 1 & 0 \\ \theta + (-1)^{n+1}\theta^* & 1 \end{pmatrix} .$$

Thus for a sufficiently large $k'' \geq 0$ there exist by 10.9 $\beta_i \in GL_{k''}(\Lambda)$, $\theta_i \in S(\Lambda^{k''})$ $(i = 1, 2)$ such that

$$\begin{aligned} &\begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_1^{*-1} \end{pmatrix} \begin{pmatrix} 1 & \theta_1 + (-1)^{n+1}\theta_1^* \\ 0 & 1 \end{pmatrix} (\alpha \oplus \sigma \oplus \ldots \oplus \sigma) \\ &= \left(\begin{pmatrix} 1 & 0 \\ \theta + (-1)^{n+1}\theta^* & 1 \end{pmatrix} \oplus \sigma \oplus \ldots \oplus \sigma \right) \begin{pmatrix} \beta_2 & 0 \\ 0 & \beta_2^{*-1} \end{pmatrix} \\ &\qquad \begin{pmatrix} 1 & \theta_2 + (-1)^{n+1}\theta_2^* \\ 0 & 1 \end{pmatrix} \;:\; H_{(-1)^n}(\Lambda^{k''}) \to H_{(-1)^n}(\Lambda^{k''}) \end{aligned}$$

so that by another application of 10.7

$$\alpha \;=\; \begin{pmatrix} 1 & 0 \\ \theta + (-1)^{n+1}\theta^* & 1 \end{pmatrix} \;=\; 1 \in M_{(-1)^n}(\Lambda) \;,$$

verifying that Φ is one-one. □

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